1 Introduction

In the first part of this book, starting with this chapter, we focus on the single hop model for wireless networks, where each source has a destination at a fixed distance from it, and each source transmits its information directly to its destination without the help of any other node in the network.

For the single hop model, in this chapter, we begin by introducing the concept of transmission capacity of wireless networks, that measures the largest number of simultaneously allowed transmissions across space, satisfying a per-user outage probability constraint. To facilitate the analysis of transmission capacity, we assume that the nodes of the wireless network are distributed uniformly randomly in space as a Poisson point process. Then we present some tools from stochastic geometry that are necessary for analysis, with some relevant examples of applications of these tools.

Using these tools/results from stochastic geometry, we then present an exact characterization of the transmission capacity of a wireless network under the Rayleigh fading model, when each node has a single antenna and uses an ALOHA protocol. The derived results reveal the dependence of several important parameters on the transmission capacity, such as outage probability constraint, path-loss exponent, rate of transmission per node, ALOHA transmission probability etc. Using the exact expression for the transmission capacity, we also find the optimal transmission probability for the ALOHA protocol that maximizes the transmission capacity. For the path-loss model, where the effects of multi-path fading are ignored, we also present a lower and upper bound on the transmission capacity that are tight in the parameters of interest. The technique used in finding the lower and upper bound has wider ramifications for cases where exact expressions on the transmission capacity cannot be found.

We then present a surprising result that even when each node transmits independently using an ALOHA protocol in a wireless network, where nodes are located uniformly randomly, the interference received from all nodes at any point in space has spatial as well as temporal correlation. The inherent randomness arising out of location of nodes gives rise to this correlation. Temporal correlation impacts the performance of ARQ type protocols and the results presented in this chapter help in analyzing the performance of ARQ type protocols in wireless networks in Chapter **??**.

Finally, we consider guard-zone based and CSMA scheduling protocols, that are *smarter* than ALOHA protocol in choosing which transmitters should be active simultaneously to maximize the transmission capacity. With the guard-zone based strategy, only those nodes that are at a distance greater than a threshold from any receiver are allowed to transmit, thus restricting the interference seen by any receiver. With CSMA protocol, each node measures the channel, and transmits depending on a function of the channel measurement. Guard-zone based strategy or CSMA protocol, however, do not lend themselves to exact analysis because of correlations across different node transmissions, and we present only an approximate characterization of their performance that is known to be accurate by extensive simulations.

2 Transmission Capacity Formulation

We begin by defining a homogenous Poisson point process (PPP) as follows, that will be used to model the location of the nodes of the wireless network.

Definition 2.1 *For compact sets $A \subset \mathbb{R}^2, B \subset \mathbb{R}^2$, with finite area $\nu(A) < \infty, \nu(B) < \infty$, a homogenous PPP Φ with density λ is a random point process with*

$$\mathbb{P}_\Phi(\#(A) = k) = \frac{(\lambda\nu(A))^k}{k!}e^{-\lambda\nu(A)}, \tag{1}$$

and

$$\mathbb{P}_\Phi(\#(A) = k, \#(B) = m) = \mathbb{P}_\Phi(\#(A) = k)\mathbb{P}_\Phi(\#(B) = m), \tag{2}$$

for $A \cap B = \phi$. The specific case of A containing no nodes $k = 0$ is called the void probability of the PPP,

$$\mathbb{P}_\Phi(\#(A) = 0) = e^{-\lambda\nu(A)}. \tag{3}$$

The first condition (1) requires that the number of points of a PPP lying in a finite region is a Poisson random variable with mean λ times the area of the region, where λ is the density or the expected number of points of PPP in an unit area. The second condition (2) states that the number of points in non-overlapping regions should be independent.

Consider the following example that shows the correspondence between PPP distributed node locations and nodes that are located uniformly randomly in a given area of interest.

Example 2.2 *Let n nodes be distributed uniformly in region A with area $\nu(A)$. Then the probability that there are k nodes in region $B \subseteq A$ is binomial with parameters $\left(n, \frac{\nu(B)}{\nu(A)}\right)$. Taking the limit in the number of nodes $n \to \infty$ while fixing the density of nodes $\frac{n}{\nu(A)} = \lambda$, the probability that there are k nodes in region B is Poisson distributed with $\lambda\nu(B)$.*

To understand the fundamental throughput limits (capacity) of a wireless network, we model the location of nodes of a wireless network to be distributed according to a Poisson point process (PPP). The PPP assumption corresponds to having nodes distributed uniformly random in a given area of interest. Even though this is not entirely accurate, assuming that nodes are distributed uniformly randomly is reasonable for some of the main examples of wireless networks such as sensor networks, where sensors are thrown randomly in a given area, or military or vehicular networks where nodes are mobile and there locations are close to being symmetrically distributed across the network. The uniformly distributed node locations assumption serves as a reasonable approximation to a realistic wireless network scenario as well as allows analytical tractability.

To be specific, we consider a wireless network located in $\mathbb{R}^2$ with the location of nodes distributed as a homogenous PPP with density λ, i.e. the expected number of nodes per unit area is λ. A PPP is characterized by two properties; the probability of the number of nodes in any area A with Lebegue measure $\nu(A)$ is Poisson distributed with parameter $\lambda\nu(A)$, and the number of nodes lying in non-overlapping areas are independent.

To avoid the overload of notation, whenever required, we let T_n and R_n denote both the node and the location of the n^{th} transmitter and receiver, respectively. Let $\Phi_T = \{T_n\}$ be the transmitter location PPP process, and $\Phi_R = \{R_n\}$ be the receiver location process. We assume that the receiver R_n corresponding to the transmitter T_n is at a fixed distance d away in a random orientation. This assumption is not really binding but made for purposes of simple exposition. Results could be extended to random distances between transmitters and receivers by taking an expectation with respect to the distance distribution.

In this chapter, we assume each transmitter and receiver to have a single antenna. Extension to multiple antennas is the subject matter of Chapter **??**. We start by considering the ALOHA protocol, where each transmitter is active with probability p independently of all other transmitters. Since an independent thinning of a PPP is also a PPP (Prop. 3.2), the active set of nodes of a wireless network with the ALOHA protocol also form a PPP, allowing analytical tractability. More sophisticated scheduling policies provide better performance, however, they entail correlation among different transmitters breaking the PPP assumption on the transmitter locations leading to more complicated analysis. We consider two such protocols in Sections 8.1 and 8.2.

