Unified PITs via the Jacobian

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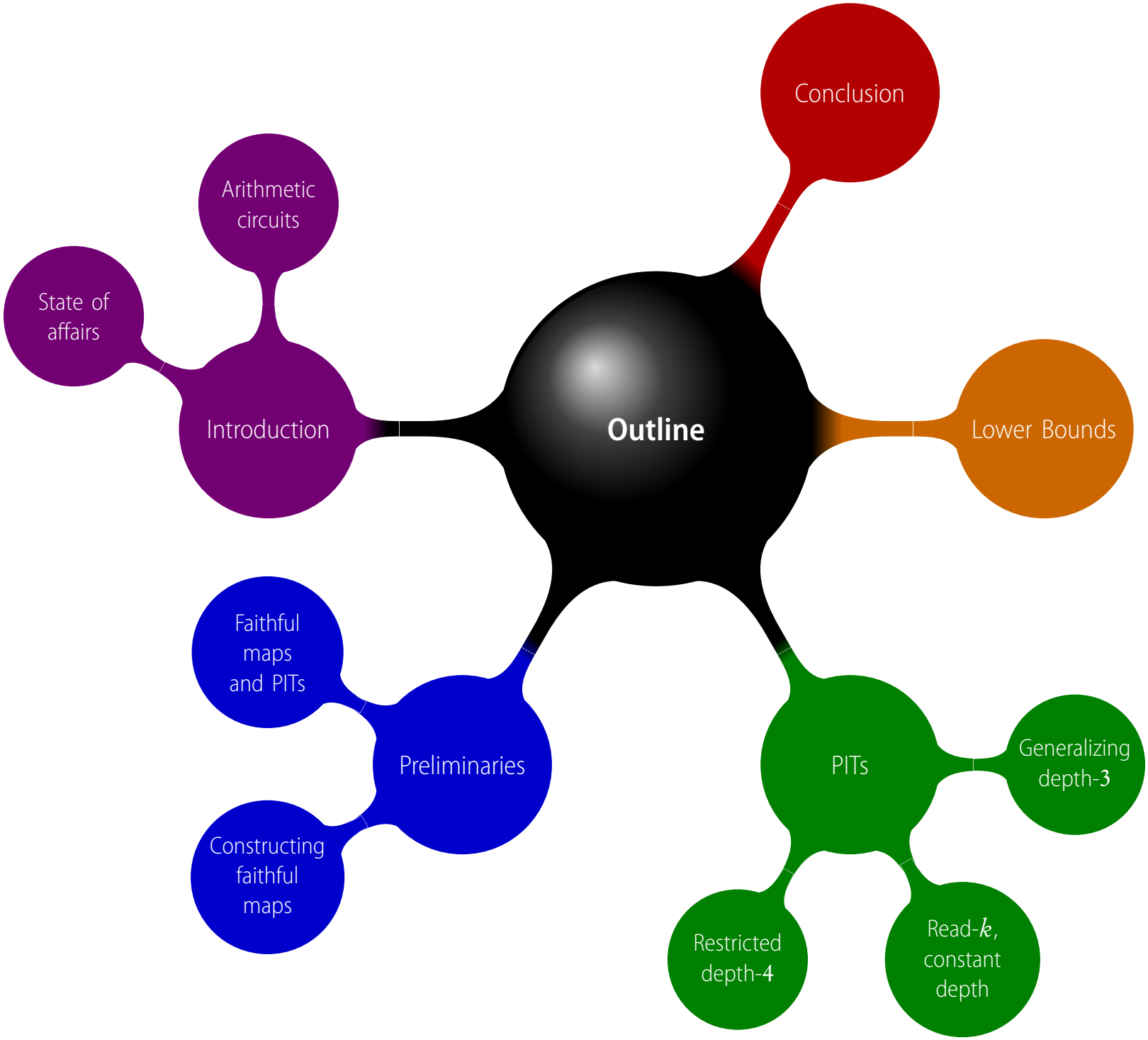
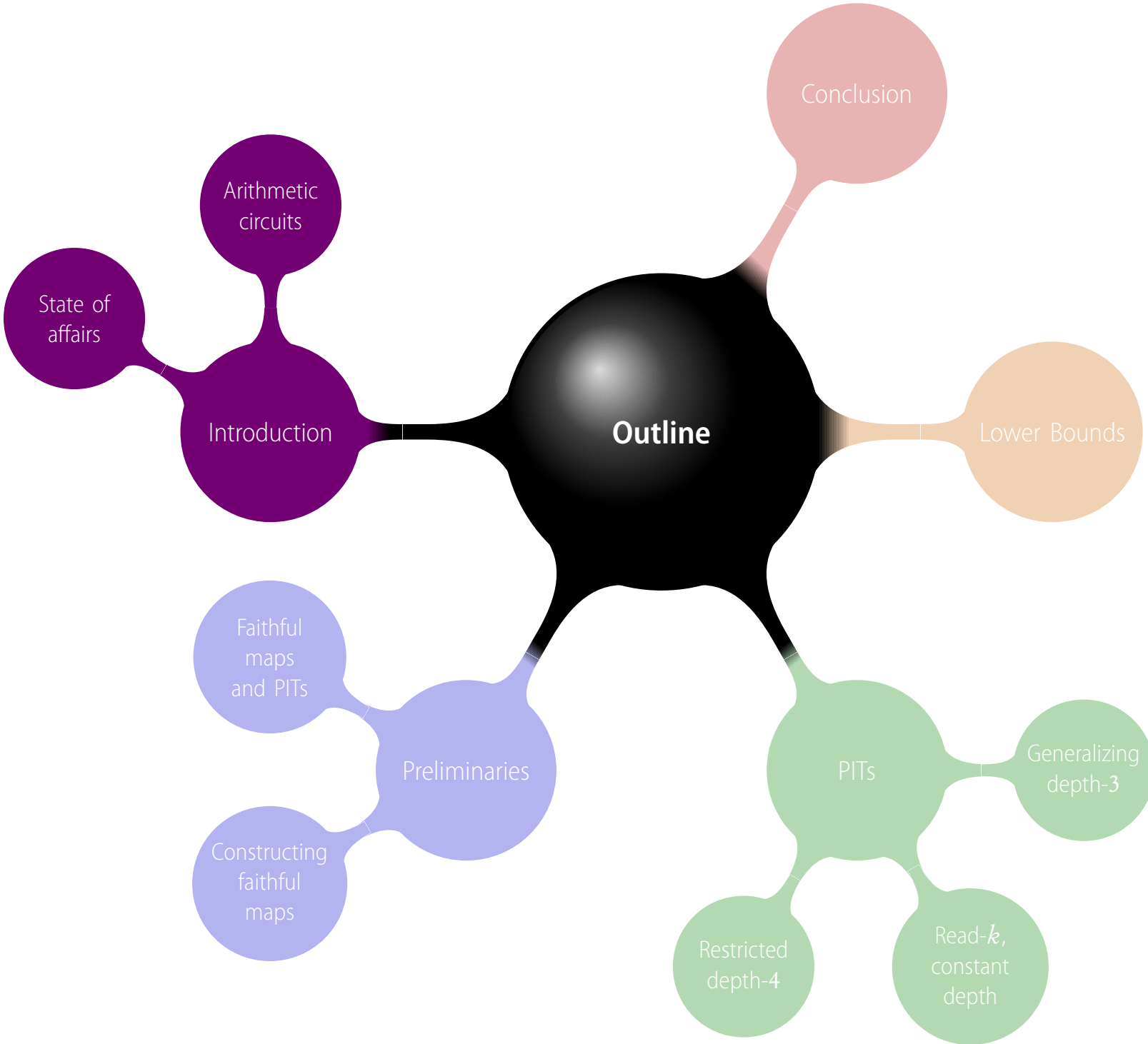

Outline
Introduction
Arithmetic circuits
State of affairs
Preliminaries
Faithful maps and PITs
Constructing faithful maps
Lower Bounds
PITs
Generalizing depth-3
Read-k, constant depth
Restricted depth-4
Conclusion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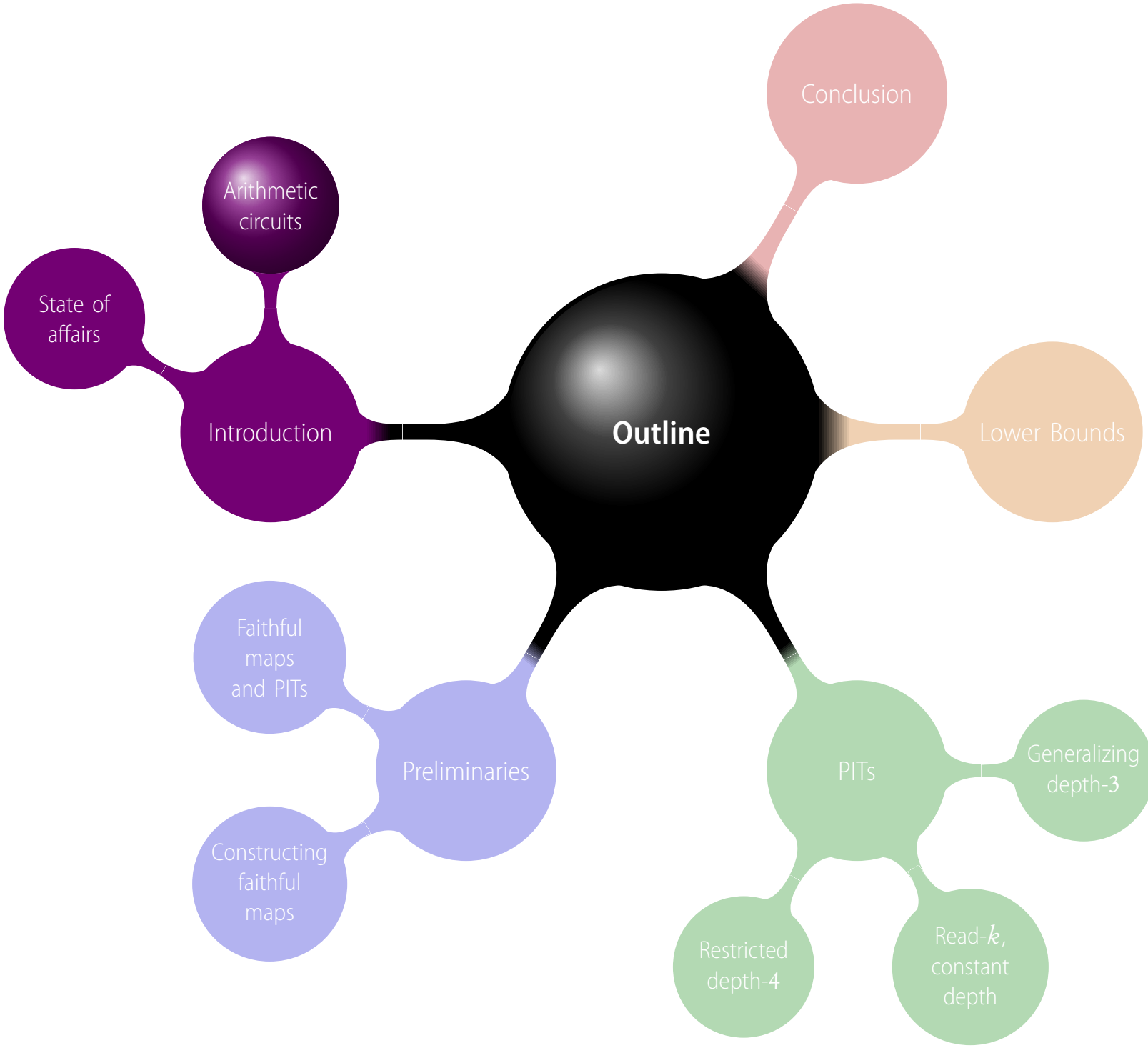
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$$
\begin{aligned}
f(x_1, x_2, x_3, x_4) \quad = \quad & 1 + x_1 + x_2 + x_3 + x_4 \\
& + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 \\
& + x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_1x_2x_3 \\
& + x_1x_2x_3x_4
\end{aligned}
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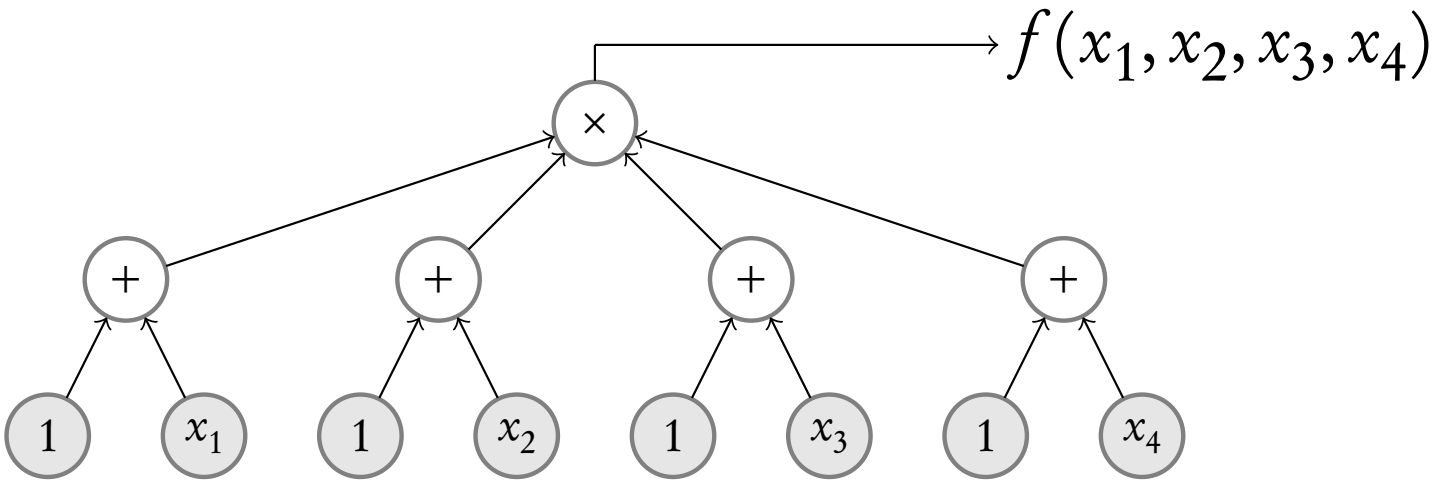
&\quad + x_1x_2x_3x_4 \\
&= (1 + x_1)(1 + x_2)(1 + x_3)(1 + x_4)
\end{aligned}
$$

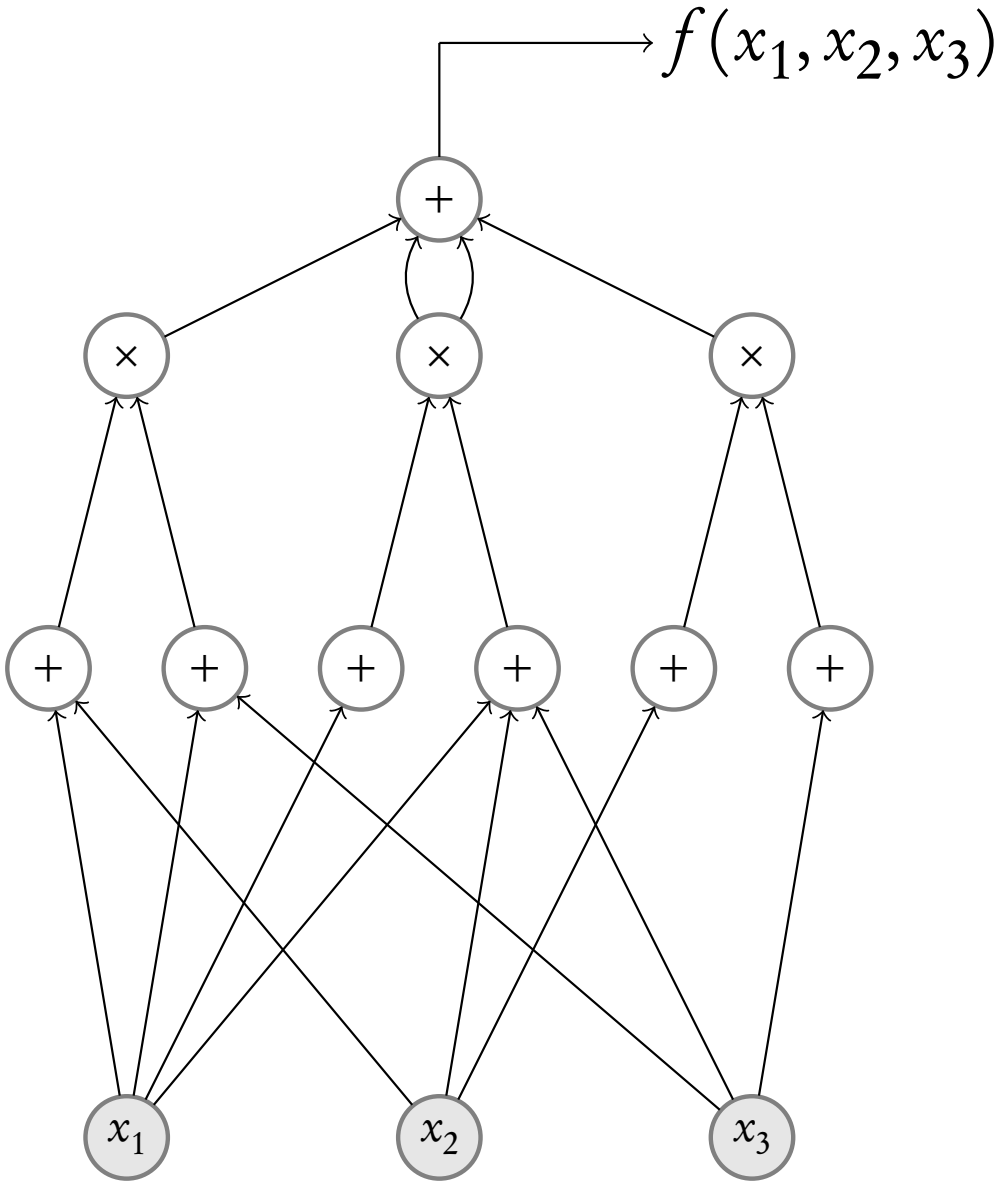
... certainly a more compact representation.

