# Algebra and Computation

## Problem Set 1

*Due date: March $1^{st}$, 2017*

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### Instructions

1. The problem set has **6 questions** with a total score of **60 points**.
2. You are welcome to collaborate with other classmates. But if you do, please mention who all you collaborated with.

   I'd suggest you discuss only after you have spent enough time thinking about the problems independently.
3. Solutions are expected as a LaTeX document. You may use this very file by obtaining the source files from megh.
4. The deadline is 1st March 2017 (Wednesday) 2359 hrs IST. For each day of delay, you lose **7 points** of your total score in this assignment. So if you plan to delay, be smart about it.

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### Questions

**Question 1.** *Let $G$ be a finite group such that let $p$ be a prime dividing the size of $G$. If $K \leq G$ with $|K| = p^r$, show that there is some Sylow $p$-subgroup $P \leq G$ such that $K \leq P$.* **(5 points)**

**Question 2.** *Let $H \trianglelefteq G$. Show that subgroups $K \leq G$ that contain $H$ are in one-to-one correspondence with subgroups $K' \leq \frac{G}{H}$.*

*Furthemore, normal subgroups $K \trianglelefteq G$ that contain $H$ are in one-to-one correspondence with normal subgroups $K' \trianglelefteq \frac{G}{H}$.* **(5 points)**

**Question 3.** *Prove the following two facts we used in Lecture 5.*

*(a).* **(5 points)** *If $G$ normalizes $H$ and $K \leq G$, show that $[GH : KH] \leq [G : K]$.*

*(b).* **(5 points)** *If $K \leq G$, show that $[G \cap H : K \cap H] \leq [G : K]$.*

**Question 4.** *Let $P$ be a $p$-group, i.e. $|P| = p^r$ where $p$ is a prime and $r > 1$ an integer.*

*(a).* **(5 points)** *Show that $P$ has a non-trivial* centre. *That is,*

$$\mathrm{Centre}(P) := \{g \in P \ : \ gh = hg \ \forall \ h \in P\}$$

*is a non-trivial subgroup of $P$.*

*(b).* **(5 points)** *Show that every subgroup $Q \leq P$ is subnormal in $P$.*

**Question 5.** *If $G \leq S_n$ is a finite group, and $h \in S_n$ define the* centralizer of $h$ in $G$ *(denoted by $\mathrm{C}_G(h)$) to be*

$$\mathrm{C}_G(h) := \{g \in G \ : \ gh = hg\}\,.$$

*Consider the task* Centralizer$(S, h)$ *where the input is a group $G = \langle S \rangle \leq S_n$ and $h \in S_n$ and the goal is to compute a generating set of $\mathrm{C}_G(h)$.*

*(a).* **(10 points)** *Show that* Centralizer *is at least as hard as* SetStab.

*(b).* **(10 points)** *Show that* SetStab *is at least as hard as* Centralizer.

**Question 6.** *Using the following hint (or not), prove that any subgroup $G \leq S_n$ has a generating set of size at most $(n-1)$, and that it can be computed efficiently given a generating set for $G$.* **(10 points)**

> *For any non-trivial permutation $g \in S_n$, let $\ell(g)$ be the smallest $i \in \{1, \ldots, n\}$ moved by $g$, i.e. $\ell(g) = \min\{i \ : \ i^g \neq i\}$.*
>
> *Given a set $A$ of permutations, define the graph $X_A = (V, E_A)$ as*
>
> $$V = \{1, \ldots, n\} \qquad \text{and} \qquad E_A = \{(i, i^g) \ : \ g \in A \ , \ i = \ell(g)\}\,.$$
>
> *If $X_A$ has no cycles in it, then of course $|A| \leq (n-1)$.*