Under this ALOHA protocol model, the received signal at receiver R_n is

$$y_n = \sqrt{P} d^{-\alpha/2} h_{nn} x_n + \sum_{m:T_m \in \Phi_T \setminus \{T_n\}} \sqrt{P} d_{mn}^{-\alpha/2} \mathbf{1}_m h_{mn} x_m + w, \tag{4}$$

where x_n is the signal transmitted from transmitter T_n with power P, d_{mn} and h_{mn} are the distance and the channel coefficient between T_m and receiver R_n, respectively, $\mathbf{1}_m$ is an indicator function that represents if T_m is active or not, and w is the AWGN with zero mean and unit variance. With the ALOHA protocol, $\mathbf{1}_m = 1$ with probability p, and 0 with probability $1-p$. In (4), $Pd_{mn}^{-\alpha}|h_{mn}|^2$ is the interference power of transmitter T_m at receiver R_n, and

$$I = \sum_{m:T_m \in \Phi_T \setminus \{T_n\}} P d_{mn}^{-\alpha} |h_{mn}|^2 \tag{5}$$

is the total interference power seen at R_0.

As stated in Assumption **??**, throughout this book, we consider the path-loss function to be $d^{-\alpha}$ for ease of exposition. Most of analysis presented in this book is applicable for more general path-loss functions such as $\frac{1}{1+d^{\alpha}}$, which compared to $d^{-\alpha}$ model does not have a singularity at 0. The $d^{-\alpha}$ path-loss function models the far field communication fairly well, but results in unbounded signal power at extremely close distances.

From (4), the signal-to-interference-plus-noise ratio (SINR) at receiver R_n is

$$\mathsf{SINR}_n = \frac{Pd^{-\alpha}|h_{nn}|^2}{\sum_{m:T_m \in \Phi_T \setminus \{T_n\}} P d_{mn}^{-\alpha} \mathbf{1}_m |h_{mn}|^2 + 1}, \tag{6}$$

and the outage probability $P^n_{out}(B)$ at receiver R_n is defined to be the event that the SINR is below a threshold $\beta(B)$ that is a function of rate of transmission B bits/sec/Hz,

$$P^n_{out}(B) = \mathbb{P}(\mathsf{SINR} \le \beta). \tag{7}$$

Remark 2.3 *With PPP distributed transmitter locations, the SINR is identically distributed for all receivers, hence we drop the index n from outage probability definition* (7)*, and represent it as $P_{out}(B)$.*

With mutual information to be equal to $\log(1 + \mathsf{SINR})$, for B bits/sec/Hz transmission rate, $\beta(B) = 2^B - 1$. For ease of notation, we just write β in place of $\beta(B)$. We consider the quality of service (QoS) requirement as the constraint on the outage probability. In particular, we assume an outage probability constraint of ϵ with transmission rate B bits/sec/Hz. The outage probability constraint of ϵ, allows on average $(1-\epsilon)B$ bits/sec/Hz of successful transmission rate between any transmitter-receiver pair. Since the transmitter density is λ nodes/m^2, on average, $(1-\epsilon)B\lambda$ bits/sec/Hz/m^2 can be transmitted in the wireless network. This is essentially the idea behind the concept of transmission capacity defined in [1] that captures the spatial capacity of the network or the number of simultaneously allowed transmissions under an outage probability constraint. The formal definition of transmission capacity is as follows.

Definition 2.4 *Assuming B bits/sec/Hz of transmission rate between any transmitter-receiver pair, and an outage probability constraint of ϵ at each receiver, let*

$$\lambda^\star = \sup\{\lambda : P_{out}(B) \le \epsilon, \ \forall n\}.$$

The transmission capacity of a wireless network with PPP distributed nodes with density $\lambda^\star$ is defined as

$$C := \lambda^\star(1-\epsilon)B \text{ bits/sec/Hz/m}^2.$$

Assumption 2.5 *To keep the problem non-trivial, we assume that the power transmitted P by each transmitter is sufficient to satisfy the outage probability constraint of ϵ in the absence of interference. In the absence of interference,*

$$\mathsf{SINR} = \mathsf{SNR} = Pd^{-\alpha}|h_{nn}|^2.$$

Thus, if h_{nn}'s are Rayleigh distributed, i.e. $|h_{nn}|^2 \sim \exp(1)$, then the outage probability without interference ($I = 0$ in (5)*) is*

$$P_{out}(B) = \mathbb{P}(\mathsf{SNR} \le \beta) = \mathbb{P}\left(d^{-\alpha}|h_{nn}|^2 \le \beta\right) = 1 - \exp\left(-\frac{\beta d^\alpha}{P}\right).$$

Thus, we assume that power P is such that $1 - \exp\left(-\frac{\beta d^\alpha}{P}\right) \le \epsilon$.

To find the transmission capacity, we first need to derive an expression for the outage probability $P_{out}(B)$ in terms of λ and B. Then optimizing over the constraint $P_{out}(B) \le \epsilon$, we can obtain $\lambda^\star$. To find the outage probability expression, we need tools from stochastic geometry which are detailed as follows.

3 Basics of Stochastic Geometry

Proposition 3.1 *A homogenous PPP is stationary, i.e. if $\Phi = \{x_1, x_2, \dots\}$ is any homogenous PPP with density λ, then $\Phi_x = \{x_1+x, x_2+x, \dots\}$ is also a homogenous PPP with density λ.*

Proof: Follows easily from Definition 2.1, since for a PPP, the probability for any number of nodes to lie in any region only depends on its Lebesgue measure/area in $\mathbb{R}^2$. □

Proposition 3.2 *If the points of a homogenous PPP are independently retained with probability p and removed with probability $1-p$, the resulting process is also a homogenous PPP with density λp. This is called the random thinning property of a PPP.*

Proof: Left as an excercise. □

Theorem 3.3 *Slivnyak's Theorem: Let Φ be any homogenous PPP. Then conditioned on $x \in \Phi$, $\mathbb{P}(f(\Phi)|x \in \Phi) = \mathbb{P}(f(\Phi \cup \{x\}))$ for any function f.*

Slivnyak's theorem is an important result that allows us to compute probabilities of events conditioned on the event that there is a point of Φ at location x, which is a zero probability event. It essentially says that conditioned on the event that there is a point of Φ at location x, the distribution is equivalent to a new process that is a union of Φ and an extra point at x.