Arithmetic Formulae



- Tree
- Leaves containing variables or constants

Arithmetic Circuits



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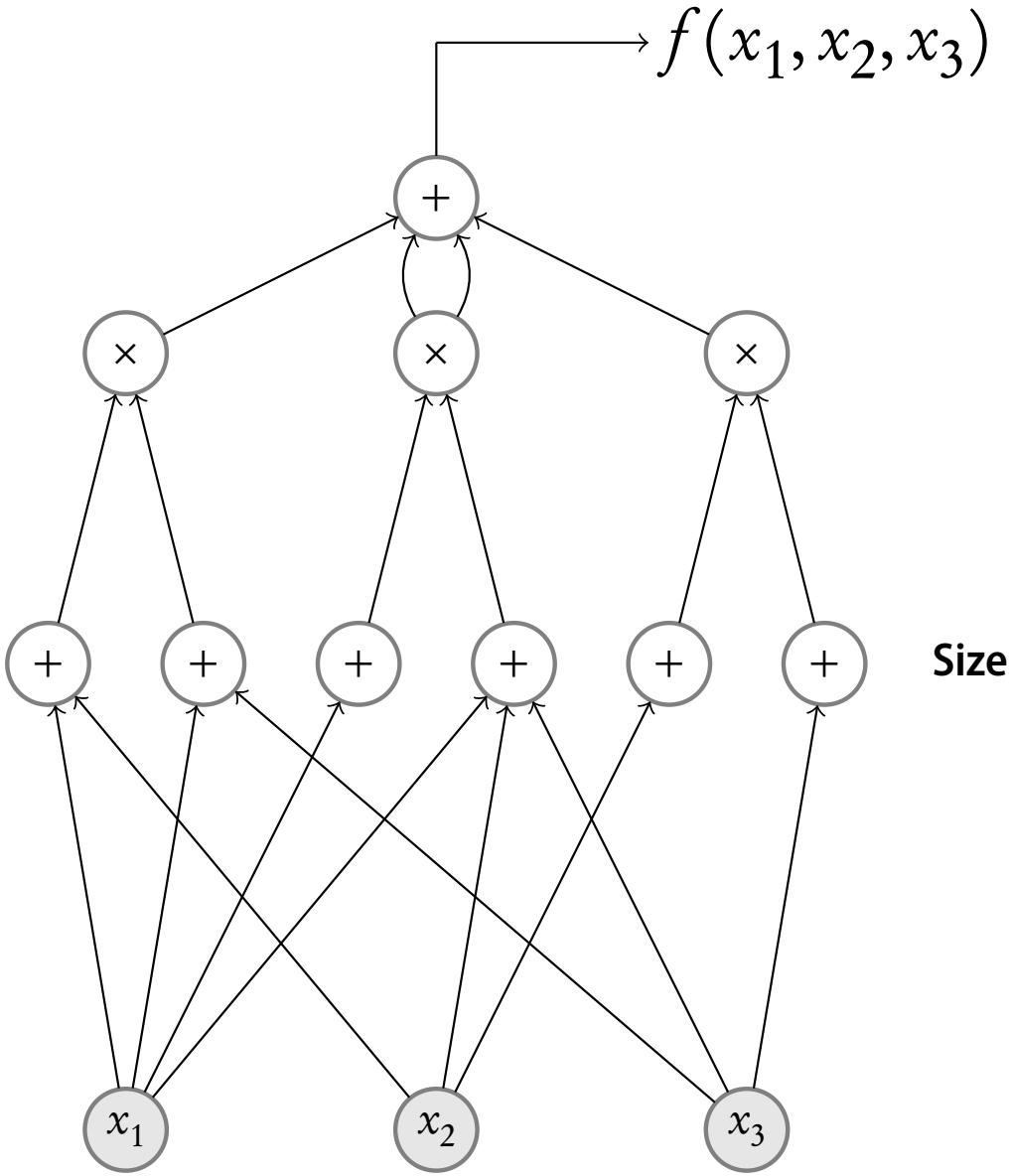
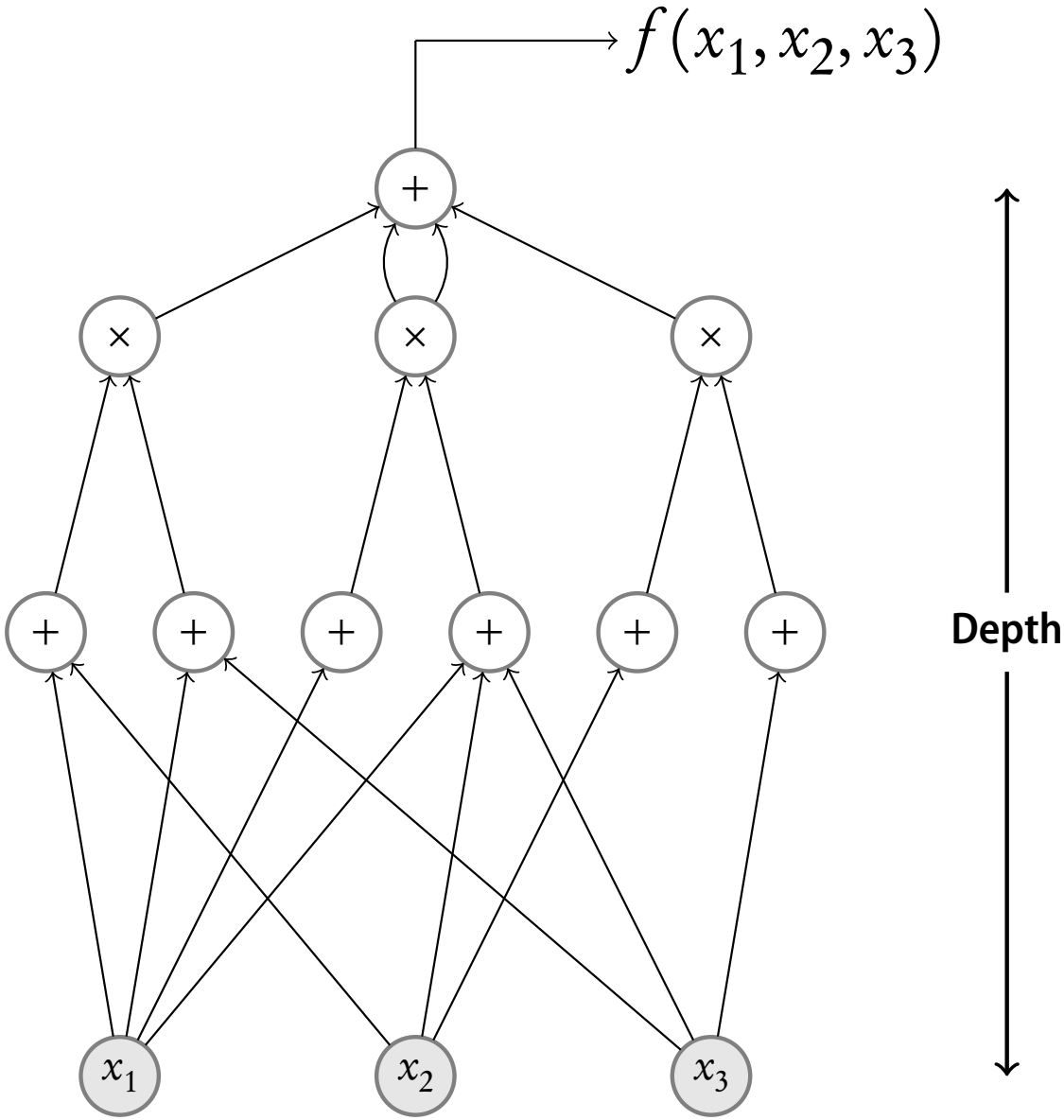
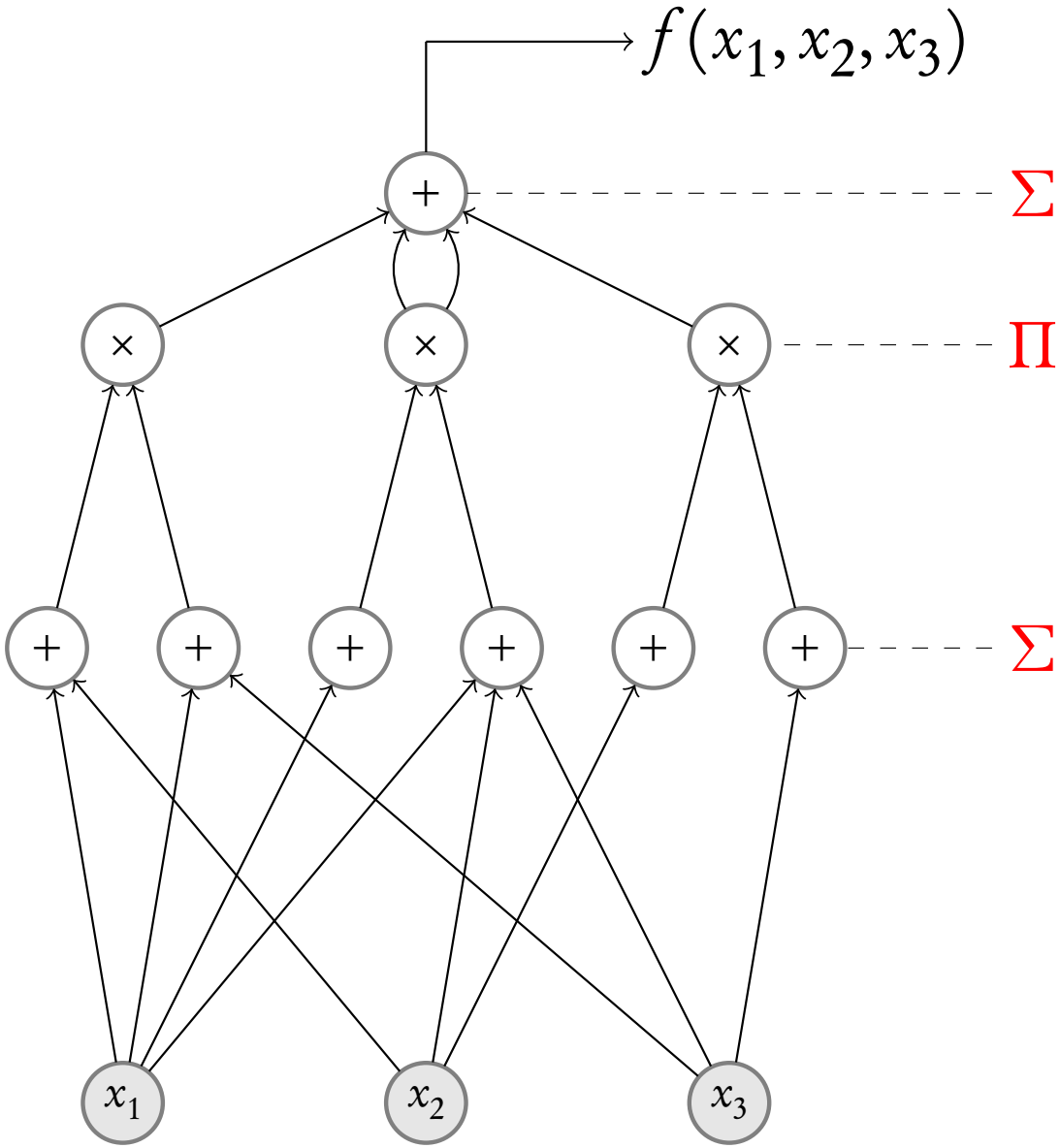


Size = number of gates

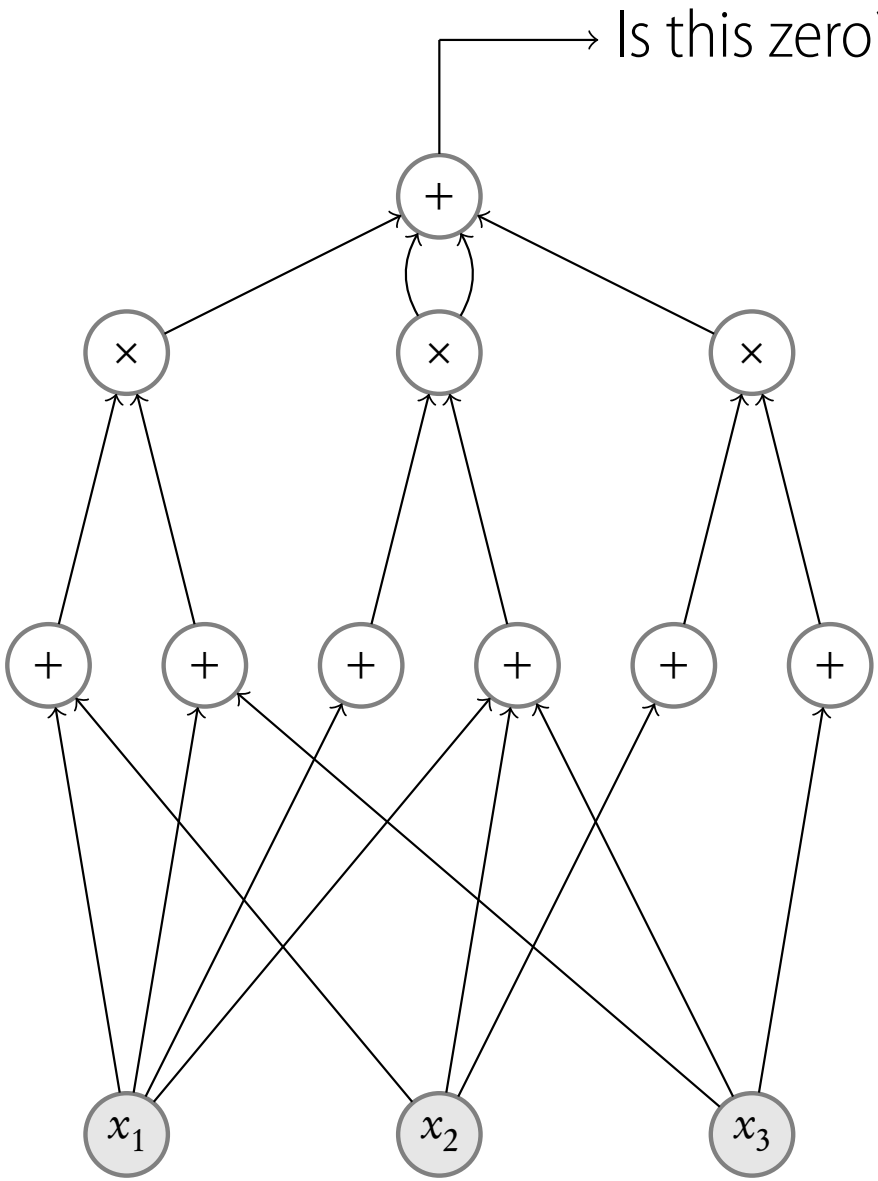
Arithmetic Circuits



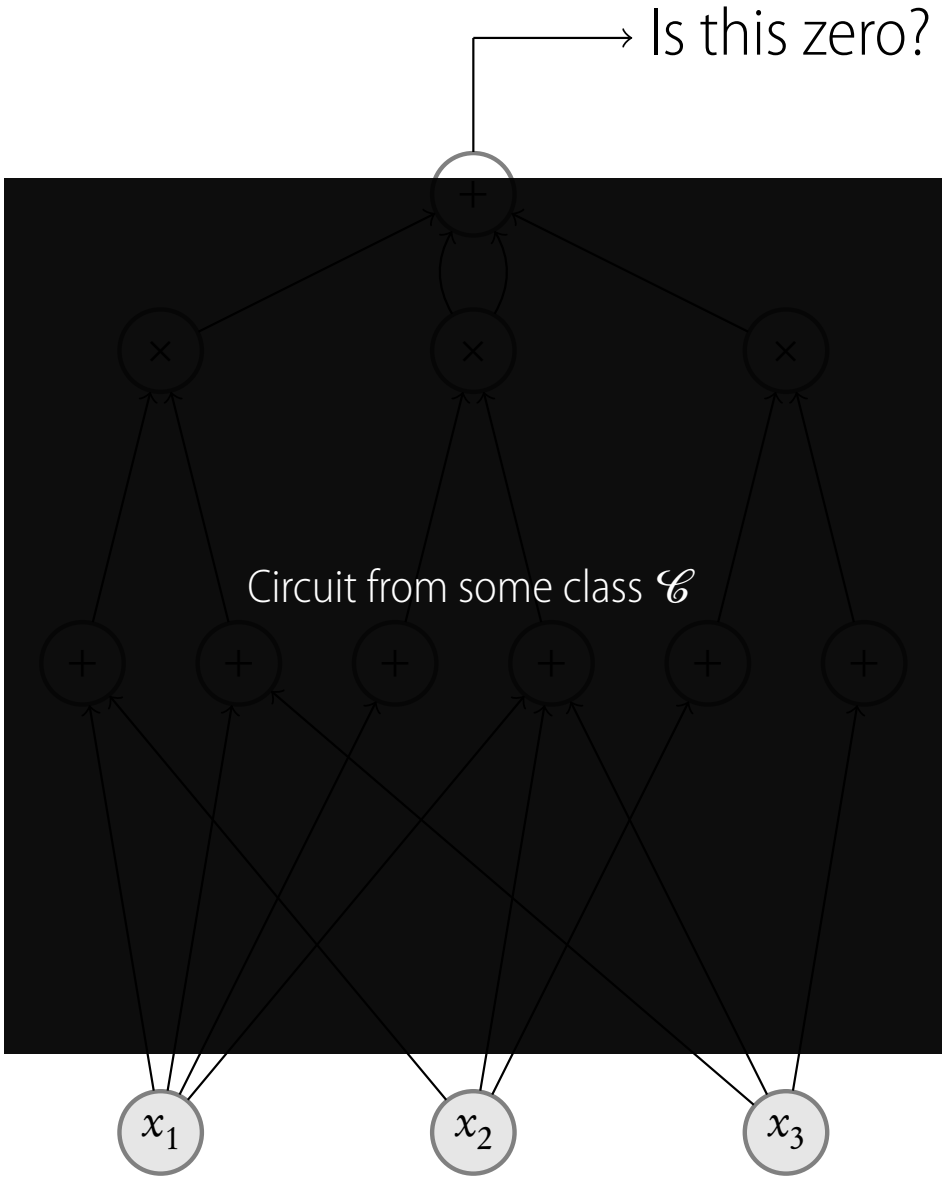
Arithmetic Circuits



Identity Testing of Arithmetic Circuits



Black-box Identity Testing of Arithmetic Circuits



The [Schwartz-Zippel-DeMillo-Lipton] Lemma

Lemma

Let f be a non-zero polynomial of degree d, and let $S \subseteq \mathbb{F}$. Then,

$$\Pr_{a_i \in S}[f(a_1, \cdots, a_n) = 0] \quad \leq \quad \frac{d}{|S|}$$

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Thus, if $|S| \geq d + 1$, then S^n contains a witness.

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Big Question: If f is computable by a small circuit, do we have polynomial sized hitting set?

Why do we care?

Part of many important results like $\mathbf{IP} = \mathbf{PSPACE}$, the $\mathbf{PCP}$ theorem, AKS primality test, etc.

Connections with lower bounds. [Kabanets-Impagliazzo03], [Agrawal05]: "Efficient PIT algorithms imply lower bounds"

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> *"For the pessimist, this indicates that derandomizing identity testing is a hopeless problem. For the optimist, this means on the contrary that to obtain an arithmetic circuit lower bound, we 'simply' have to prove a good upper bound on identity testing."*
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> *- [Kayal-Saraf09]*

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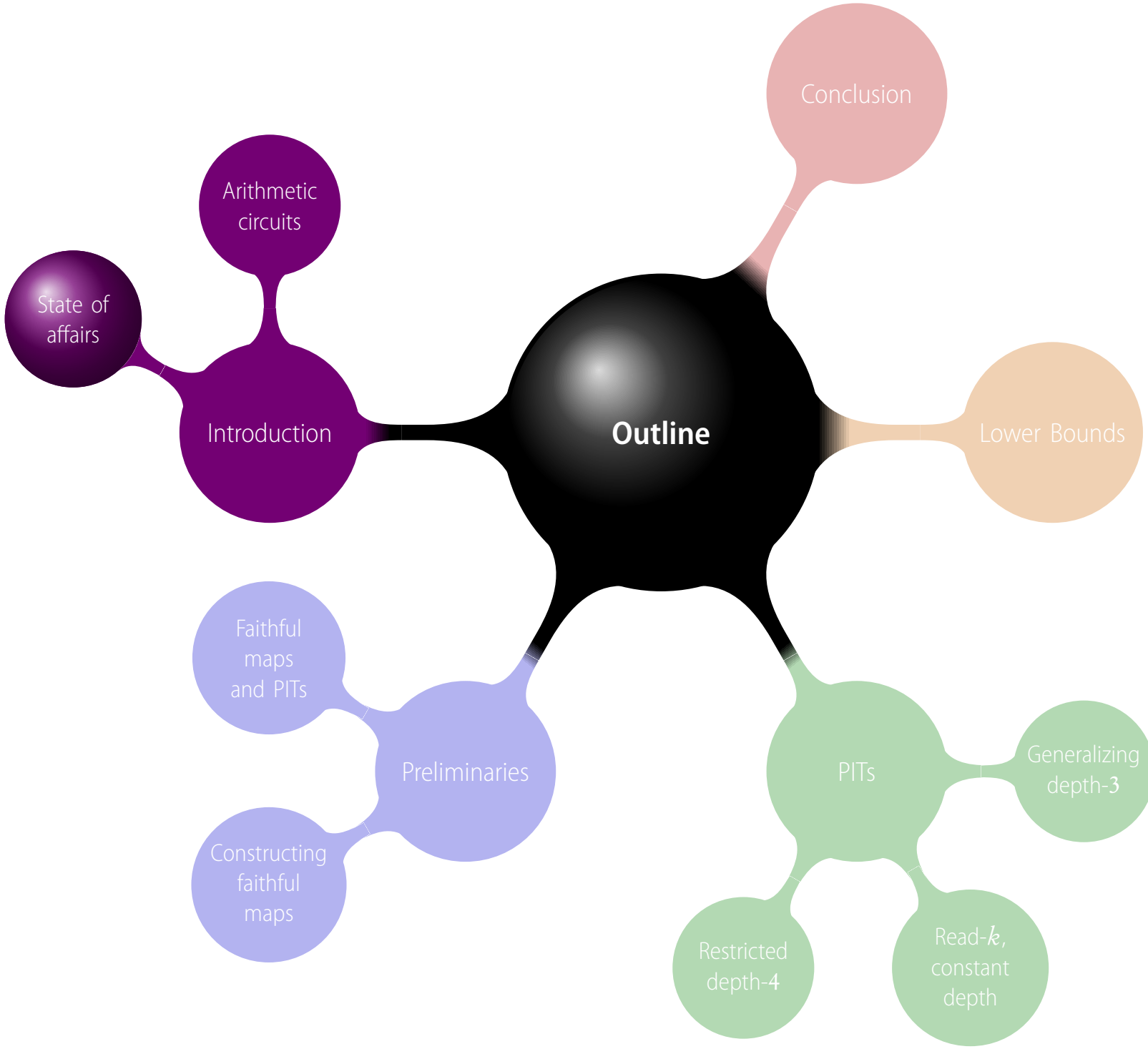
Of course, it is a natural problem!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State of affairs

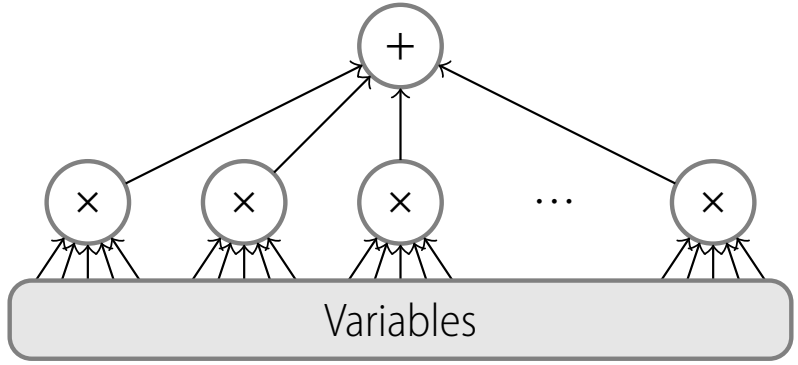
"If you can't solve a problem, then there is an easier problem you can solve: find it."

- George Pólya

Identity tests of restricted types of circuits:

- Formulae:
  1. Bounded depth formulae?
  2. Bounded read formulae?

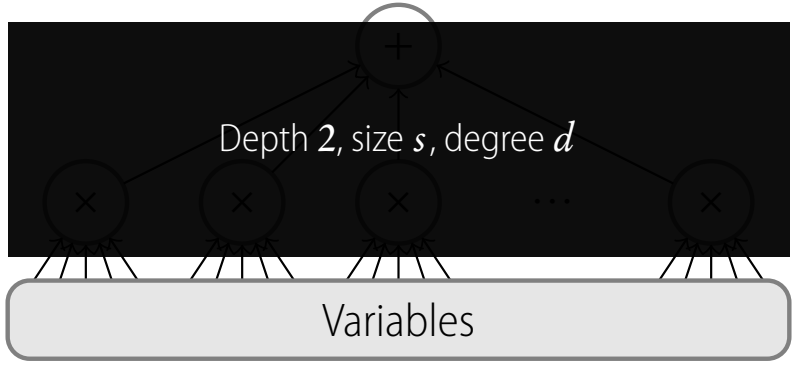
State of affairs for $\Sigma\Pi$ *Circuits*



$$f \quad = \quad \sum_{i=1}^{\text{poly}} \text{monomial}_i$$

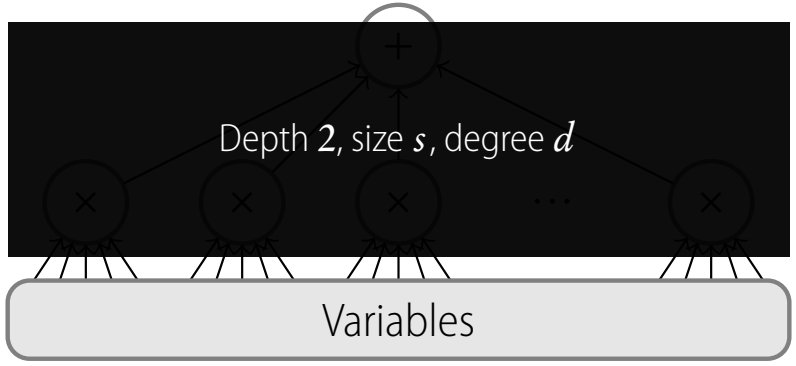
Depth **2** is easy (sparse polynomials)

State of affairs for $\Sigma\Pi$ *Circuits*



Black-box not-too-hard as well.

State of affairs for $\Sigma\Pi$ *Circuits*

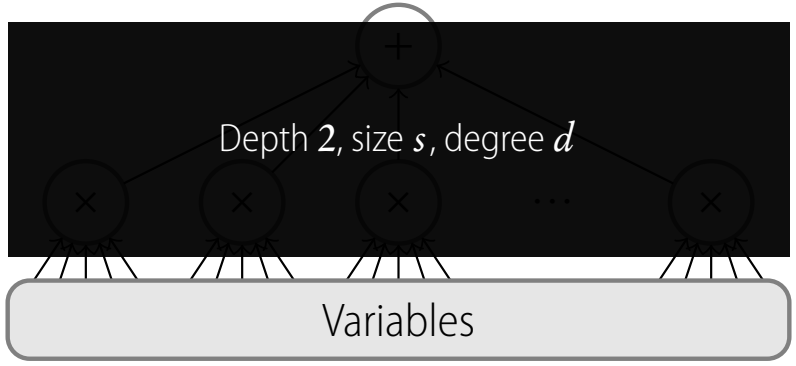


Black-box not-too-hard as well.

$$\Phi: \quad x_i \mapsto u^{(d+1)^i}$$

Works, but exponential degree

State of affairs for $\Sigma\Pi$ Circuits



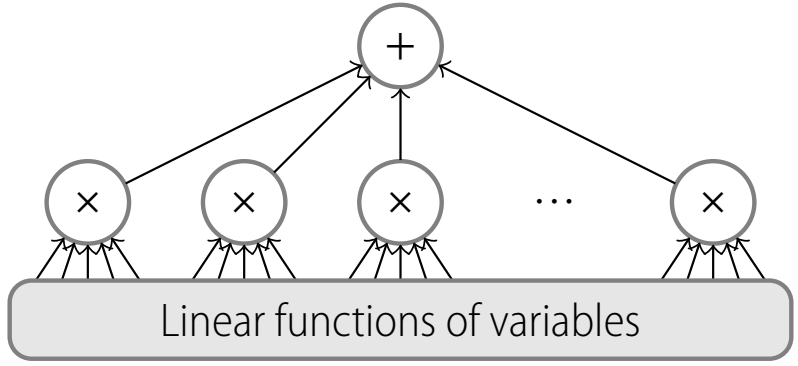
Black-box not-too-hard as well.

$$\Phi_r : \quad x_i \mapsto u^{(d+1)^i \bmod r}$$

Not too many bad r's

Hint: u^a and u^b collide if and only if $r \mid (a-b)$

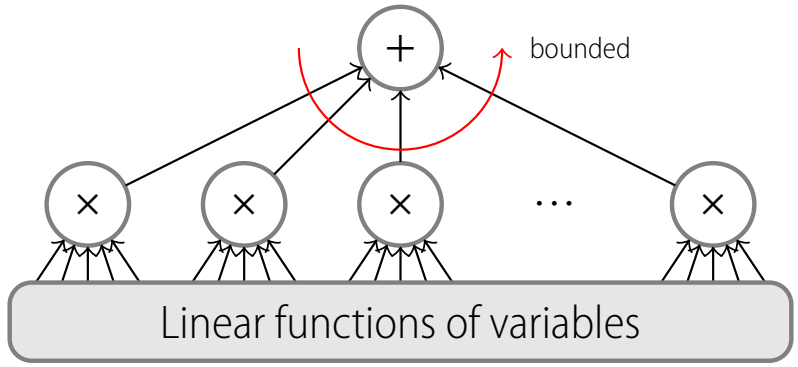
State of affairs for $\Sigma\Pi\Sigma$ *Circuits*



$$f \quad = \quad \sum_{i=1}^{k} \ell_{i1} \cdots \ell_{id}$$

PIT for even depth **3** circuits is open.

State of affairs for $\Sigma\Pi\Sigma(k)$ *Circuits*



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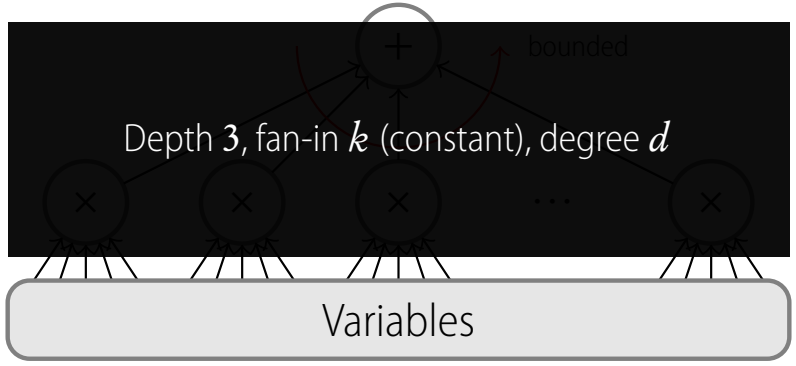
[KayalSaxena07] : PIT in time $\mathbf{poly}(s^k)$

State of affairs for $\Sigma\Pi\Sigma(k)$ *Circuits*



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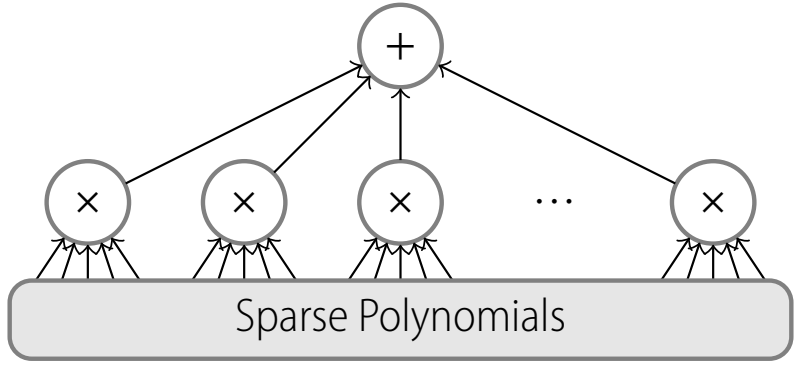
[KayalSaxena07] : PIT in time $\mathbf{poly}(s^k)$

[SaxenaSeshadri11]: Black-box PIT in time $\mathbf{poly}(s^k)$

State of affairs for $\Sigma\Pi\Sigma\Pi$ *Circuits*

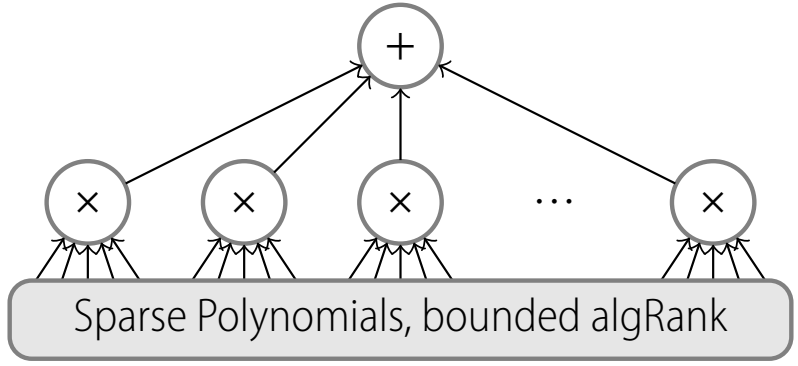


$$f \quad = \quad \sum_{i=1}^{\text{poly}} g_{i1} \cdots g_{id}$$

[AgrawalVinay08] : Black-box PIT for depth **4** implies $n^{O(\log n)}$ black-box PIT for any depth!

Depth **4** is (almost) as hard as the general case.

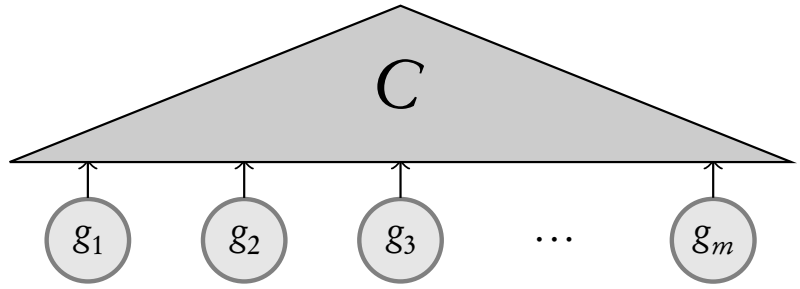
State of affairs for $\Sigma\Pi\Sigma\Pi$ *Circuits*



$$f \quad = \quad \sum_{i=1}^{\text{poly}} g_{i1} \cdots g_{id} \quad \text{with } \mathbf{algRank}\left\{ g_{ij} \right\} \leq k$$

[BeeckenMittmannSaxena11]: Polynomial time black-box PIT

State of affairs for $\Sigma\Pi\Sigma\Pi$ *Circuits*



$$f \quad = \quad C(g_1, \cdots, g_m) \quad \text{with } \mathbf{algRank}\{g_i\} \leq k$$

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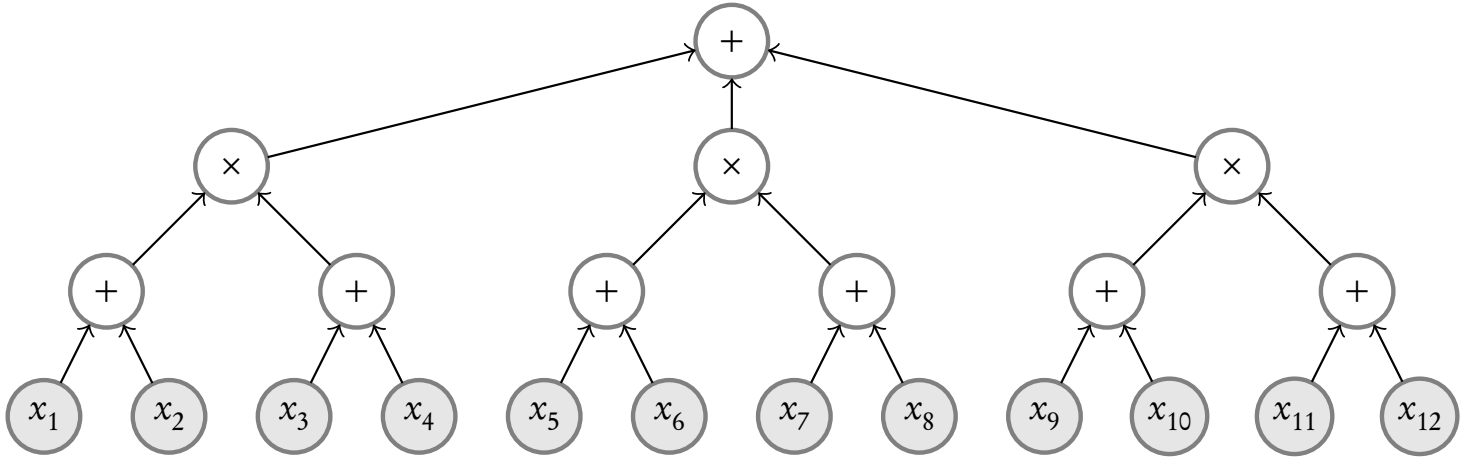
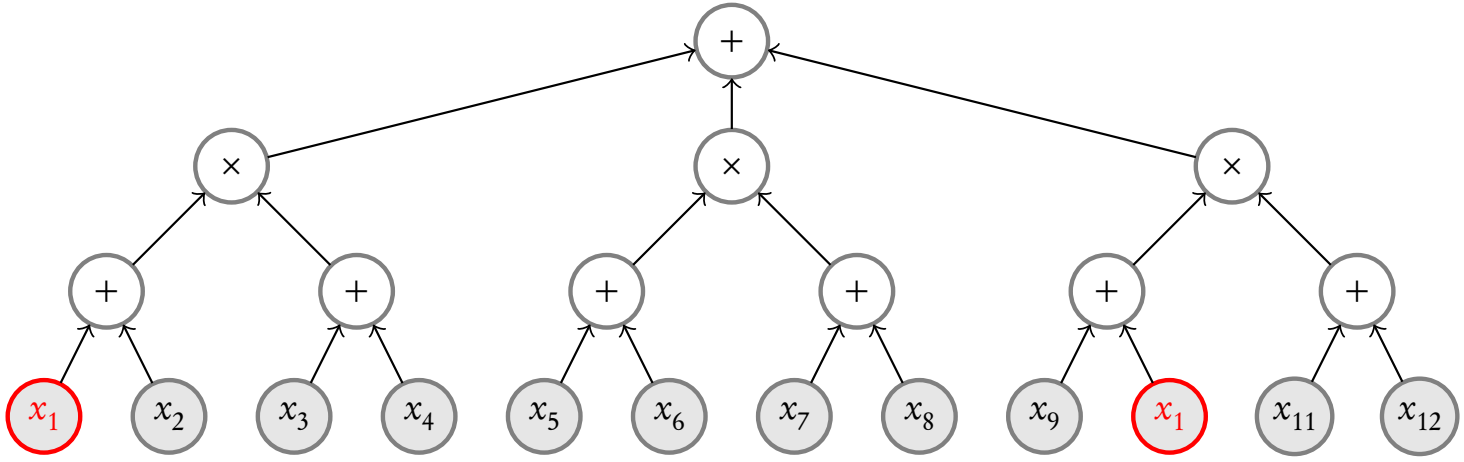
State of affairs for bounded read formulae