The utility of Slivnyak's theorem is illustrated by the following example.

Example 3.4 *Let $B \subseteq \mathbb{R}^2$ be a compact set containing the origin o. Let Φ be a PPP, and let $o \in \Phi$, i.e. there is a point of the PPP at the origin. Then*

$$\mathbb{P}(\#_\Phi(B) = k|o \in \Phi) = \mathbb{P}(\#_{\Phi \cup \{o\}}(B) = k) = \mathbb{P}(\#_\Phi(B) = k-1).$$

Note that without Slivnyak's theorem, the same answer can be derived by conditioning on the event that a point of Φ is in $\mathbf{B}(o, r)$ and letting $r \to 0$.

Definition 3.5 *Let $\mathcal{G}$ be the family of all non-negative, bounded measurable functions $g : \mathbb{R}^d \to \mathbb{R}$ on $\mathbb{R}^d$ whose support $\{x \in \mathbb{R}^d : g(x) > 0\}$ is bounded. Let $\mathcal{F}$ be the family of all functions $f = 1 - g$, for $g \in \mathcal{G}$, $0 \le g \le 1$. Then the probability generating functional for a point process $\Phi = \{x_n\}$ is defined as*

$$PGF(f) = \mathbb{E}\left\{\prod_{x_n \in \Phi} f(x_n)\right\}.$$

Theorem 3.6 *For a homogenous PPP Φ with density λ, the probability generating functional is given by*

$$PGF(f) = \exp^{-\int(1-f(x))\lambda dx}.$$

Theorem 3.6 is very useful for deriving the outage probability in a PPP distributed wireless network by allowing us to compute the expectation of a product of functions over the entire PPP. We will make use of Theorem 3.6 quite frequently in this book.

Theorem 3.7 *Campbell's Theorem : For any measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and for a homogenous PPP Φ with density λ,*

$$\mathbb{E}\left\{\sum_{n:x_n\in\Phi} f(x_n)\right\} = \int_{\mathbb{R}^d} f(x)\lambda dx.$$

Using Campbell's Theorem one can show that the interference received at the origin (or any other point using stationarity) from all points of the PPP with the path-loss model of $x^{-\alpha}$ is unbounded for any α as follows.

Example 3.8 *Let $I = \sum_{n:x_n\in\Phi} x_n^{-\alpha}|h_n|^2$ be the inteferfence received at the origin from all points of the PPP, where the fading gains $|h_n|^2$ are i.i.d. with mean 1. Then from Campbell's Theorem*

$$\mathbb{E}\left\{I\right\} = \mathbb{E}\left\{\sum_{n:x_n\in\Phi} x_n^{-\alpha}|h_n|^2\right\} = \mathbb{E}\left\{\sum_{n:x_n\in\Phi} x_n^{-\alpha}\right\} = \int_{\mathbb{R}^2} x^{-\alpha}\lambda dx, \quad (8)$$

since $\mathbb{E}\{|h_n|^2\} = 1$. Since the path-loss function $x^{-\alpha}$ has a singularity at 0, the expected interference $\mathbb{E}\left\{I\right\}$ is unbounded for any value of α. However, if we can avoid interference coming from a small disc of radius ϵ around the origin, $\mathbf{B}(0, \epsilon)$, i.e. somehow inhibit all points of the PPP within $\mathbf{B}(0, \epsilon)$, then $\mathbb{E}\left\{I\right\}$ will be finite. We will use this technique of inhibition to compute the expected interference for finding meaningful bounds on the transmission capacity.

Definition 3.9 *Marked Poisson Process: A marked PPP Φ_M is obtained by attaching a mark m_n to each point of $x_n \in \Phi$, where Φ is a PPP, and $\Phi_M = \{(x_n, m_n) : x_n \in \Phi\}$. Marks could represent the power transmitted by point x_n, its color, shape of any other characterstic. The marks m_n belong to set M with distribution $\mathcal{M}$, such that for any bounded set $A \subset \mathbb{R}^2$, $\#\left(\Phi_M(A \times \mathsf{M})\right) < \infty$ almost surely, i.e. the number of points of Φ_M lying in a bounded region are finite.*

Next Theorem allows us to make a correspondence between a marked PPP Φ_M and a PPP defined over $\mathbb{R}^2 \times \mathsf{M}$.

Theorem 3.10 *Marking Theorem : The following statements are equivalent*

- *a marked PPP Φ_M, where if conditioned on the points $\{x_n\}$ of the PPP Φ, the marks m_n of the marked PPP Φ_M with marks in M, are independent, with probability distribution $\mathcal{M}$ on M,*
- *a PPP on $\mathbb{R}^2 \times \mathsf{M}$ with measure $\Lambda(A \times B) = \lambda\nu(A) \times \mathcal{M}(B)$, where $\nu(A)$ is the Lebesgue measure of A.*

We next present three examples to illustrate how Marking Theorem is useful for analyzing PPP distributed ad hoc networks.

Example 3.11 *Finding the distribution of the L^{th} strongest interferer in a PPP. Let Φ be a PPP, and let $I_n = x_n^{-\alpha}|h_n|^2$ be the interference power of the n^{th} (unordered) transmitter $x_n \in \Phi$ at the origin, where $|h_n|^2$ are i.i.d. We want to find the distribution of the L^{th} largest interference power.*

Let us define I_n to be mark corresponding to $x_n \in \Phi$, and consider the marked PPP as

$$\Phi_M = \{(x_n, I_n) : x_n \in \Phi\}.$$

Note that $I_n \in \mathbb{R}^+$ are independent given x_n, since $|h_n|^2$ are independent. Thus, from the Marking Theorem, Φ_M is equivalent to a PPP Ψ on $\mathbb{R}^2 \times \mathbb{R}^+$ with an appropriate density measure $\Lambda(B)$ on subsets B of $\mathbb{R}^2 \times \mathbb{R}^+$. Let us define

$$\mathcal{B}(g) = \{(x_n, I_n) : I_n > g\},$$

to be the set of points x_n of Φ such that the interference they cause at the origin is more than some threshold g. Note that $\mathcal{B}(g) \subset \Psi$ is the set of points lying in a subset of $\mathbb{R}^2 \times \mathbb{R}^+$, and hence the number of points in $\mathcal{B}(g)$ is Poisson distributed with mean equal to the density measure of the subset corresponding to $\mathcal{B}(g)$, i.e.