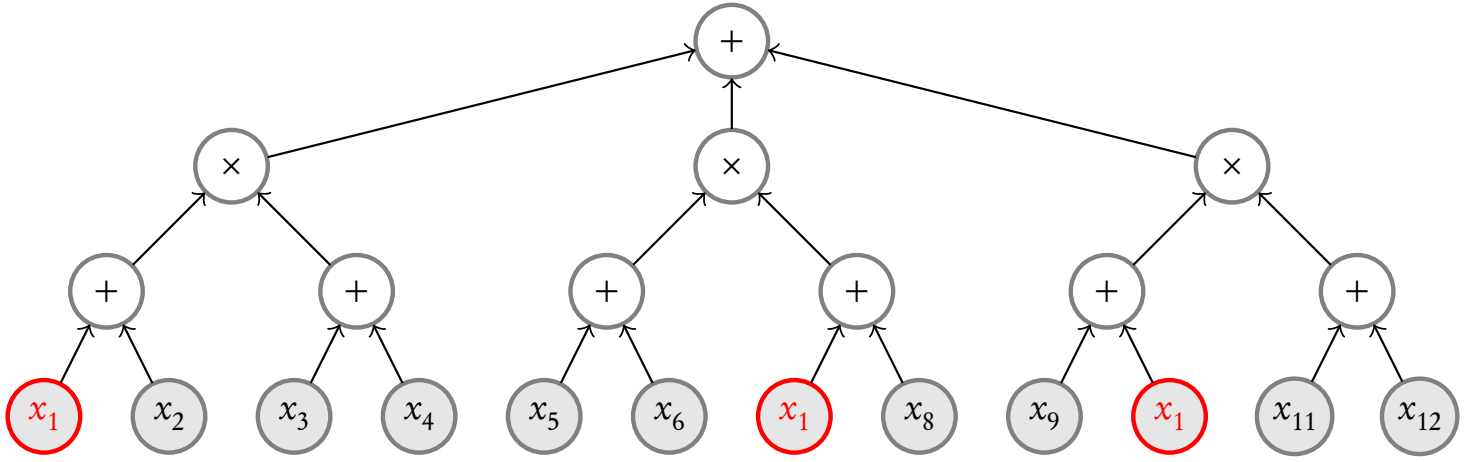
Read-**1** formula

State of affairs for bounded read formulae



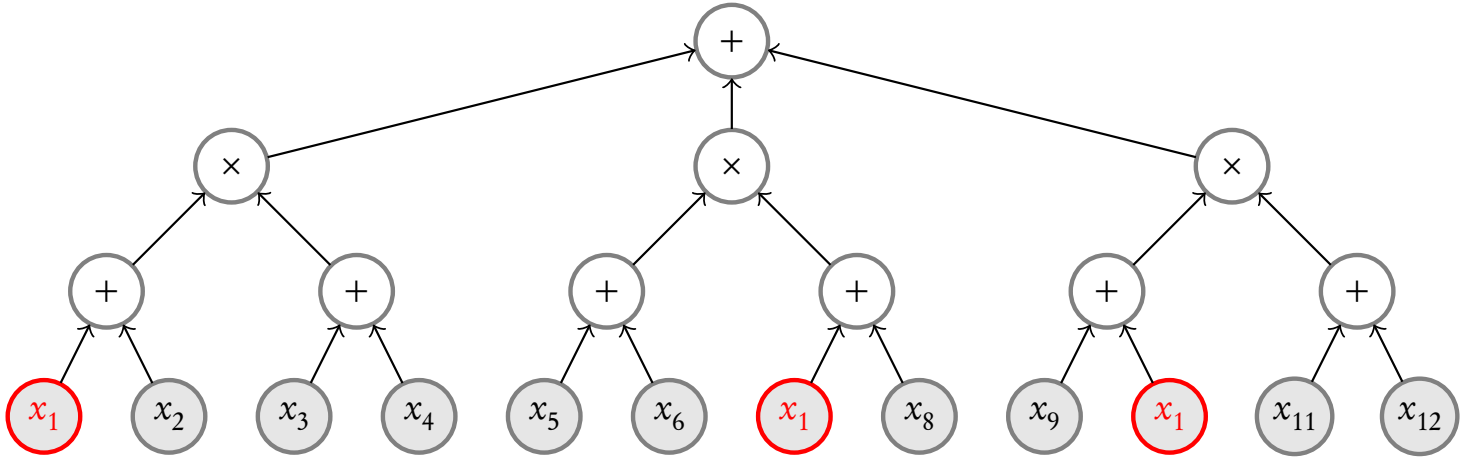
Read-**2** formula

State of affairs for bounded read formulae



Read-**3** formula

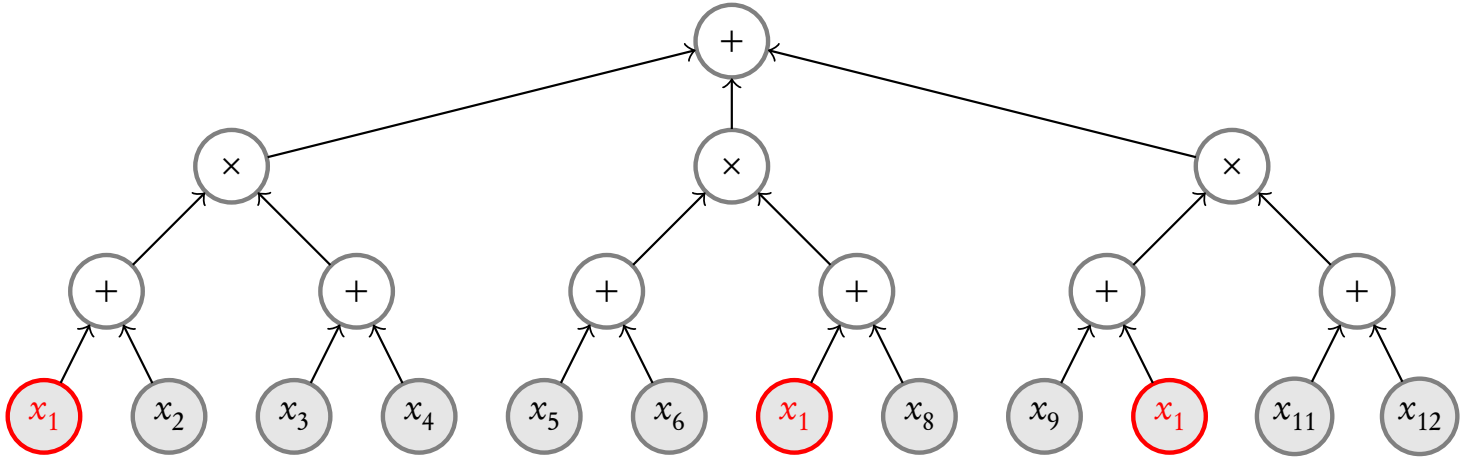
State of affairs for bounded read formulae



Read-k formula

Status of PIT:

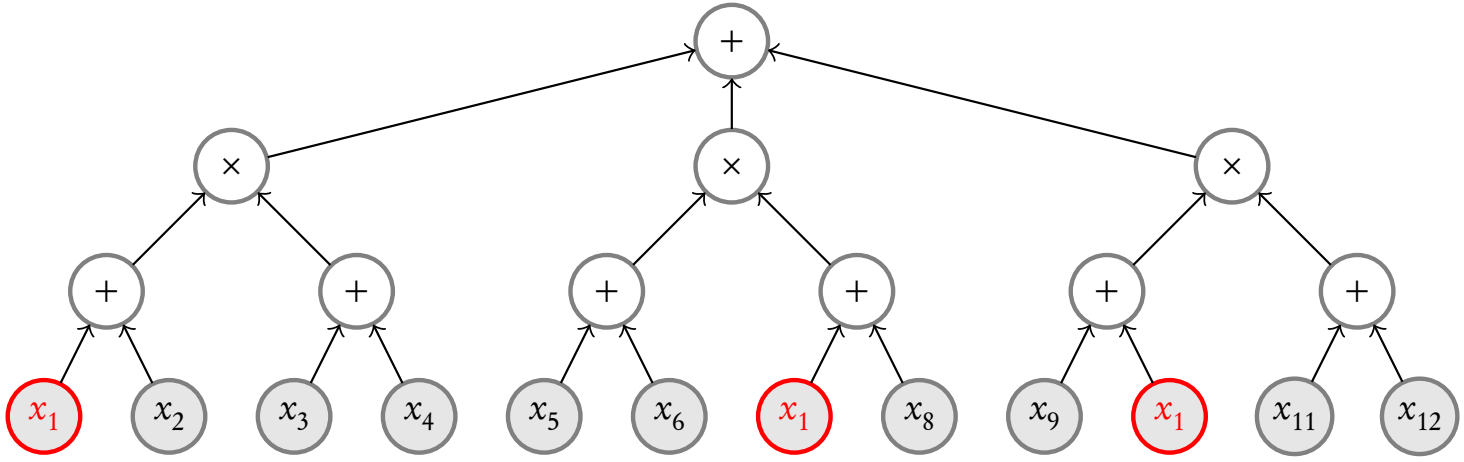
State of affairs for bounded read formulae



Read-k formula

Status of PIT: Open!

State of affairs for bounded read formulae



Read-k multilinear formula

Status of PIT:

- [SarafVolkovich11]: Polytime black-box PIT for multilinear $\Sigma\Pi\Sigma\Pi(k)$.
- [Anderson-vanMelkebeek-Volkovich11]: Polytime black-box PIT for constant depth, multilinear, read-k formulae.
  Quasi-poly black-box PIT for arbitrary depth, multilinear read-k formulae, and polynomial time non-blackbox PIT.

Summary of results

Model	**Best known PIT**	**Idea**
$\Sigma\Pi\Sigma(k)$	s^k black-box	CRT over local rings
bounded $\mathbf{algRank}$ $\Sigma\Pi\Sigma\Pi$	Polytime black-box	Jacobian
$\Sigma\Pi\Sigma\Pi(k)$ multilinear	s^{k^3} black-box	sparsity bounds
multilinear read-k	Quasi-poly black-box	shattering, fragmentation under partial derivatives

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$T_1 + \cdots + T_k \stackrel{?}{=} 0$	s^k black-box	CRT over local rings
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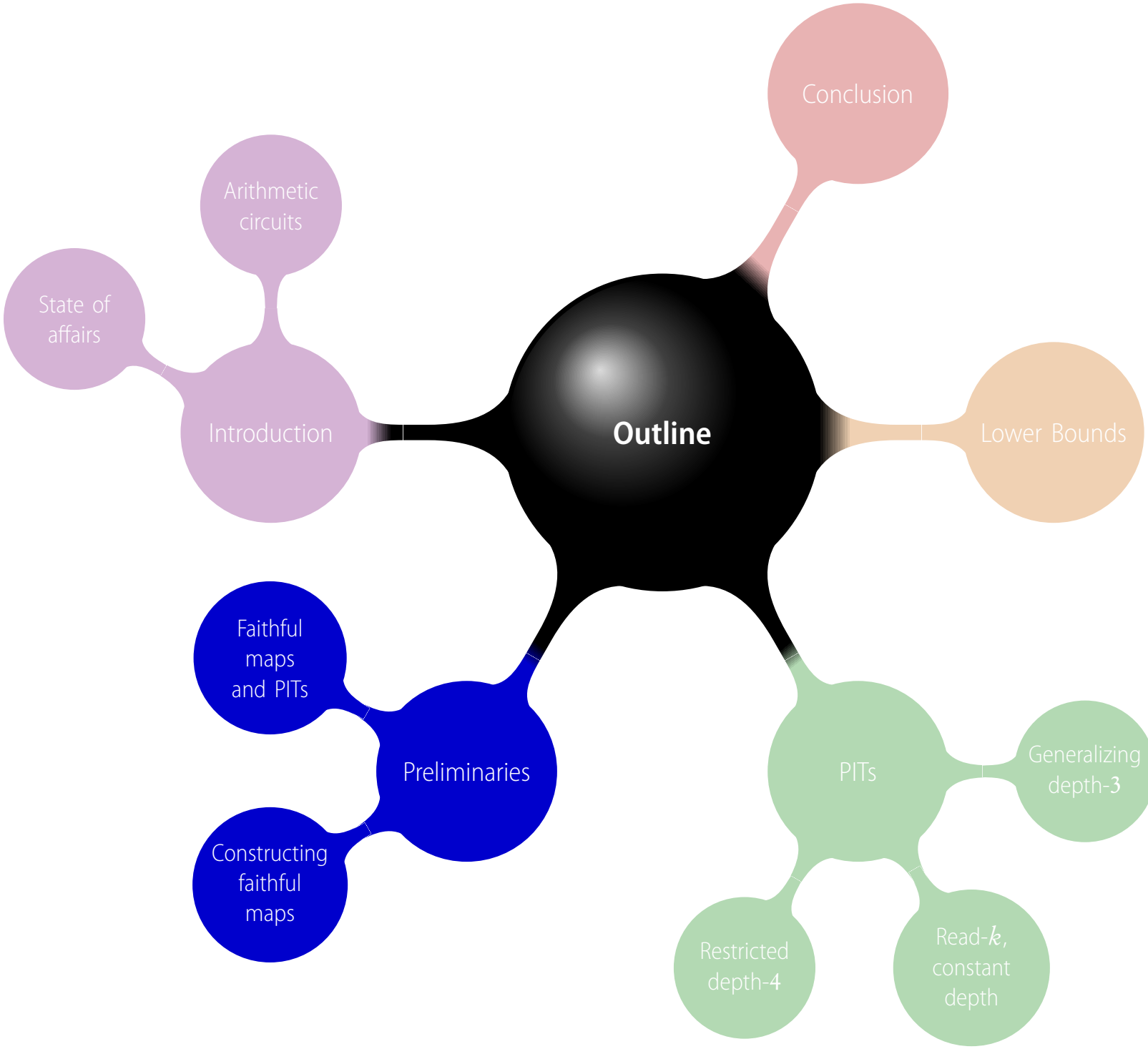
... and some lower bounds

*: $\text{char}(\mathbb{F}) = 0$ or large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the maximum number of linearly independent ℓ_{ij}'s

Rank of a $\Sigma\Pi\Sigma$ circuit

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the maximum number of linearly independent ℓ_{ij}'s

If $\text{rank}(C)$ is "small" (say less than k):

1. Construct a linear transformation $\Psi : \mathbb{F}[x_{[n]}] \mapsto \mathbb{F}[y_{[k]}]$ such that $\dim\left\{\ell_{ij}\right\} = \dim\left\{\Psi(\ell_{ij})\right\}$.
2. Show that this preserves non-zeroness of C.
3. Use [DLSZ] on $\Psi(C)$ to get a hitting set of size $(d+1)^k$.

Rank of a ΣΠΣΠ circuit

$$C = \sum_i \prod_j f_{ij}$$

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3. Use [DLSZ] on $\Psi(C)$ to get a hitting set of size $(d+1)^k$.

Rank of a ΣΠΣΠ circuit

$$C = \sum_i \prod_j f_{ij}$$

$\text{rank}(C) \stackrel{\text{def}}{=} \text{algRank}\left\{f_{ij}\right\}$

the maximum number of algebraically independent f_{ij}'s

If $\text{rank}(C)$ is "small" (say less than k):

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If $\text{rank}(C)$ is "small" (say less than k):

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Definition

$\{f_1, \cdots, f_m\}$ are algebraically independent if there is no non-trivial polynomial relation between them. That is,

$$H(f_1, \cdots, f_m) = 0 \quad \Longleftrightarrow \quad H = 0$$

Formal definitions

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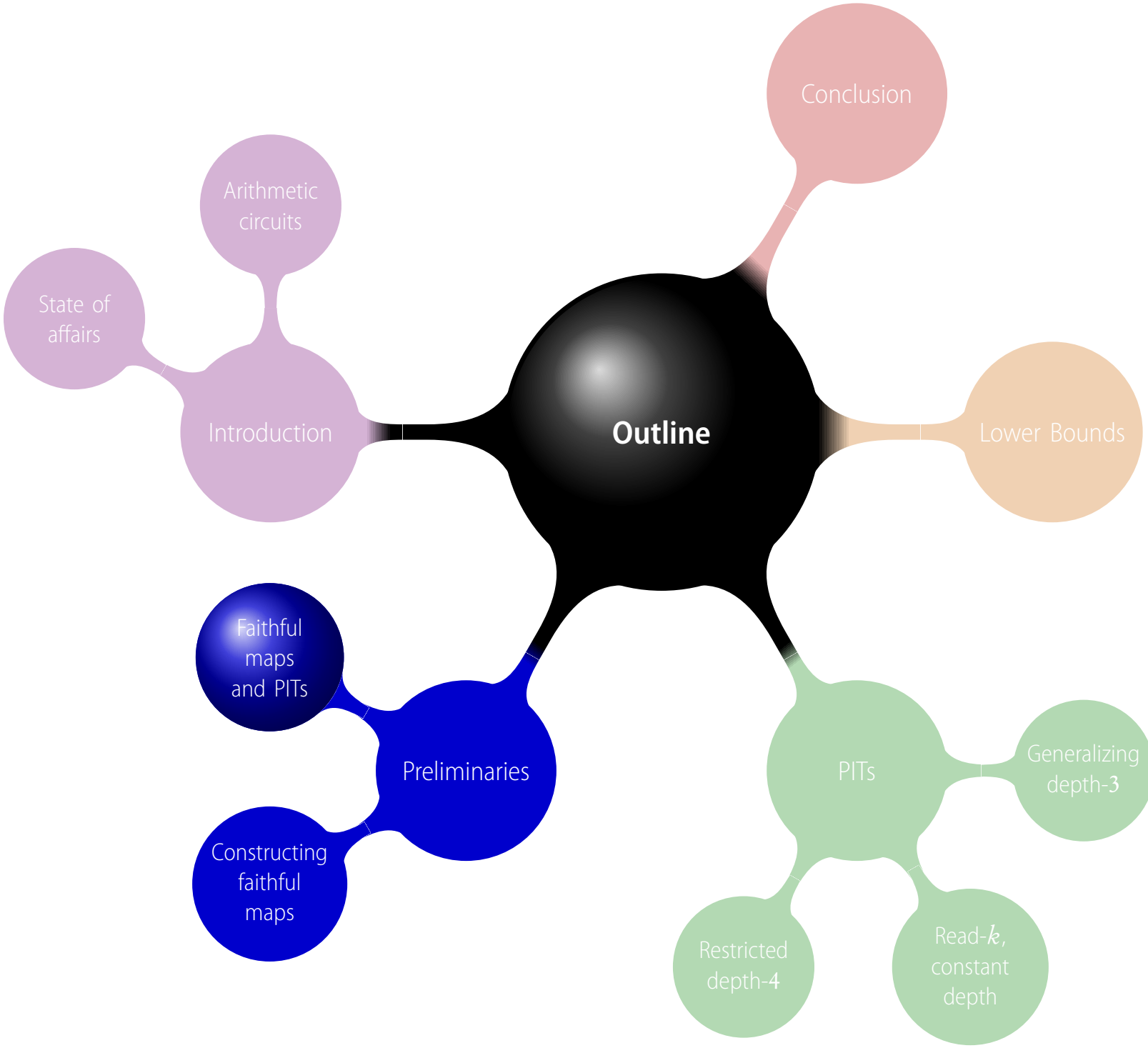
A map $\Psi : \mathbb{F}[x_1, \cdots, x_n] \to \mathbb{F}[y_1, \cdots, y_k]$ is faithful for $\{f_1, \cdots, f_m\}$ if

$$\text{algRank}\{f_1, \cdots, f_m\} \quad = \quad \text{algRank}\{\Psi(f_1), \cdots, \Psi(f_m)\}$$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Faithful maps preserve non-zeroness

Theorem (Beecken-Mittmann-Saxena)

If $\Psi : \mathbb{F}[x_1, \cdots, x_n] \to \mathbb{F}[y_1, \cdots, y_k]$ *is faithful for* $\{f_1, \cdots, f_m\}$*, then for any* C

$$C(f_1, \cdots, f_m) \neq 0 \quad \textit{if and only if} \quad \Psi(C(f_1, \cdots, f_m)) \neq 0$$

Proof.