$$\begin{aligned}
\Lambda(\mathcal{B}(g)) &= \lambda \mathbb{P}(x^{-\alpha}|h|^2 > g), \\
&= \lambda \mathbb{E}_{|h|^2}\left\{\int_{0, x\in\mathbb{R}^2}^{(|h|^2/g)^{1/\alpha}} dx\right\}, \\
&\stackrel{(a)}{=} \lambda \mathbb{E}_{|h|^2}\left\{\int_{0, x\in\mathbb{R}}^{(|h|^2/g)^{1/\alpha}} 2\pi x dx\right\}, \\
&= \lambda \int_0^\infty \int_{0, x\in\mathbb{R}}^{(|h|^2/g)^{1/\alpha}} 2\pi x dx f_h(h) dh,
\end{aligned}$$

where we get the term $2\pi r$ in (a) by changing the integration from $\mathbb{R}^2$ to $\mathbb{R}^1$. Hence the cumulative distribution function $F_L(g)$ of the L^{th} strongest interferer is equal to the probability that there are less than or equal to $L-1$ points in $\mathcal{B}(g)$.

Marking Theorem can also be used to obtain many other interesting results as described in the next two examples.

Example 3.12 *Consider a random disc process on $\mathbb{R}^2$ whose centers form a PPP Φ with density λ, and where for each point $x_n \in \Phi$ an independent random radius r_n (mark) is selected with distribution $F_{\mathsf{R}}(\mathsf{r}) = \mathbb{P}(\mathsf{R} \le \mathsf{r})$. From the Marking Theorem, this process is equivalent to a PPP on $\mathbb{R}^2 \times \mathbb{R}^+$ with density measure $\Lambda(A \times [0,\ \mathsf{r}]) = \lambda\nu(A)F_{\mathsf{R}}(\mathsf{r})$ for compact $A \subset \mathbb{R}^2$ and $\mathsf{r} > 0$.*

A disc $\mathbf{B}(x, \mathsf{r})$ contains the origin o if and only if (x, r) belongs to set $S = \{(x, \mathsf{r}) : x \in \mathbf{B}(o, \mathsf{r})\}$, i.e. x lies in the circle with center as origin and radius r. Since S is a subset of $\mathbb{R}^2 \times \mathbb{R}^+$, the number of points lying in S (also the number of discs containing the origin) are Poisson distributed with mean that is equal to the density measure of S. From the Marking Theorem the density measure of S is

$$\Lambda(S) = \lambda \int_S F_{\mathsf{R}}(d\mathsf{r})dx = \lambda \int_0^\infty \nu(\mathbf{B}(o, \mathsf{r}))F_{\mathsf{R}}(d\mathsf{r})dx = \lambda\mathbb{E}(\pi\mathsf{R}^2).$$

Example 3.12 is taken from the lecture notes of Gustavo De Veciana, ECE Dept., The University of Texas at Austin. Next, we present an application of the Marking Theorem to prove the random thinning property of the PPP (Prop. 3.2).

Example 3.13 *Random Thinning: With each point x_n of a PPP Φ on $\mathbb{R}^2$ associate a mark $m_n \in \{0,1\}$ with $\mathbb{P}(m_n = 1) = p$ independently of all other points $x_m, m \neq n$. Then from the Marking Theorem, this marked PPP is a equivalent to a PPP on $\mathbb{R}^2 \times \{0,1\}$, with density measure*

$$\Lambda(A \times \{y\}) = \lambda\nu(A)[yp + (1-y)(1-p)]$$

for compact $A \subset \mathbb{R}^2$ and $y \in \{0,1\}$. Define a subset

$$S = \{(A, m_n) : m_n = 1\} \subset \mathbb{R}^2 \times \{0,1\}$$

that corresponds to the thinned version of the original PPP Φ. The number of points in S is Poisson distributed with density measure $\Lambda(S) = \lambda\nu(A)p$.

Now with relevant background of stochastic geometry tools at hand, we proceed towards analyzing the outage probability (7), and consequently the transmission capacity. Note that the outage probability (7) is invariant to the choice of any transmitter-receiver pair because of the stationarity of the PPP. Thus, without any loss of generality, we consider the pair (T_0, R_0) for the transmission capacity analysis. From (6), the SINR for the (T_0, R_0) pair is

$$\mathsf{SINR} = \frac{Pd^{\alpha}|h_{00}|^2}{\sum_{m:T_m\in\Phi_T\setminus\{T_0\}} Pd_{m0}^{-\alpha}\mathbf{1}_m|h_{m0}|^2 + 1}.$$

Remark 3.14 *Since we have considered a typical transmitter-receiver pair (T_0, R_0), to derive the outage probability $\mathbb{P}(\mathsf{SINR} \le \beta)$, we need to know the distribution of interference I received by R_0, where*

$$I := \sum_{m:T_m\in\Phi_T\setminus\{T_0\}} Pd_{m0}^{-\alpha}\mathbf{1}_m|h_{m0}|^2,$$

conditioned on a transmitter being located at T_0.

From Slivnyak's Theorem (Theorem 3.3), we know that conditioned on the event that there is a transmitter at T_0, the equivalent point process is $\Phi_T^0 = \Phi_T \cup \{T_0\}$. Thus, now working with Φ_T^0, for communication between transmitter T_0 and receiver R_0, all transmitters belonging to $\Phi_T^0\setminus\{T_0\} = \Phi_T$ are interferers. Thus, the conditional interference seen at the receiver R_0 is $I^0 := \sum_{m:T_m\in\Phi_T} Pd_{m0}^{-\alpha}\mathbf{1}_m|h_{m0}|^2$.

Therefore, the conditional distribution of interference I^0 seen at receiver R_0, is the sum of interferences from all points of the homogenous PPP with density λ, which is also called as the shot-noise process [2]. Thus, the conditional outage probability $P_{out}(B)$ is equal to

$$\begin{aligned}\mathbb{P}(\mathsf{SINR} \le \beta) &= \mathbb{P}\left(\frac{Pd^{\alpha}|h_{00}|^2}{\sum_{m:T_m\in\Phi_T\setminus\{T_0\}} Pd_{m0}^{-\alpha}\mathbf{1}_m|h_{m0}|^2 + 1} \,\middle|\, T_0 \textit{ is a transmitter}\right), \\ &= \mathbb{P}\left(\frac{Pd^{\alpha}|h_{00}|^2}{\sum_{m:T_m\in\Phi_T} Pd_{m0}^{-\alpha}\mathbf{1}_m|h_{m0}|^2 + 1}\right). \qquad (9)\end{aligned}$$

Now we are ready to derive a closed form expression for the outage probability, and consequently the transmission capacity, as described in the next section.

4 Rayleigh Fading Model

In this section, we consider the received signal model of (4), when the fading channel magnitudes h_{nm} are i.i.d. exponentially distributed (Rayleigh fading) across different users n, m. Rayleigh fading is the most popular fading model for analyzing wireless communication systems, and represents the scenario of richly scattered fading environment. In Section 5, we will consider just the path-loss model with no fading, i.e. $h_{nm} = 1$, that models the line-of-sight communication, and find tight bounds on the transmission capacity.

4.1 Derivation of Transmission Capacity

Theorem 4.1 *The outage probability at a typical receiver R_0 with PPP distributed transmitter locations with density λ is*

$$P_{out}(B) = 1 - \exp\left(-\frac{d^\alpha\beta}{P}\right)\exp\left(-\lambda p c \beta^{\frac{2}{\alpha}} d^2\right),$$

where $c = \frac{2\pi\Gamma(\frac{2}{\alpha})\Gamma(1-\frac{2}{\alpha})}{\alpha}$, and Γ is the Gamma function. Hence, under an outage probability constraint of ϵ, the transmission capacity is

$$C = \frac{-\ln(1-\epsilon) - \frac{d^\alpha\beta}{P}}{c\beta^{\frac{2}{\alpha}}d^2}(1-\epsilon)B \ \textit{bits/sec/Hz/m}^2.$$

Remark 4.2 *Recall from Assumption 2.5, $-\ln(1-\epsilon) > \frac{d^\alpha\beta}{P}$. Thus, the transmission capacity is always non-negative.*

Proof: From the outage probability definition (7) and conditional SINR distribution with a transmitter located at T_0 (9),

$$\begin{aligned}
P_{out}(B) &= \mathbb{P}\left(\frac{Pd^{-\alpha}|h_{00}|^2}{\sum_{m:\ T_m\in\Phi} P\mathbf{1}_m d_{m0}^{-\alpha}|h_{m0}|^2 + 1} \le \beta\right), \\
&\stackrel{(a)}{=} 1 - \mathbb{E}\left\{\exp\left(-\frac{d^\alpha\beta}{P}\left(\sum_{m:\ T_m\in\Phi} P\mathbf{1}_m d_{m0}^{-\alpha}|h_{m0}|^2 + 1\right)\right)\right\}, \\
&= 1 - \exp\left(-\frac{d^\alpha\beta}{P}\right)\mathbb{E}\left\{\exp\left(-d^\alpha\beta\sum_{m:\ T_m\in\Phi}\mathbf{1}_m d_{m0}^{-\alpha}|h_{m0}|^2\right)\right\},
\end{aligned}$$

where (a) follows by taking the expectation with respect to $|h_{00}|^2$, where $|h_{00}|^2 \sim \exp(1)$. We have to take expectation with respect to $|h_{m0}|^2 \sim \exp(1)$, and ALOHA protocol's indicator variable $\mathbf{1}_m$ that is 1 with probability p and 0 otherwise. We first take the expectation with respect to $|h_{m0}|^2$ that are i.i.d. $\forall\ m$, and obtain

$$P_{out}(B) \quad = \quad 1 - \exp\left(-\frac{d^\alpha \beta}{P}\right) \mathbb{E}\left\{ \prod_{m:\ T_m \in \Phi} \left(\frac{1}{1 + \mathbf{1}_m d^\alpha \beta d_{m0}^{-\alpha}} \right) \right\}$$

Now we take the expectation with respect to the ALOHA protocol indicator function $\mathbf{1}_m$. Note that

$$\frac{1}{1 + \mathbf{1}_m d^\alpha \beta d_{m0}^{-\alpha}} = \frac{\mathbf{1}_m}{1 + d^\alpha \beta d_{m0}^{-\alpha}} + 1 - \mathbf{1}_m.$$

Thus,

$$\mathbb{E}\left\{ \frac{1}{1 + \mathbf{1}_m d^\alpha \beta x^{-\alpha}} \right\} = \frac{p}{1 + d^\alpha \beta d_{m0}^{-\alpha}} + 1 - p. \tag{10}$$

Hence,

$$\begin{aligned}
P_{out}(B) \quad &\stackrel{(b)}{=} \quad 1 - \exp\left(-\frac{d^\alpha \beta}{P}\right) \mathbb{E}\left\{ \prod_{x \in \Phi} \left(\frac{p}{1 + d^\alpha \beta x^{-\alpha}} + 1 - p \right) \right\}, \\
&\stackrel{(c)}{=} \quad 1 - \exp\left(-\frac{d^\alpha \beta}{P}\right) \exp\left(-\lambda \int_{\mathbb{R}^2} 1 - \left(\frac{p}{1 + d^\alpha \beta x^{-\alpha}} + 1 - p \right) dx \right), \\
&= \quad 1 - \exp\left(-\frac{d^\alpha \beta}{P}\right) \exp\left(-\lambda \int_{\mathbb{R}^2} \frac{p d^\alpha \beta x^{-\alpha}}{1 + d^\alpha \beta x^{-\alpha}} dx \right), \\
&= \quad 1 - \exp\left(-\frac{d^\alpha \beta}{P}\right) \exp\left(-2\pi\lambda \int_0^\infty \frac{p d^\alpha \beta x^{-\alpha}}{1 + d^\alpha \beta x^{-\alpha}} x dx \right), \\
&\stackrel{(d)}{=} \quad 1 - \exp\left(-\frac{d^\alpha \beta}{P}\right) \exp\left(-\lambda p c \beta^{\frac{2}{\alpha}} d^2 \right),
\end{aligned}$$

where (b) follows by replacing the distance of the m^{th} interferer d_{m0} by x, (c) follows by using the probability generating function of the PPP (Theorem 3.6), and finally in (d) $c = \frac{2\pi\Gamma(\frac{2}{\alpha})\Gamma(1-\frac{2}{\alpha})}{\beta}$ and Γ is the Gamma function. The transmission capacity expression follows easily by using the outage probability constraint of $P_{out}(B) \leq \epsilon$.
$\square$

In Theorem 4.1, we derived the exact expression for the transmission capacity using the probability generating functional of the PPP. The probability generating functional allowed us to compute the expectation of a product of functions over the entire PPP. Also the fact that the channel coefficients are Rayleigh distribution was instrumental in allowing us to derive the closed form distribution using the probability generating functional. Instead of using the probability generating functional, an alternate way to compute an exact expression for the transmission capacity is via the use of the Laplace transform of the shot-noise process as described in [2].