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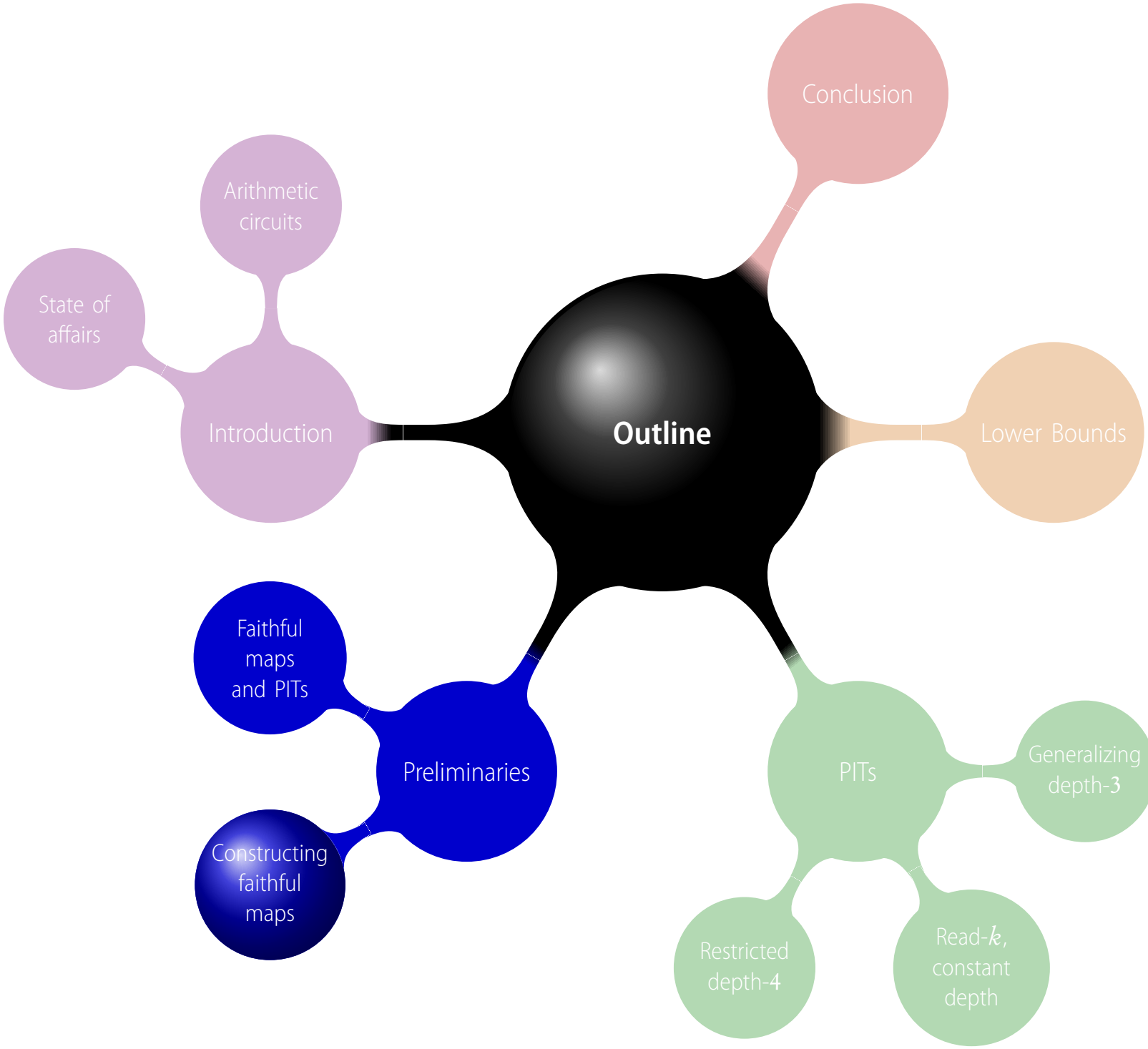
□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Answer: Use the Jacobian!

The Jacobian

$$\mathscr{J}_{x_1,\cdots,x_n}(f_1,\cdots,f_m)=\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m\times n}$$

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Theorem (Jacobi Criterion)

If $\mathrm{char}(\mathbb{F}) = 0$ *or "large enough",*

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How do we use this to construct faithful maps?

Revisiting $\Sigma\Pi\Sigma$ circuits:

$$C = \sum\prod \ell_{ij}$$

Say $\dim \left\{\ell_{ij}\right\}_{i,j} = k$.

How do preserve the rank in a blackbox fashion?

Rank preserving maps

Lemma (GabizonRaz05)

Given n, k, there is a set of $O(nk^2)$ of linear transformations $\{\Psi_t\} : \mathbb{F}^n \to \mathbb{F}^k$ such that for any subspace $V \subset \mathbb{F}^n$ of dimension k, there is at least one Ψ_t that is an isomorphism between V and $\mathbb{F}^k$.

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What do we need?

$$\begin{bmatrix} \partial_{x_1} f_1 & \cdots & \partial_{x_n} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_m & \cdots & \partial_{x_n} f_m \end{bmatrix}_{m \times n}$$

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How does the Jacobian evolve?

Evolution of the Jacobian under homomorphism

$$\frac{\partial \Psi(f)}{\partial y} = \frac{\partial}{\partial y} f(\overline{\Psi(x)})$$

Evolution of the Jacobian under homomorphism

$$\begin{aligned}
\frac{\partial \Psi(f)}{\partial y} &= \frac{\partial}{\partial y} f(\overline{\Psi(x)}) \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \left[\overline{\Psi(x)}\right] \quad \cdot \quad \frac{\partial \Psi(x_i)}{\partial y}
\end{aligned}$$

Evolution of the Jacobian under homomorphism

$$\begin{aligned}\frac{\partial \Psi(f)}{\partial y} &= \frac{\partial}{\partial y} f(\overline{\Psi(x)}) \\ &= \sum_{i=1}^{n} \Psi\left(\frac{\partial f}{\partial x_i}\right) \cdot \frac{\partial \Psi(x_i)}{\partial y}\end{aligned}$$

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$$\begin{bmatrix} \partial_{y_1}\Psi(f_1) & \cdots & \partial_{y_k}\Psi(f_1) \\ \vdots & \ddots & \vdots \\ \partial_{y_1}\Psi(f_m) & \cdots & \partial_{y_k}\Psi(f_m) \end{bmatrix} =$$

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$$\Psi: \quad x_i \quad \mapsto \quad \sum_{j=1}^{k} y_j t^{ij} \quad + \Phi(x_i)$$

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If $\operatorname{rank}\left(\mathscr{J}(f_1,\cdots,f_m)\right) = \operatorname{rank}\left(\Phi \circ \mathscr{J}(f_1,\cdots,f_m)\right)$, we are done.

Recipe for constructing faithful maps

$$\mathscr{J}(f_1,\cdots,f_m)=$$

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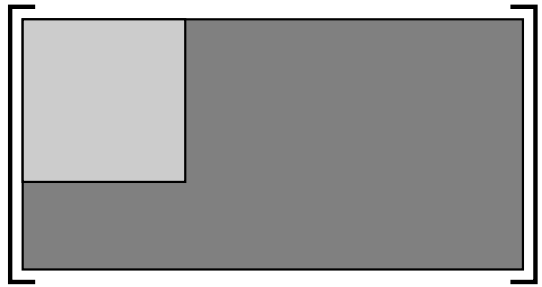

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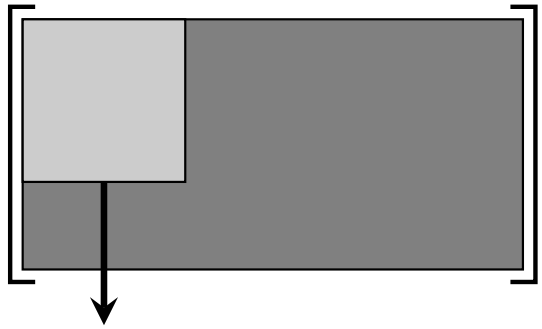


$$J=\begin{vmatrix} \partial_{x_1} f_1 & \cdots & \partial_{x_k} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_k & \cdots & \partial_{x_k} f_k \end{vmatrix}$$

Recipe for constructing faithful maps

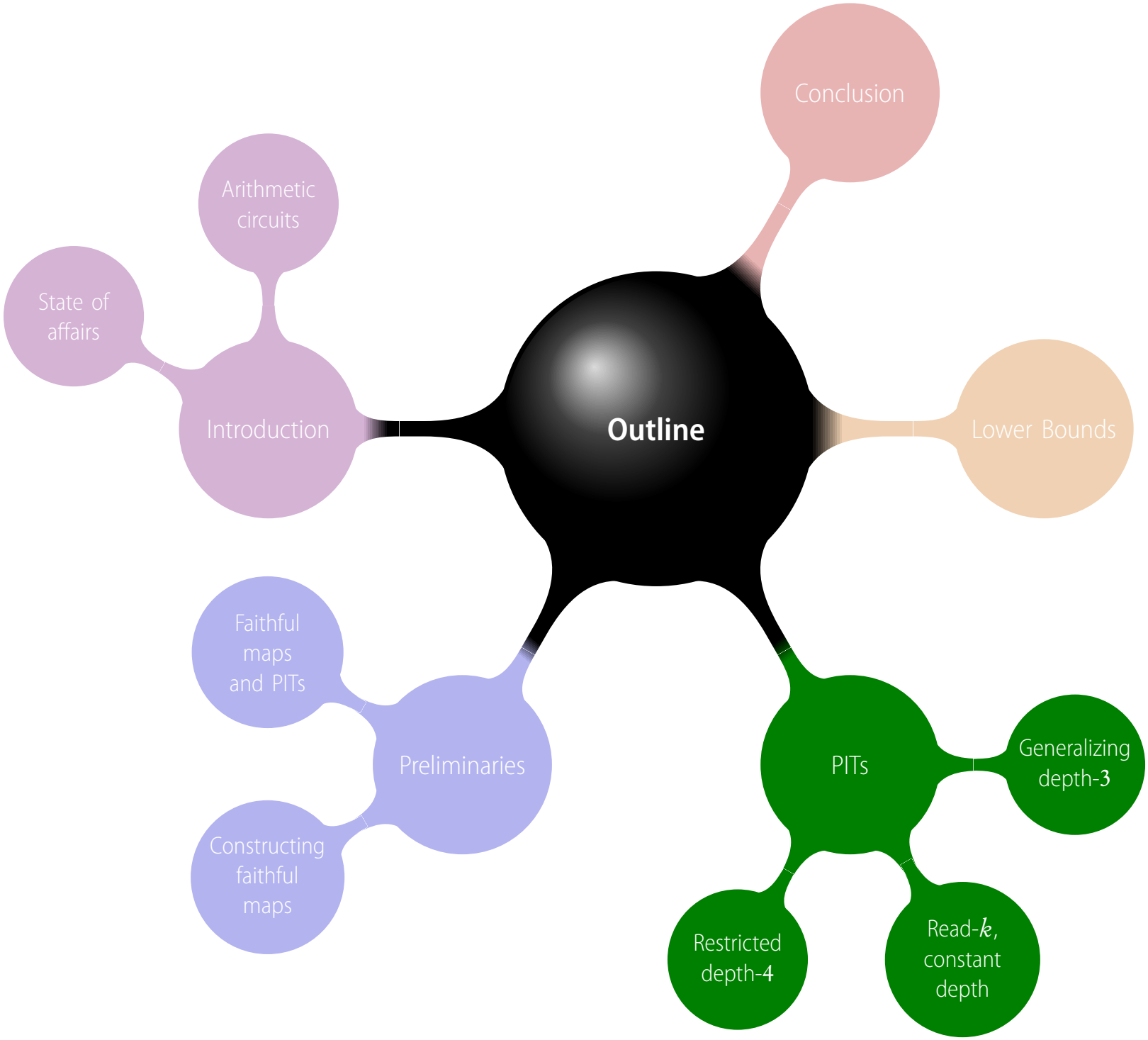
$$\mathcal{J}(f_1,\cdots,f_m) = \left[\begin{array}{c} \\ \\ \\ \end{array}\right]$$

$$J = \left|\begin{array}{ccc} \partial_{x_1} f_1 & \cdots & \partial_{x_k} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_k & \cdots & \partial_{x_k} f_k \end{array}\right| \longleftarrow \text{Preserve this determinant}$$

Lemma (Composition Lemma)

Let Φ *be a map such that* $\Phi(J) \neq 0$. *Then the map* Ψ *is faithful to* $\{f_1,\cdots,f_m\}$:

$$\Psi : x_i \quad \mapsto \quad \sum_{j=1}^{k} y_j t^{ij} \quad + \quad \Phi(x_i)$$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Preliminaries
Faithful maps and PITs
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Lower Bounds
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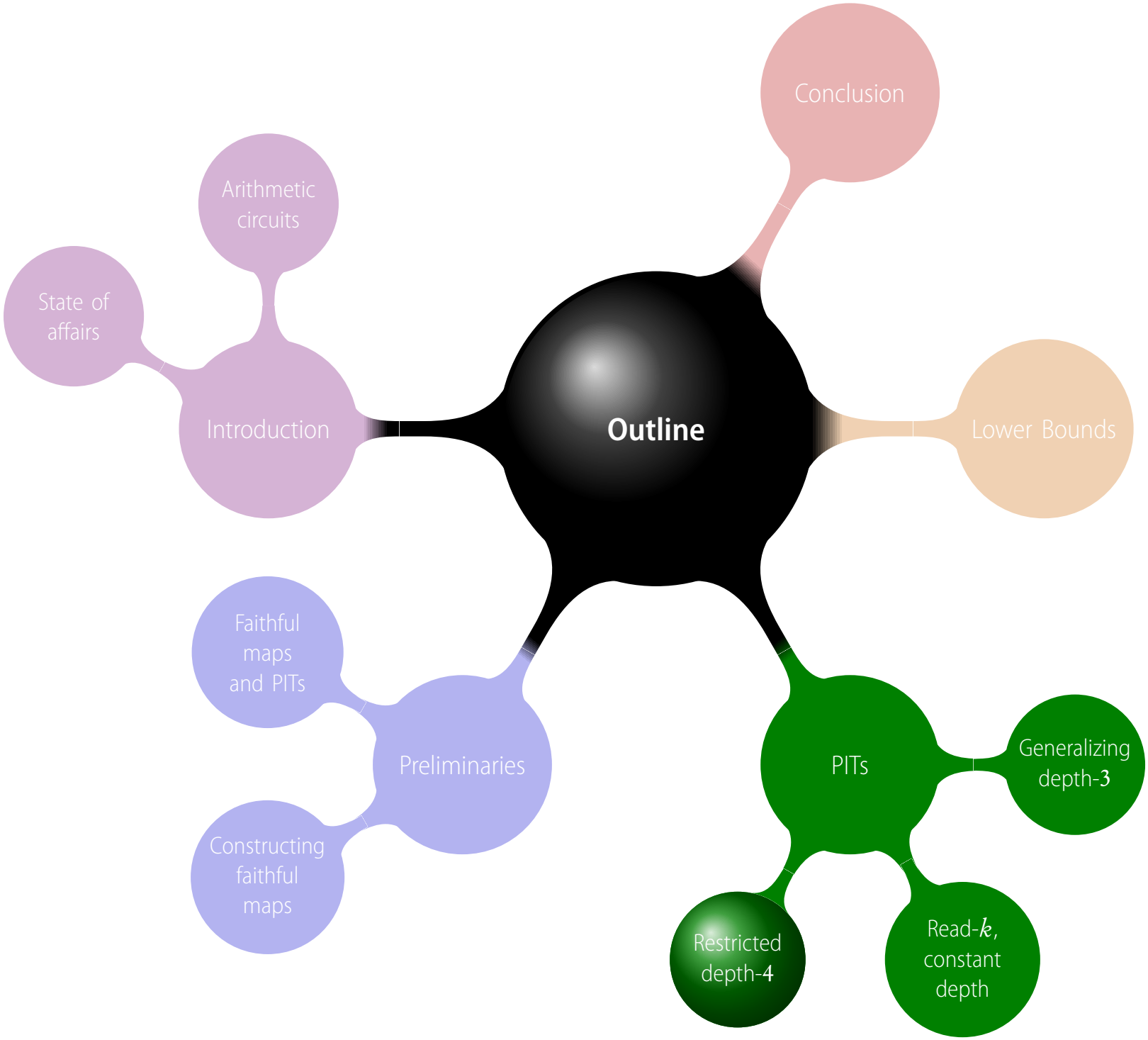
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[BeeckenMittmannSaxena]'s PIT

Theorem

There is a black-box PIT for circuits of the form $C(f_1, \cdots, f_m)$ *where each* f_i *is "sparse" and* $\mathrm{algRank}\{f_1, \cdots, f_m\} \leq k$.

Proof.