The derived expression reveals the exact dependence of critical parameters such as the outage probability constraint ϵ, rate B, and distance between transmitter and receiver d on the transmission capacity. To see how does the transmission capacity scales with the outage probability constraint ϵ, it is useful to look at the regime of small values of ϵ that corresponds to an extremely strict outage probability constraint. For small values of ϵ, using the Taylor series expansion of $\log$, the transmission capacity is seen to be directly proportional to the outage probability constraint of ϵ. So tightening the outage probability constraint leads to a linear fall in the transmission capacity.

To reveal the interplay between distance d between any transmitter-receiver pair and the transmission capacity, we look at the interference limited regime, where the interference power at any receiver is much large than the AWGN power i.e.

$$I = \sum_{T_m \in \phi} P d_{m0}^{-\alpha} |h_{m0}|^2 >> 1,$$

and we can ignore the AWGN contribution safely without losing accuracy. Ignoring the AWGN term that gives rise to $-d^{\alpha}\beta$ term in the numerator of the transmission capacity expression in Theorem 4.1, the transmission capacity scales as $\Theta(d^{-2})$. Thus, Theorem 4.1 reveals an interesting spatial packing relationship, where the transmission capacity can be interpreted as the packing of as many simultaneous spatial transmissions where each transmission occupies an area of $\Theta(d^2)$. This packing relationship is similar to the $\sqrt{n}$ scaling of throughput capacity result of [3] (discussed in Chapter **??**), where n nodes are distributed uniformly in an unit area. Throughput capacity measures the sum of the rate of successful transmissions achievable between all pairs of nodes simultaneously with high probability. In this section's model, n corresponds to the density λ and each communication happens over a fixed distance d, while in the throughput capacity model, the expected distance between each source-destination pair is a constant. Thus, the transport capacity, i.e. capacity multiplied with the distance is same for the transmission capacity and throughput capacity metric, since from Theorem 4.1, $\lambda \propto \frac{1}{d^2}$, and transport capacity is $\lambda d = \sqrt{n}$, similar to the transport capacity of order $\sqrt{n}$ with the transport capacity metric.

After deriving the transmission capacity with the Rayleigh fading model, next, we consider the path-loss model, where the multi-path fading component is ignored. At first, this might appear futile, and to make it appear even more ridiculous, we will only obtain bounds on the transmission in contrast to an exact expression. The real advantage of the presented bounding techniques is that they are extremely useful in analyzing the transmission capacity of many advanced signal processing techniques, where exact expressions for transmission capacity cannot be found.

5 Path-Loss Model

In this section, we consider a slightly restrictive path-loss model , that does not account for multi-path fading, and where the received signal at a receiver R_n is given by

$$y_n = \sqrt{P} d^{-\alpha/2} x_n + \sum_{m: T_m \in \Phi_T \setminus \{T_n\}} \sqrt{P} d_{mn}^{-\alpha/2} x_m + w, \tag{11}$$

where compared to (4), $h_{nm} = 1, \ \forall, \ n, m,$ and we have absorbed the ALOHA parameter $\mathbf{1}_m$ into the density of the PPP Φ_T which is now equal to $p\lambda$ (Prop. 3.2). Generally, with a simplified model, the analysis becomes easier. Finding an exact expression for the transmission capacity is one exception, where it is known only for the Rayleigh fading model and not for the path-loss model. Tight lower and upper bounds on the transmission capacity [1] are however known with the path-loss model, and described in this section.

From (11), the SINR at receiver R_n is

$$\mathsf{SINR} = \frac{Pd^{-\alpha}}{\sum_{m:T_m\in\Phi_T\backslash\{T_n\}} Pd_{mn}^{-\alpha}\mathbf{1}_m + 1}. \tag{12}$$

As before, we consider a typical transmitter-receiver pair (T_0, R_0) and similar to (9), the outage probability conditioned on the event that there is a transmitter at T_0 is given by

$$\begin{aligned} P_{out}(B) &= \mathbb{P}\left(\frac{Pd^{-\alpha}}{\sum_{m:T_m\in\Phi_T\backslash\{T_n\}} Pd_{mn}^{-\alpha} + 1} \le \beta \,\middle|\, T_0 \in \Phi_T\right), \\ &\stackrel{(a)}{=} \mathbb{P}\left(\frac{Pd^{-\alpha}}{\sum_{m:T_m\in\Phi_T} Pd_{mn}^{-\alpha} + 1} \le \beta\right), \\ &\stackrel{(b)}{=} \mathbb{P}\left(I \ge \frac{d^{-\alpha}}{\beta} - \frac{1}{P}\right), \end{aligned} \tag{13}$$

where (a) follows from the Slivnyak's Theorem, and (b) follows by defining the total interference from Φ_T, as $I := \sum_{m:T_m\in\Phi_T} d_{mn}^{-\alpha}$.

5.1 Upper Bound on the Transmission Capacity

Theorem 5.1 *The transmission capacity with the path-loss model is upper bounded by*

$$C_{ub} = \frac{\epsilon}{\pi}\kappa^{\frac{2}{\alpha}}(1-\epsilon)R + \Theta\left(\epsilon^2\right) \textit{ bits/sec/Hz/m}^2,$$

as $\epsilon \to 0$*, where* $\kappa = \frac{d^{-\alpha}}{\beta} - \frac{1}{P}$.

Proof: Let $\kappa = \frac{d^{-\alpha}}{\beta} - \frac{1}{P}$. Then the interference power received from transmitter T_m located at a distance of $\kappa^{-\frac{1}{\alpha}}$ from R_0 is κ, which by definition of κ is sufficient to cause outage from (13), since $I > \kappa$.