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$$\mathcal{J}(f_1, \cdots, f_m) = \left[\begin{array}{c} \\ \\ \\ \end{array}\right]$$

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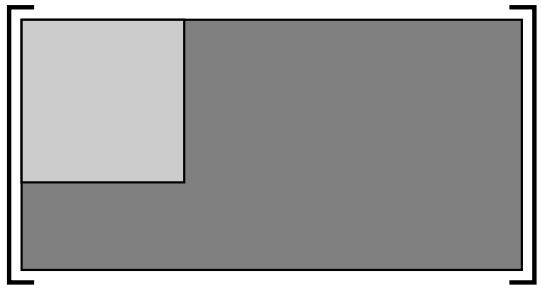
$\mathcal{J}(f_1, \cdots, f_m) =$ 

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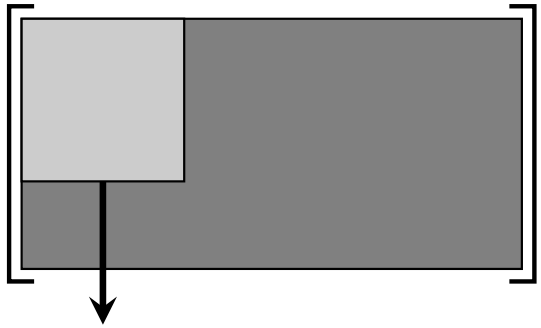

$$J = \begin{vmatrix} \partial_{x_1} f_1 & \cdots & \partial_{x_k} f_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_k & \cdots & \partial_{x_k} f_k \end{vmatrix}$$

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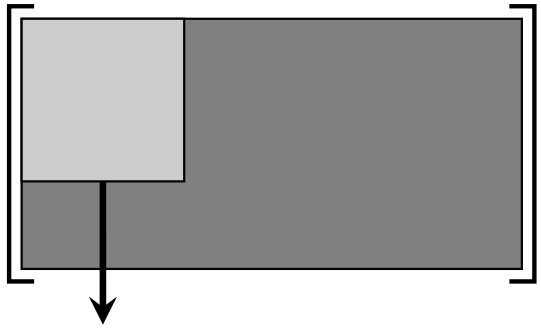

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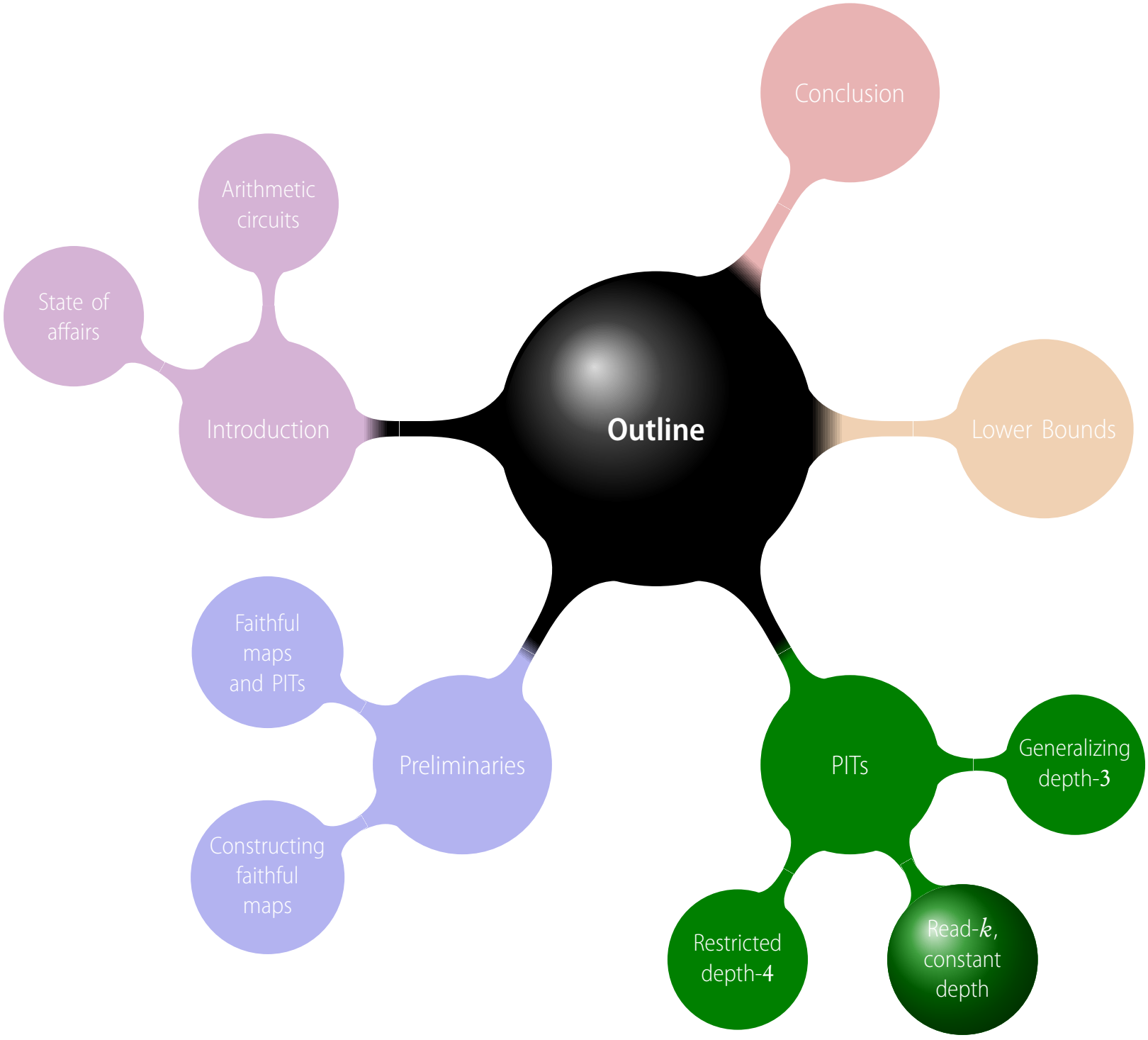
which is a sparse poly!

□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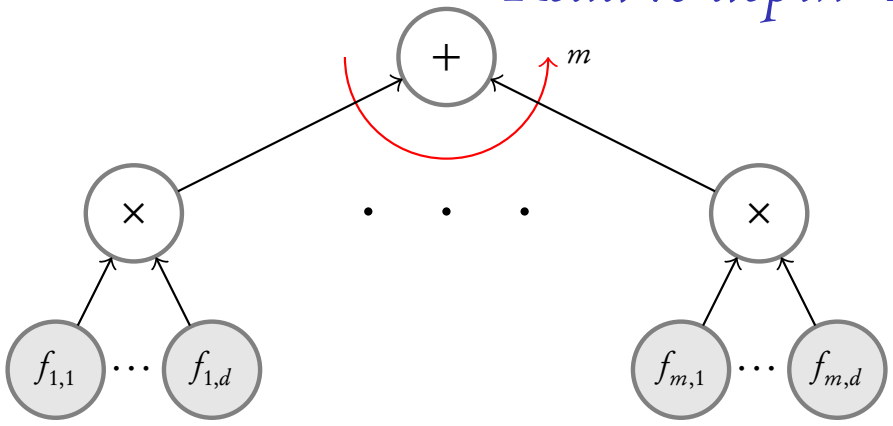
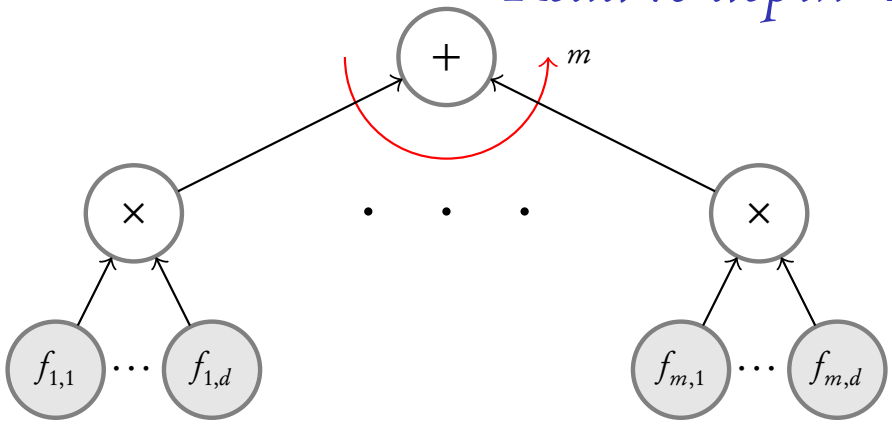
Read-k depth-4 formulae



Read-k depth-4 formulae



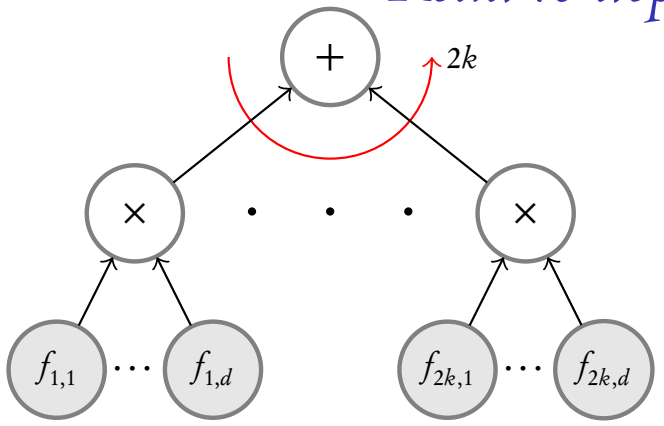
Observation

If $C(x_1, \cdots, x_n) \neq 0$, *then there exists an* i *such that*

$$C(x_1, \cdots, x_i + 1, \cdots, x_n) - C(x_1, \cdots, x_i, \cdots, x_n) \neq 0$$

In fact, any x_i that C non-trivially depends on.

Read-k depth-4 formulae



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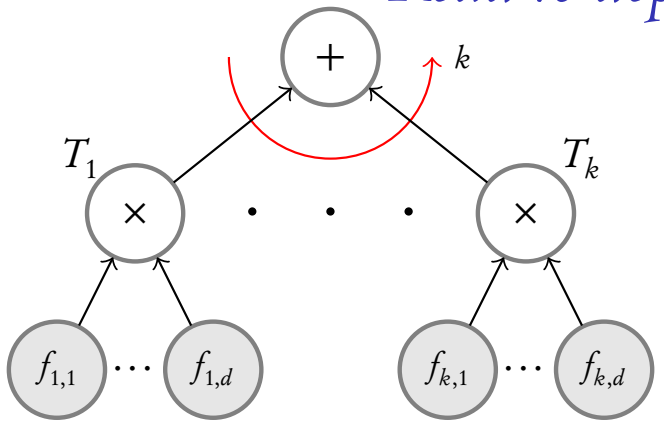

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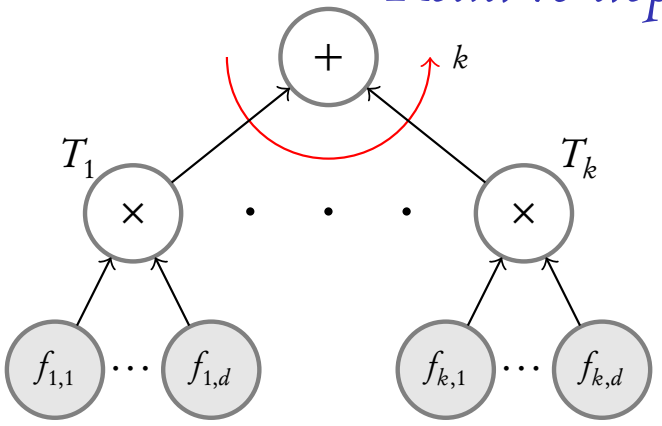

$$\mathcal{J}(T_1,\cdots,T_k) = \left[\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right]$$

$$J = \begin{vmatrix} \partial_{x_1} T_1 & \cdots & \partial_{x_r} T_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} T_r & \cdots & \partial_{x_r} T_r \end{vmatrix}$$

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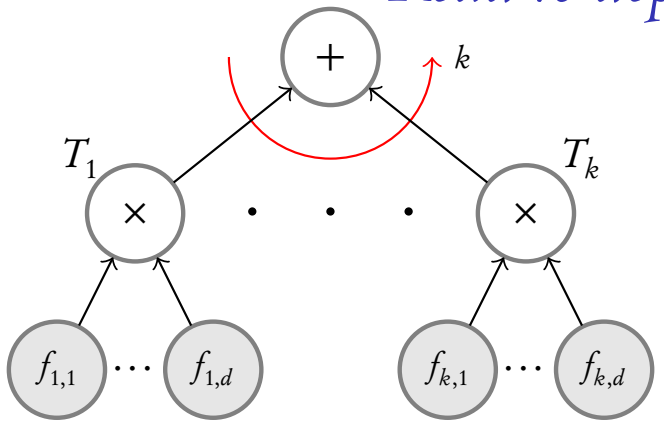

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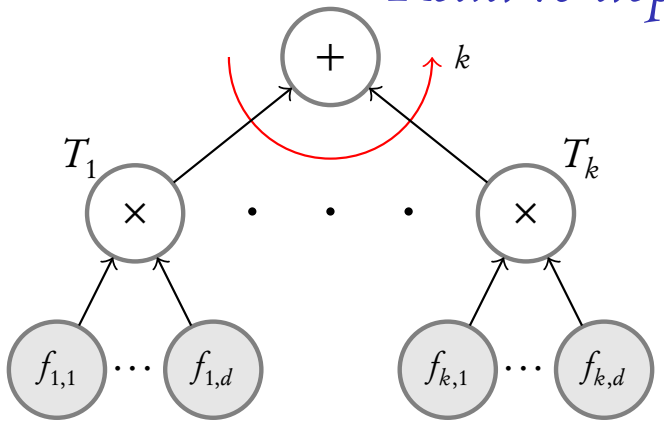

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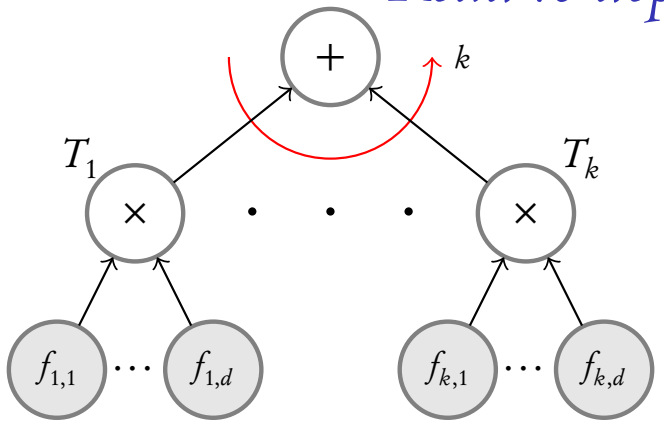
Observation

At most rk *of the* f_{ij}*'s depend on* $x_1, \cdots, x_r$.

Read-k depth-4 formulae



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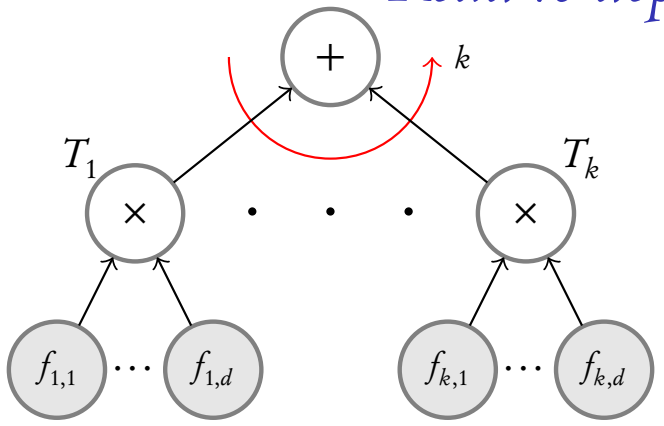
... a product of sparse polys!

□

Read-k depth-4 formulae



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$\Psi_r : x_i \mapsto u^{d^i \bmod r}$ preserves J

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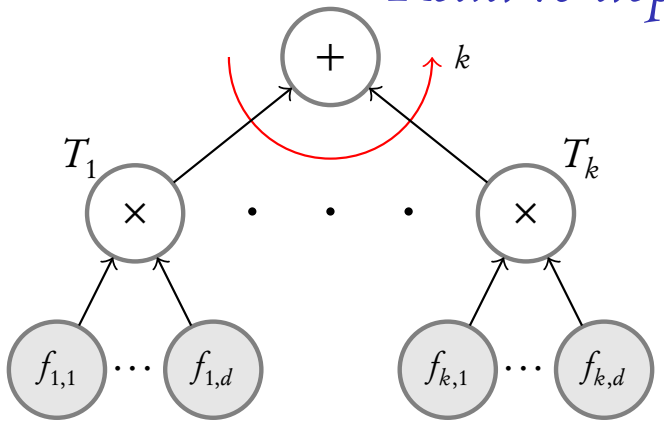

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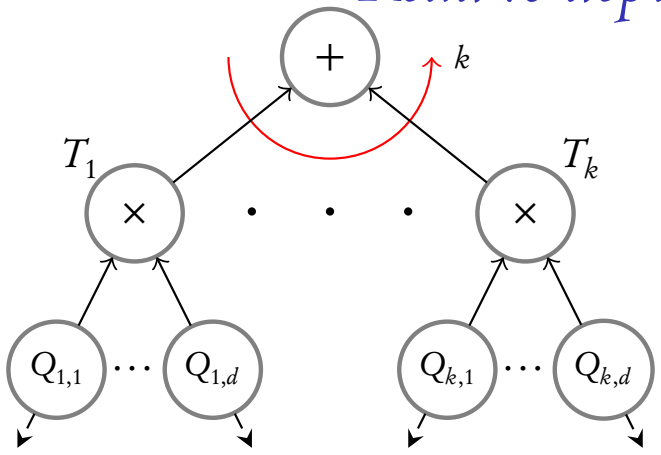
... a product of sparse polys!

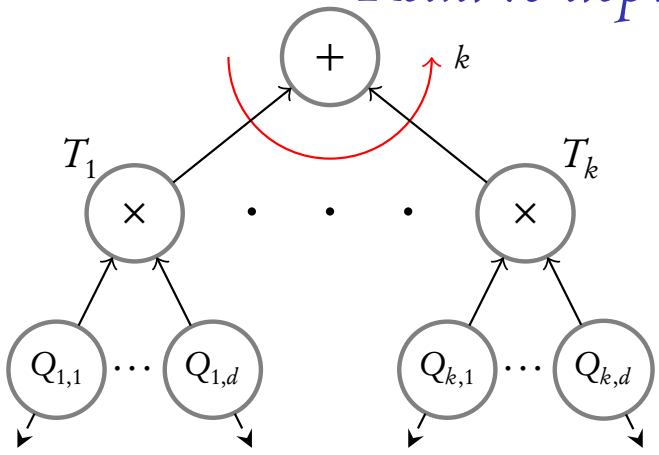
$\Psi_r : x_i \mapsto u^{d^i \bmod r}$ preserves J

$\Phi_r : x_i \mapsto \sum_{j=1}^{k} y_j t^{ij} + u^{d^i \bmod r}$ is a black-box PIT

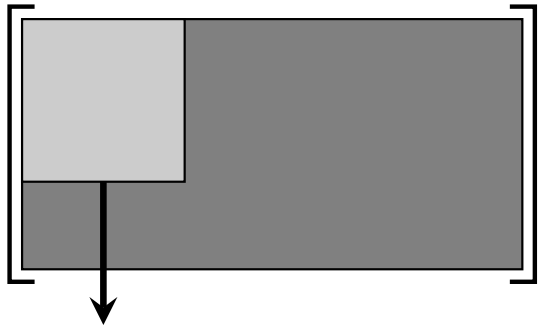
Read-k depth-D formulae



Read-k depth-D formulae



$$\mathscr{J}(T_1,\cdots,T_k)=$$



$$J \quad = \quad \begin{vmatrix} \partial_{x_1} T_1 & \cdots & \partial_{x_r} T_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} T_r & \cdots & \partial_{x_r} T_r \end{vmatrix}$$

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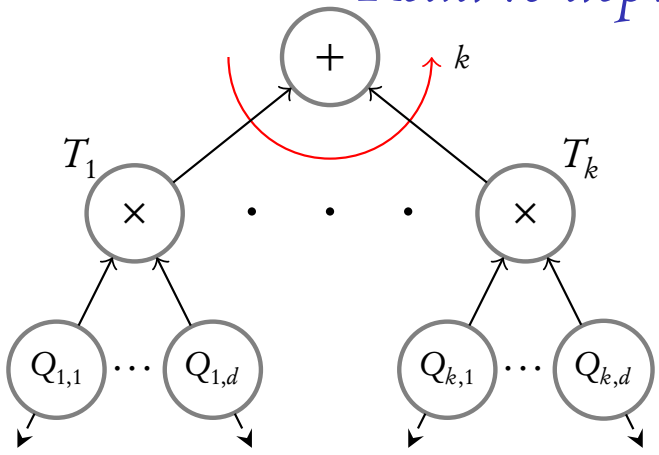

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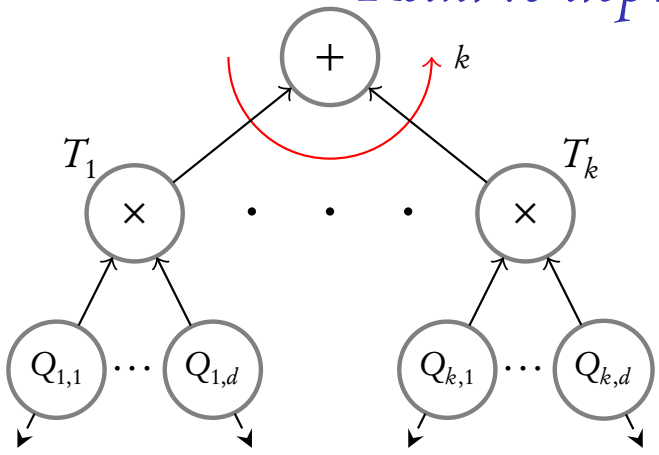