To lower bound the outage probability (upper bound the transmission capacity), we assume that there is at least one transmitter from Φ_T in the disc with radius $\kappa^{-\frac{1}{\alpha}}$ centered at R_0. Consequently, the total interference $I > \kappa$, and the outage is guaranteed. Thus, $P_{out}(B) \ge \mathbb{P}\left(\#(\mathbf{B}(R_0, \kappa^{-\frac{1}{\alpha}})) > 0\right)$. From the void probability of the PPP, we know that

$$\mathbb{P}\left(\#(\mathbf{B}(R_0, \kappa^{-\frac{1}{\alpha}})) > 0\right) = 1 - \exp^{-\lambda\pi\kappa^{\frac{2}{\alpha}}}.$$

With the given outage probability constraint of $P_{out}(B) \le \epsilon$, we find an upper bound on the largest permissible λ to be $\lambda_{ub} = -\ln(1-\epsilon)\frac{1}{\pi}\kappa^{-\frac{2}{\alpha}}$. Expanding, we get

$$\lambda_{ub} = \frac{\epsilon}{\pi}\kappa^{\frac{2}{\alpha}} + \Theta\left(\epsilon^2\right). \tag{14}$$

Corresponding upper bound on the transmission capacity follows immediately by noting that $C_{ub} = \lambda_{ub}(1-\epsilon)R$. $\square$

Keeping ALOHA probability p separately in (13), we get $\lambda_{ub} = \frac{\epsilon}{\pi p}\kappa^{\frac{2}{\alpha}} + \Theta\left(\epsilon^2\right)$ and the upper bound on the transmission capacity identical to (14), since $C_{ub} = p\lambda_{ub}(1-\epsilon)R$.

5.2 Lower Bound on the Transmission Capacity

Theorem 5.2 *The transmission capacity with the path-loss model is lower bounded by*

$$C_{lb} = \left(1 - \frac{1}{\alpha}\right)\frac{\epsilon}{\pi}\kappa^{\frac{2}{\alpha}}(1-\epsilon)R + \Theta\left(\epsilon^2\right) \textit{ bits/sec/Hz/m}^2.$$

Proof: Finding this lower bound is slightly more involved than the upper bound. We divide $\mathbb{R}^2$ into two regions, near field $\mathbf{B}(R_0, s)$ and far field $\mathbb{R}^2 \backslash \mathbf{B}(R_0, s)$ for some s that will be chosen later. Let us define two events

$$\mathcal{E}_N = \left\{\#(\mathbf{B}(R_0, s) > 0)\right\},$$

that corresponds to having at least one transmitter in disc $\mathbf{B}(R_0, s)$, and

$$\mathcal{E}_F = \left\{ \sum_{m: T_m \in \Phi_T, T_m \in \mathbb{R}^2 \backslash \mathbf{B}(R_0, s)} d_{mn}^{-\alpha} > \kappa \right\},$$

that corresponds to the case that the interference received from transmitters lying outside of $\mathbf{B}(R_0, s)$ is more than κ, and hence sufficient to cause outage.

Let $\mathcal{E} = \mathcal{E}_N \cup \mathcal{E}_F$. Then note that for $s \leq \kappa^{\frac{-1}{\alpha}}$, if outage happens (13), i.e. $I > \kappa$, then either there is a transmitter in $\mathbf{B}(R_0, s)$ or the interference from transmitters lying outside $\mathbf{B}(R_0, s)$ is more than κ. Thus, outage implies either of the two events $\mathcal{E}_N$ or $\mathcal{E}_F$ are true. Hence we have

$$P_{out}(B) \leq \mathbb{P}(\mathcal{E}).$$

Moreover, because of the PPP property, events $\mathcal{E}_N$ and $\mathcal{E}_F$ are independent since they are defined over non-overlapping regions. Thus, we bound $\mathbb{P}(\mathcal{E})$ to get a lower bound on the transmission capacity.

$$\begin{aligned} \mathbb{P}(\mathcal{E}) &= \mathbb{P}(\mathcal{E}_N \cup \mathcal{E}_F), \\ &= \mathbb{P}(\mathcal{E}_N) + \mathbb{P}(\mathcal{E}_F) - \mathbb{P}(\mathcal{E}_N)\mathbb{P}(\mathcal{E}_F), \end{aligned}$$

where we have used the independence of $\mathcal{E}_N$ and $\mathcal{E}_F$. Now we work under the constraint $\mathbb{P}(\mathcal{E}) \leq \epsilon$ that implies $P_{out}(B) \leq \epsilon$. We can write $\mathbb{P}(\mathcal{E}) \leq \epsilon$ equivalently as $\mathbb{P}(\mathcal{E}_N) \leq \epsilon_1$ and $\mathbb{P}(\mathcal{E}_F) \leq \epsilon_2$ such that $\epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2 \leq \epsilon$. Defining

$$\lambda_N = \sup\{\lambda | \mathbb{P}(\mathcal{E}_N) \leq \epsilon_1\},$$

and

$$\lambda_F = \sup\{\lambda | \mathbb{P}(\mathcal{E}_F) \leq \epsilon_2\},$$

we get the lower bound on the optimal density $\lambda^\star$ to be

$$\lambda^\star \geq \sup_{\epsilon_1 \geq 0, \epsilon_2 \geq 0, \epsilon_1+\epsilon_2-\epsilon_1\epsilon_2 \leq \epsilon} \{\inf\{\lambda_N, \lambda_F\}\}. \tag{15}$$

From (14), we know that

$$\lambda_N = \frac{\epsilon}{\pi} s^{-\frac{2}{\alpha}} + \Theta\left(\epsilon^2\right). \tag{16}$$

For computing λ_F, we make use of the Chebyeshev's inequality and get

$$\mathbb{P}(\mathcal{E}_F) = \mathbb{P}\left(\sum_{m:T_m\in\Phi_T, T_m\in\mathbb{R}^2\backslash\mathbf{B}(R_0,s)} d_{mn}^{-\alpha} > \kappa\right) \leq \frac{\mathsf{var}}{(\kappa-\mathsf{m})^2}, \tag{17}$$

where

$$\mathsf{var} = Var\left(\sum_{m:T_m\in\Phi_T, T_m\in\mathbb{R}^2\backslash\mathbf{B}(R_0,s)} Pd_{mn}^{-\alpha}\right) = \frac{\pi}{\alpha-1}s^{2(1-\alpha)}\lambda,$$

and

$$\mathsf{m} = \mathbb{E}\left\{\sum_{m:T_m\in\Phi_T, T_m\in\mathbb{R}^2\backslash\mathbf{B}(R_0,s)} Pd_{mn}^{-\alpha}\right\} = \frac{\pi}{\alpha-2}s^{2-\alpha}\lambda,$$

computed directly using the Campbell's Theorem (Theorem 3.7).