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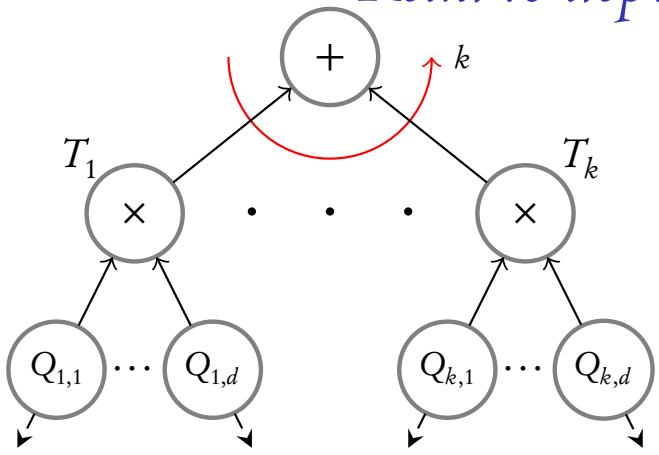
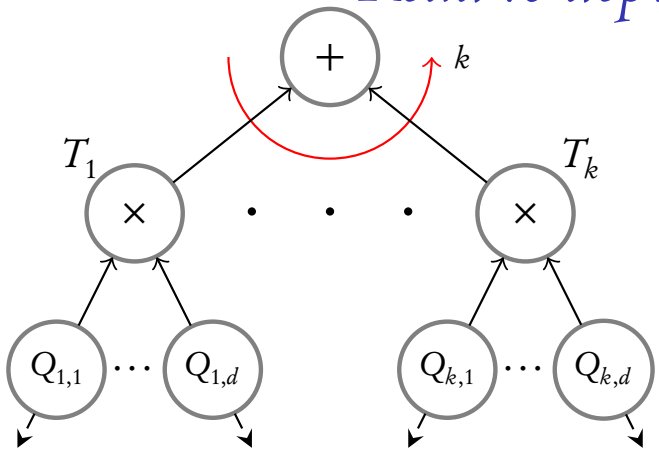
Function of "few" Q_{ij}'s

Read-k depth-D formulae



$$J = \begin{vmatrix} \partial_{x_1} T_1 & \cdots & \partial_{x_r} T_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} T_r & \cdots & \partial_{x_r} T_r \end{vmatrix} = \underbrace{\left(\prod Q_{ij}\right) \cdot \begin{vmatrix} \partial_{x_1} T'_1 & \cdots & \partial_{x_r} T'_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} T'_r & \cdots & \partial_{x_r} T'_r \end{vmatrix}}_{\text{Product of functions of "few" } Q_{ij}\text{'s}}$$

Read-k depth-D formulae



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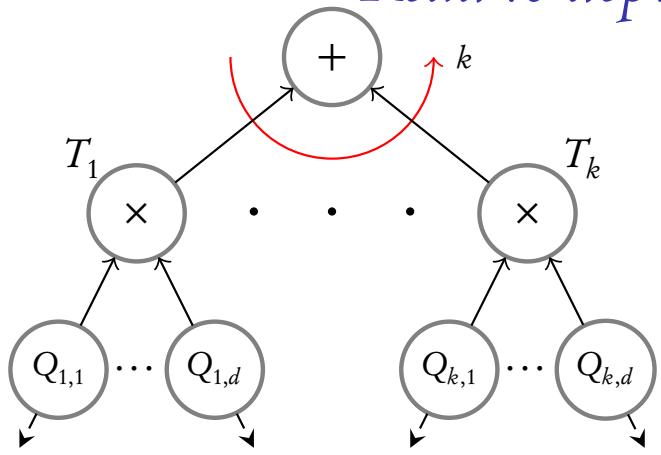
To preserve non-zeroness of C it suffices to preserve non-zeroness of J.

Read-k depth-D formulae



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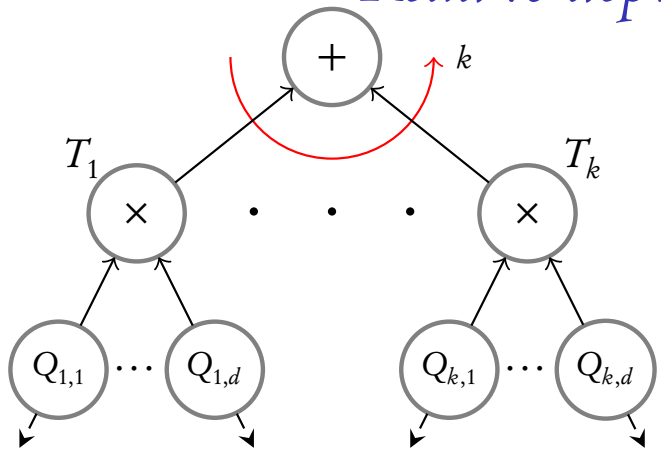
To preserve non-zeroness of C it suffices to preserve non-zeroness of J.
Hence, suffices to preserve the Jacobian of the Q_{ij}'s.

Read-k depth-D formulae



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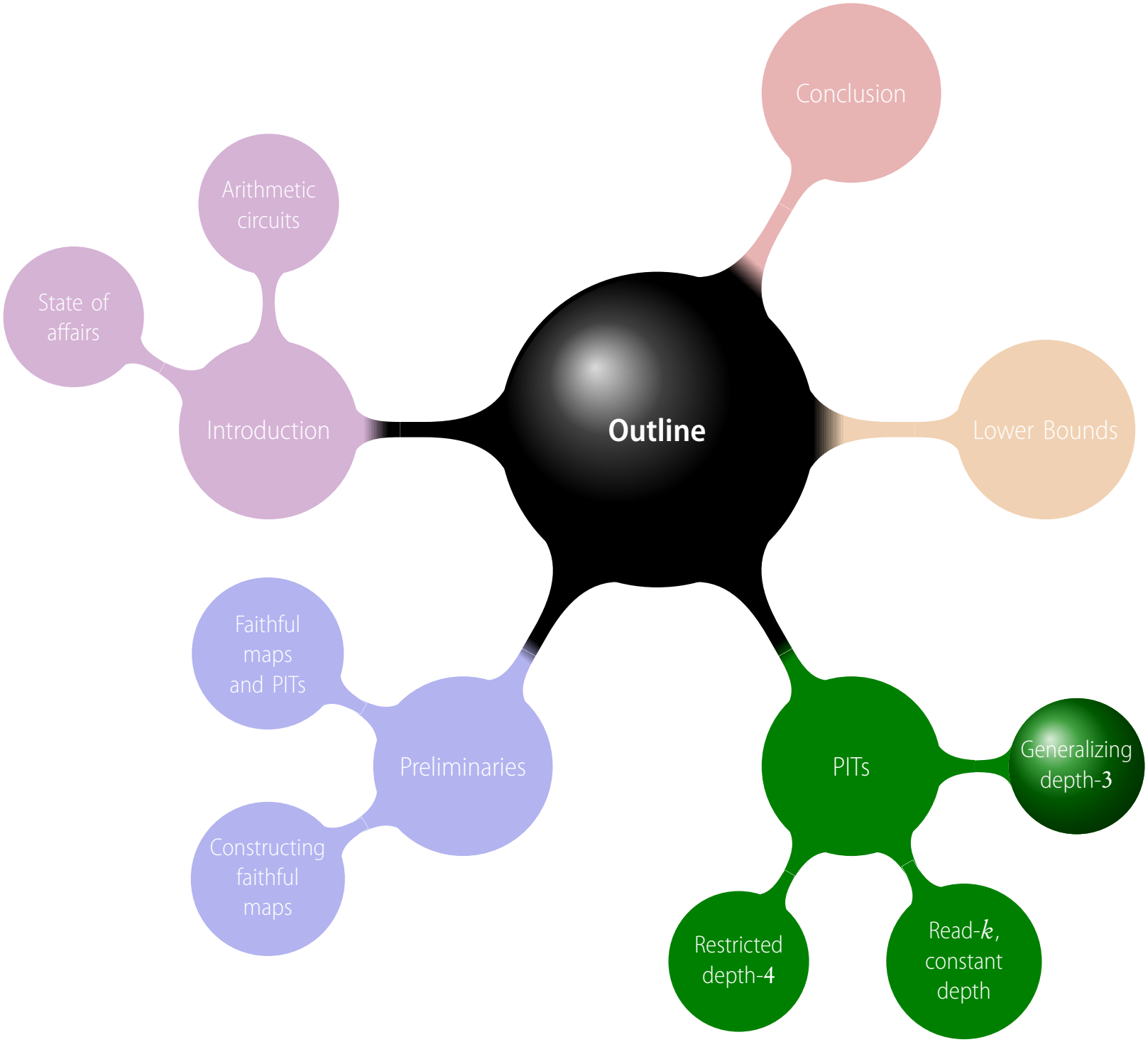
To preserve non-zeroness of C it suffices to preserve non-zeroness of J.
Hence, suffices to preserve the Jacobian of the Q_{ij}'s. Recurse! □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Generalizing $\Sigma\Pi\Sigma(k)$ *PITs*

$$f \quad = \quad C(T_1, \cdots, T_k) \qquad \text{where } T_i = \prod_{j=1}^{d} \ell_{ij}$$

Generalizing $\Sigma\Pi\Sigma(k)$ *PITs*

$$J(T_1, \cdots, T_m) = \begin{vmatrix} \partial_{x_1} T_1 & \cdots & \partial_{x_r} T_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} T_r & \cdots & \partial_{x_r} T_r \end{vmatrix}$$

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Lemma

$$\partial_x P \cdot Q = PQ \cdot \left(\frac{\partial_x P}{P} + \frac{\partial_x Q}{Q} \right)$$

Generalizing $\Sigma\Pi\Sigma(k)$ *PITs*

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Lemma

$$\partial_x P \cdot Q = PQ \cdot \left(\frac{\partial_x P}{P} + \frac{\partial_x Q}{Q} \right)$$

Lemma

$$\det \begin{bmatrix} \mathbf{b_1} + \mathbf{b'_1} \\ \mathbf{a_2} \\ \vdots \\ \mathbf{a_n} \end{bmatrix} = \det \begin{bmatrix} \mathbf{b_1} \\ \mathbf{a_2} \\ \vdots \\ \mathbf{a_n} \end{bmatrix} + \det \begin{bmatrix} \mathbf{b'_1} \\ \mathbf{a_2} \\ \vdots \\ \mathbf{a_n} \end{bmatrix}$$

Generalizing $\Sigma\Pi\Sigma(k)$ *PITs*

$$
\begin{aligned}
J(T_1,\cdots,T_m) &= \begin{vmatrix} \partial_{x_1} T_1 & \cdots & \partial_{x_r} T_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} T_r & \cdots & \partial_{x_r} T_r \end{vmatrix} \\
&= T_1 \cdots T_k \cdot \sum_{\ell_i \in T_i} \frac{J(\ell_1,\cdots,\ell_k)}{\ell_1 \cdots \ell_k}
\end{aligned}
$$

Generalizing $\Sigma\Pi\Sigma(k)$ *PITs*

$$
\begin{aligned}
J(T_1, \cdots, T_m) &= \begin{vmatrix} \partial_{x_1} T_1 & \cdots & \partial_{x_r} T_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} T_r & \cdots & \partial_{x_r} T_r \end{vmatrix} \\
&= T_1 \cdots T_k \cdot \sum_{\ell_i \in T_i} \frac{\alpha_L}{\ell_1 \cdots \ell_k}
\end{aligned}
$$

Generalizing $\Sigma\Pi\Sigma(k)$ *PITs*

$$
\begin{aligned}
J(T_1, \cdots, T_m) &= \begin{vmatrix} \partial_{x_1} T_1 & \cdots & \partial_{x_r} T_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} T_r & \cdots & \partial_{x_r} T_r \end{vmatrix} \\
&= T_1 \cdots T_k \cdot \sum_{\ell_i \in T_i} \frac{\alpha_L}{\ell_1 \cdots \ell_k}
\end{aligned}
$$

A similar analysis (slightly simpler due to lack of multiplicities)

$$C \quad = \quad T \cdot \sum \frac{\alpha_L}{\ell_1 \cdots \ell_k}$$

$$C \quad = \quad T \cdot \sum \frac{\alpha_i}{\ell_i}$$

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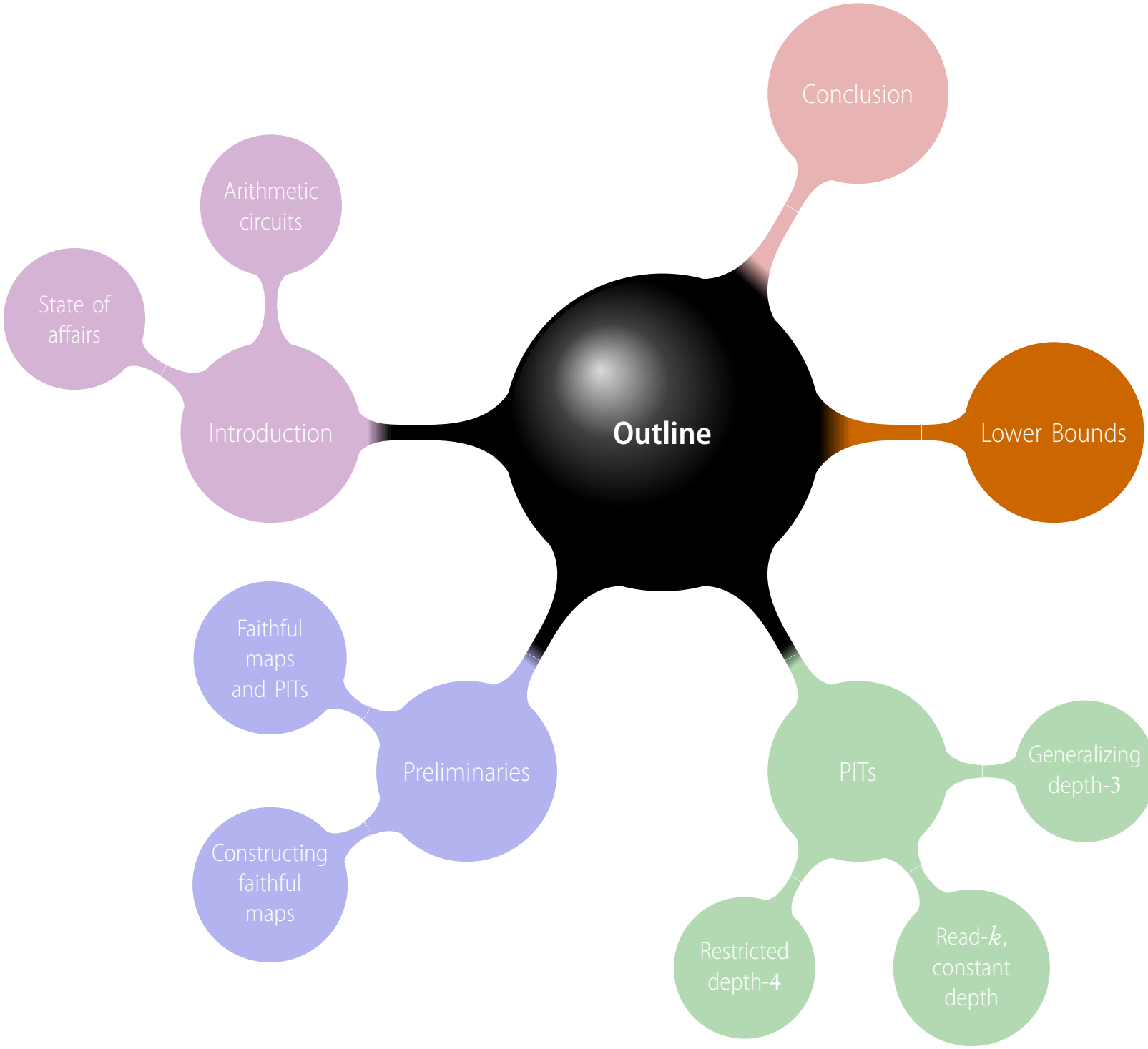
...recurse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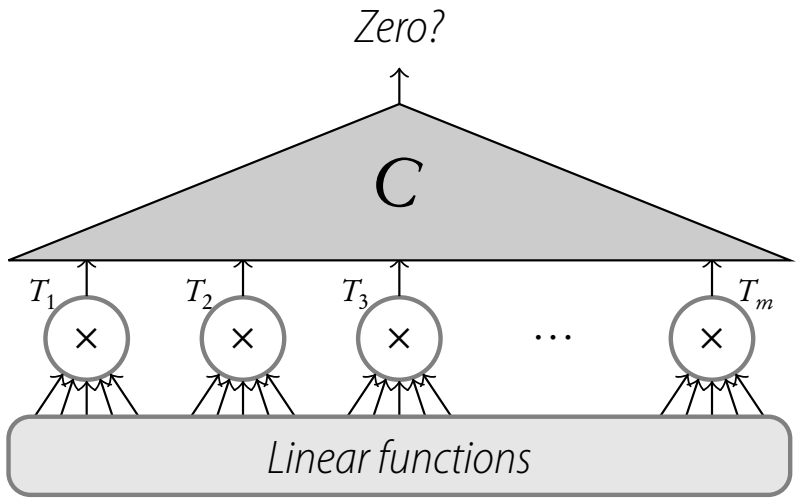
General philosophy

If you have black-box PITs for a class $\mathscr{C}$, then you have determinant/permanent lower bounds for (almost) $\mathscr{C}'$.

[KabanetsImpagliazzo03], [Agrawal05], [DvirShpilkaYehudayoff08] etc...

PITs & Lower bounds

Theorem



Black-box PIT if $\mathbf{algRank}\{T_1, \cdots, T_m\} = O(1)$.

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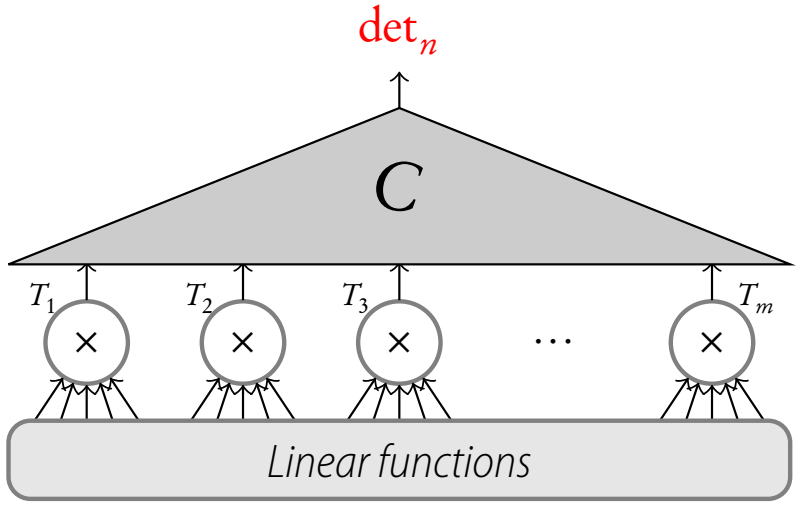
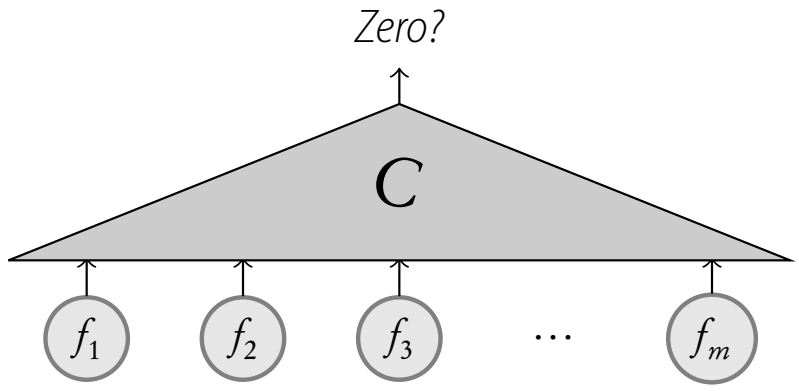


Then, $\text{algRank}\{T_1, \cdots, T_m\} = \Omega(n)$.

Theorem



Black-box PIT if $\mathbf{algRank}\{f_1, \cdots, f_m\} = O(1)$.

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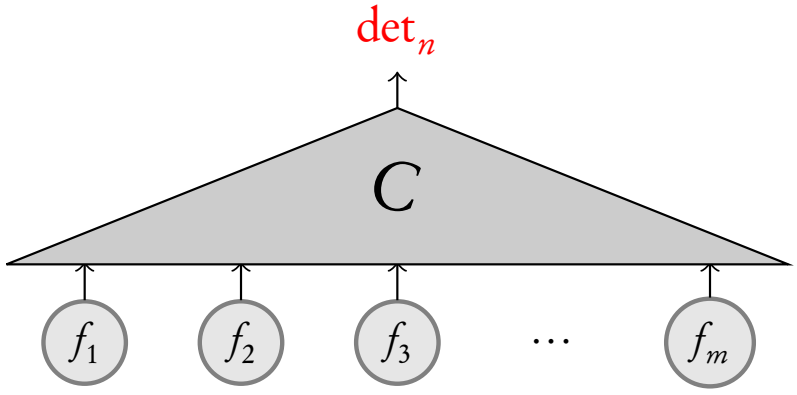
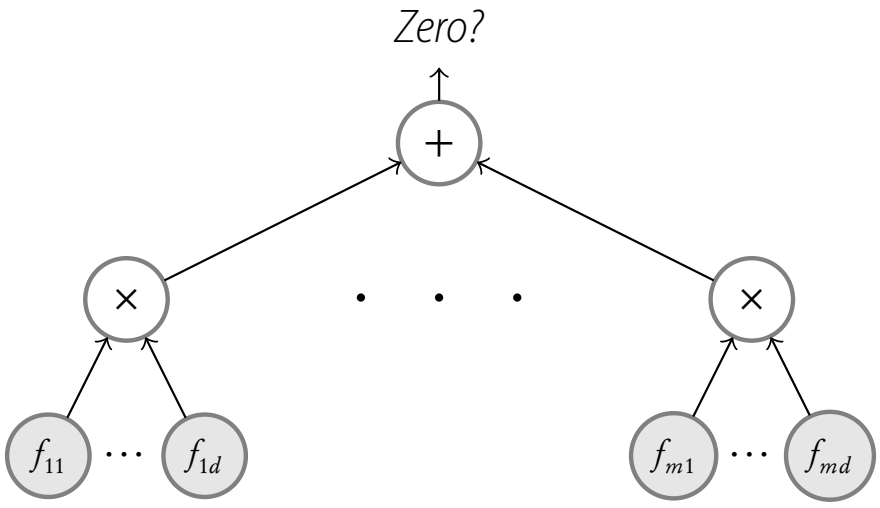


If $\text{algRank}\{f_1, \cdots, f_m\} = k$, *then* $\text{size}(f_i) \geq 2^{n/k^2}$

PITs & Lower bounds

Theorem



Black-box PIT if at most $O(1)$ *of the* f_i*'s depend on any* x_j.

PITs & Lower bounds

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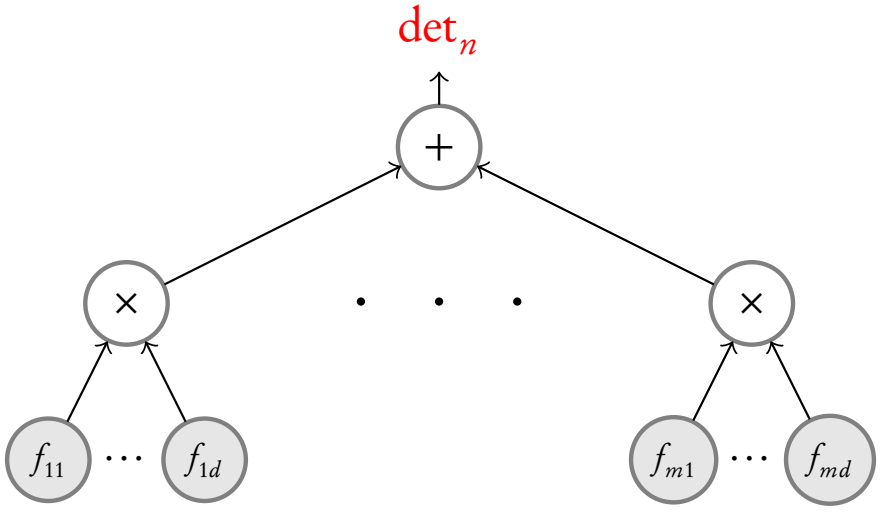

If at most k *of the* f_i*'s depend on any* x_j*, then* $\text{size}(f_i) \geq 2^{n/k^3}$

PITs & Lower bounds

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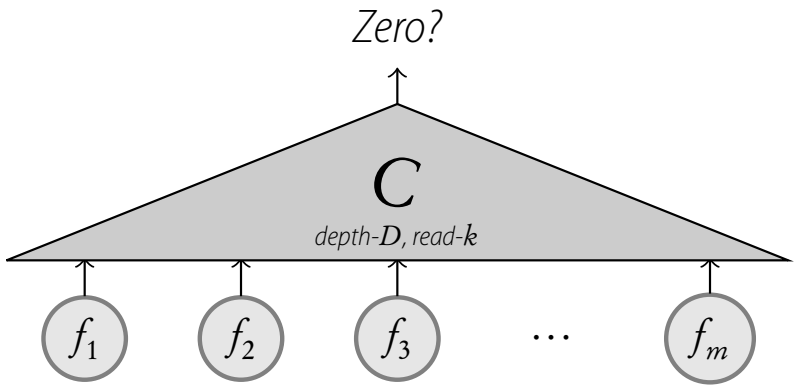

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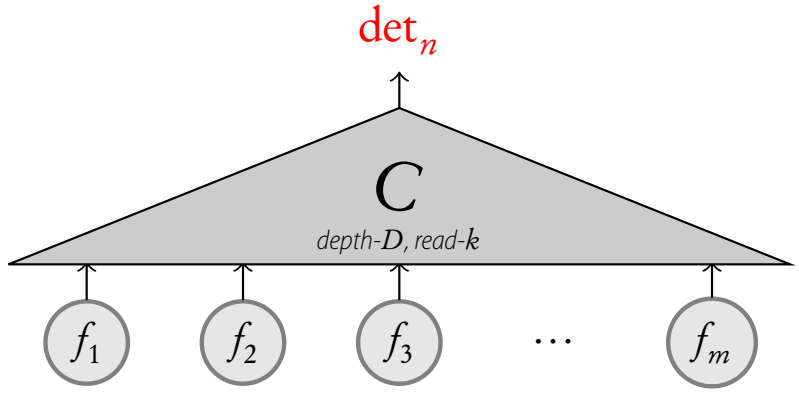

If at most $O(1)$ *of the* f_i*'s depend on any* x_j*, then* $\text{size}(f_i) \geq 2^{\Omega(n)}$*, assuming a conjecture about determinants is true.*

An illustrative example

Theorem

If $\det_n = C(f_1, \cdots, f_k)$, *then one of the* f_i*'s has* $2^{n/k^2}$ *monomials .*

Proof

$$\mathscr{J}(\det_n, f_1, \cdots, f_k) \quad =$$

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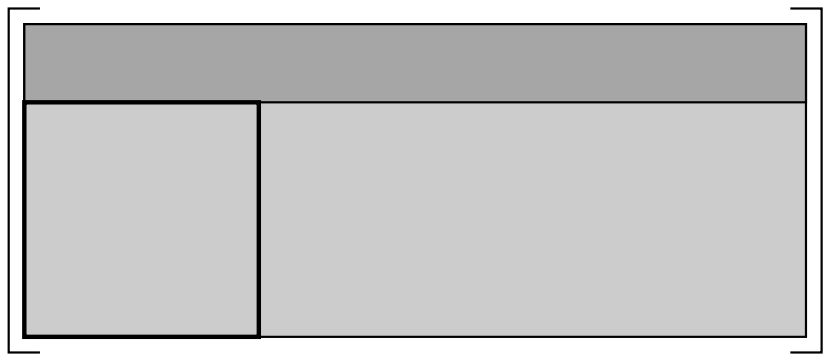

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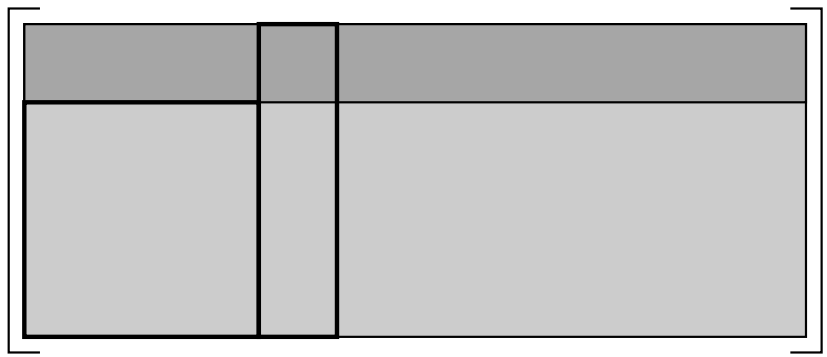

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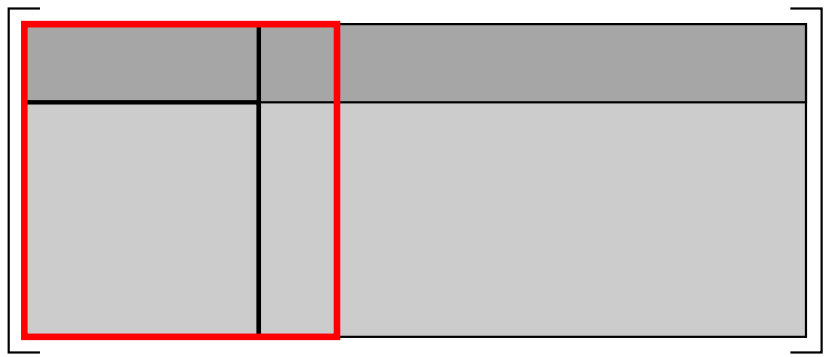

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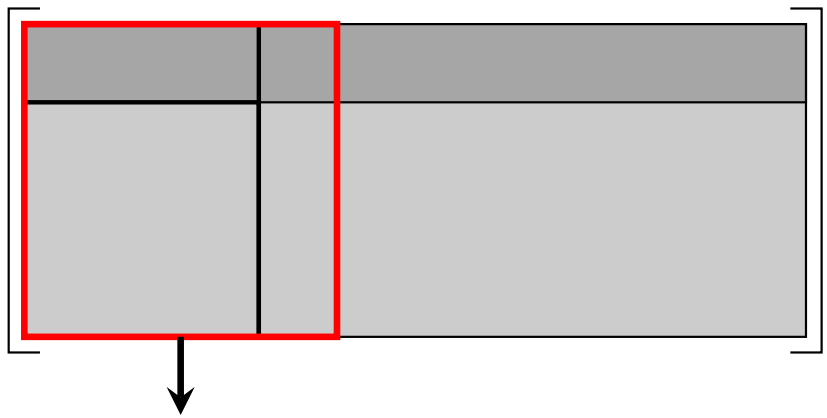

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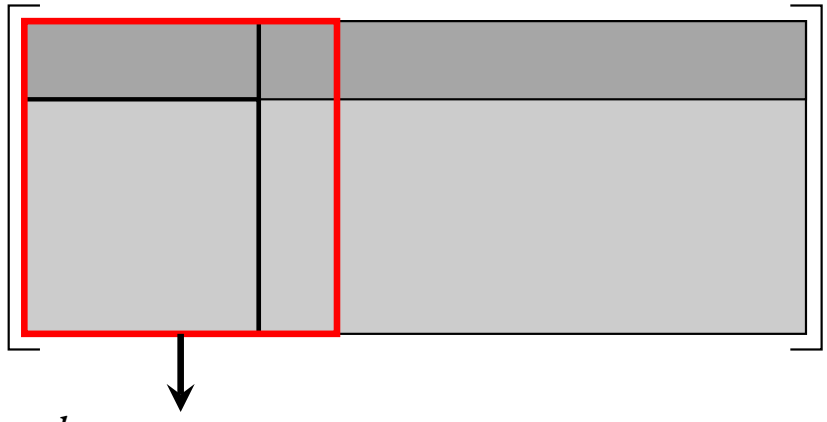

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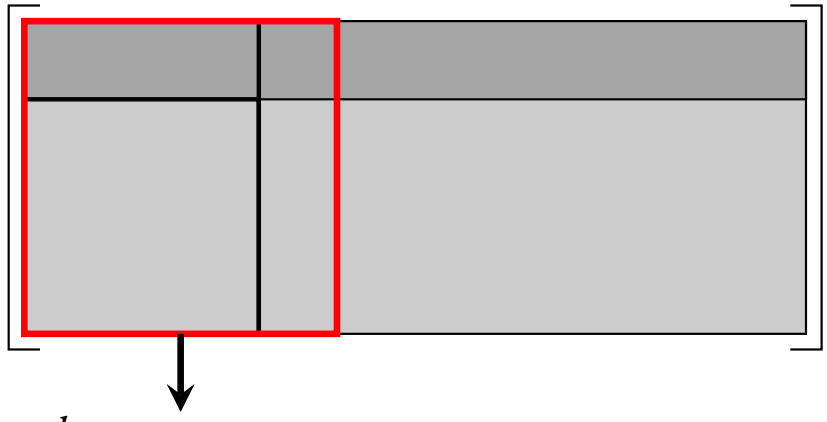

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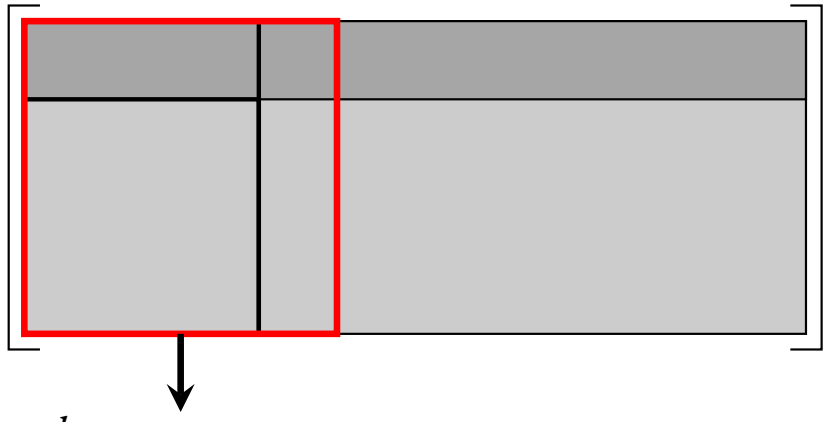

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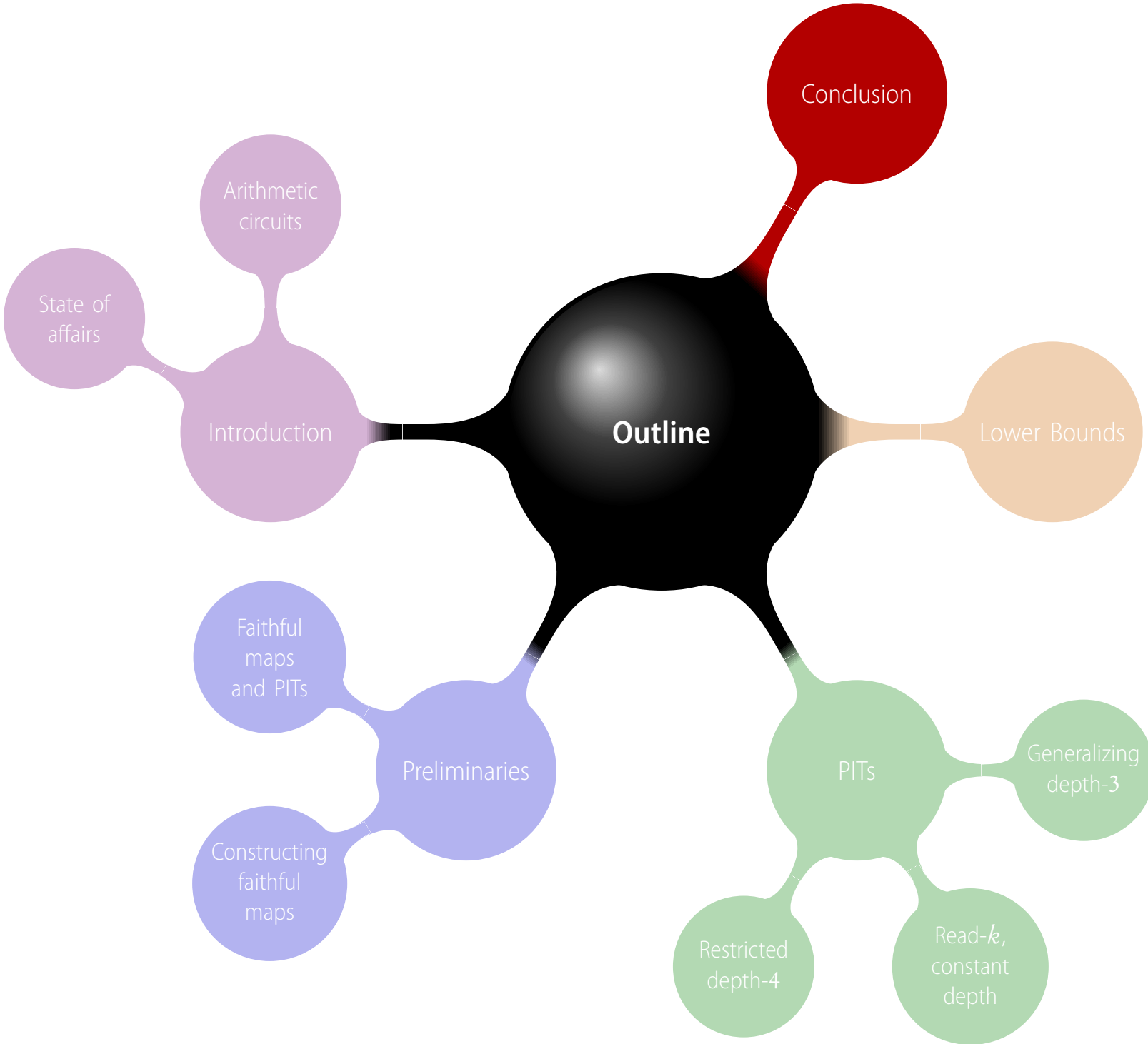
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Concluding Remarks

- Generalizes all known polynomial time black-box PITs for sub-classes of constant depth formulae.
- Unified approach.
- Simpler proofs.

Open Problems

Models where PIT is known but not hit by the Jacobian (yet!)

- Arbitrary depth, read-k formulae

  However, Jacobian gives a quasipoly blackbox test for arbitrary depth read-1 formulae

- Diagonal circuits: $\ell_1^d + \cdots + \ell_m^d \stackrel{?}{=} 0$

  Polynomial time non-blackbox known [Kayal09,Saxena08]

Others problems

- PIT for bounded fan-in depth-4 circuits? With constant degree sparse polyomials?
- PITs for polynomials with low dimension partial-derivative space?
- Conjecture on independence of minors
- Fields of small characteristic

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