By equating the bound on $\mathbb{P}(\mathcal{E}_F)$ (17) to ϵ_2, and keeping the dominant terms, we get

$$\lambda_F = \frac{(\alpha-1)\kappa^2}{\pi^2} s^2(\alpha-1)\epsilon_2 + \Theta(\epsilon_2^2) \tag{18}$$

From (15), we know that for a given ϵ_1, ϵ_2 pair, the optimal solution satisfies $\lambda_N = \lambda_F$. Equating $\lambda_N = \lambda_F$ from (16) and (18), we get

$$s = \left(\frac{\epsilon_1}{\epsilon_2}\right)^{\frac{1}{2\alpha}} ((\alpha-1)\kappa)^{\frac{-1}{2\alpha}}.$$

Thus, for a given ϵ_1, ϵ_2 pair, by substituting for s, we get

$$\lambda_N = \lambda_F = (\alpha-1)^{\frac{1}{\alpha}} \kappa^{\frac{2}{\alpha}} \frac{1}{\pi} \epsilon_1^{1-\frac{1}{\alpha}} \epsilon_2^{\frac{1}{\alpha}}. \tag{19}$$

Moreover, for small outage probability constraint ϵ, $\epsilon_2 = \epsilon - \epsilon_1 + \Theta(\epsilon^2)$, and to get the lower bound we need to solve,

$$\max_{\epsilon=\epsilon_1+\epsilon_2} \lambda_N.$$

Using (19), the optimum is $\epsilon_1^\star = \left(1-\frac{1}{\alpha}\right)\epsilon$ and $\epsilon_2^\star = \frac{\epsilon}{\alpha}$, and we get the required lower bound (15) on $\lambda^\star$ as

$$\lambda_N = \lambda_{lb} = \left(1-\frac{1}{\alpha}\right)\kappa^{\frac{2}{\alpha}} \frac{1}{\pi}\epsilon + \Theta(\epsilon^2). \tag{20}$$

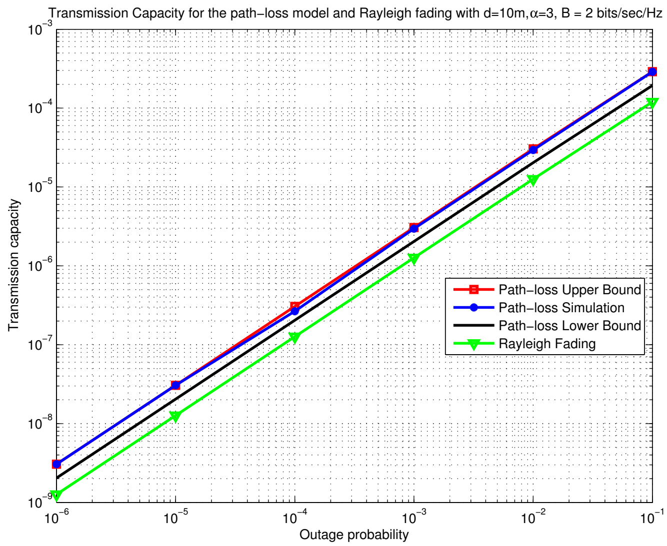

Figure 1: Transmission Capacity with Rayleigh fading and path-loss model with ALOHA protocol.

Required lower bound on the transmission capacity $C_{lb} = \lambda_{lb}(1-\epsilon)R$. $\square$

The derived upper and lower bound on the transmission capacity for the path-loss model have the exact same scaling in terms of the parameters κ (that depends on d, and β) and ϵ (the outage probability constraint), and only differ in constants. Thus, this technique of dividing the overall interference into two regions, and using simple void probability expressions and Chebyeshev's inequality is powerful enough to derive meaningful expressions for the transmission capacity. These bounding technique will come handy, when we analyze more complicated protocols and techniques such multiple antennas, ARQ protocol etc.

Similar to the Rayleigh fading model, comparing the upper and lower bounds derived in Theorems 5.1 and 5.2, using the definition of κ, it is clear that the transmission capacity is inversely proportional to d^2, where d is the distance between each transmitter and receiver. So operationally, the transmission capacity exhibits the same spatial packing behavior with or without taking fading into account.

In Fig. 1, we plot the transmission capacity with respect to the outage probability constraint ϵ for both the Rayleigh fading and path-loss models. For the path-loss model, we plot both the derived upper and lower bound in addition to the simulated performance. We see that the upper bound is very close to the simulated performance. We can also see that there is some performance degradation in transmission capacity while including the multi-path fading that is Rayleigh distributed.

For both the Rayleigh fading and path-loss model, the transmission capacity expressions (in Theorem 4.1, Theorem 5.1, and Theorem 5.2) are independent of the ALOHA probability of access p. This happens since we have constrained the outage probability to be below ϵ, which in turn gives an upper bound on the effective density of the PPP λp, and the transmission capacity loses its dependence on p. If we define the transmission capacity as the product of the density of the PPP and the success probability of any node, which we call as *goodput*, then we can unravel the dependence of p on network performance which is presented in next section.

6 Optimal ALOHA Transmission Probability

Let the network goodput of a wireless network be defined as

$$\mathsf{G} = \lambda(1 - P_{out}(B))B \text{ bits/sec/Hz/m}^2,$$

by accounting for λ concurrent transmissions per meter square at rate B bits/sec/Hz with outage probability $P_{out}(B)$.

Then from Theorem 4.1 using the expression for $P_{out}(B)$, we have

$$\mathsf{G} = \lambda p \exp\left(-\frac{d^\alpha \beta}{P}\right) \exp\left(-\lambda p c \beta^{\frac{2}{\alpha}} d^2\right), \tag{21}$$

where $c = \frac{2\pi\Gamma(\frac{2}{\alpha})\Gamma(1-\frac{2}{\alpha})}{\beta}$. Ignoring, the AWGN contribution $\exp\left(-\frac{d^\alpha\beta}{P}\right)$, we get

$$\mathsf{G} = \lambda p \exp\left(-\lambda p c \beta^{\frac{2}{\alpha}} d^2\right). \tag{22}$$

Differentiating G with respect to p and equating it to 0, the optimal ALOHA access probability is $p^\star = \min\left\{1, \frac{1}{\lambda c \beta^{\frac{2}{\alpha}} d^2}\right\}$, and

$$\mathsf{G}^* = \begin{cases} \frac{1}{e c \beta^{\frac{2}{\alpha}} d^2} & \text{if } \lambda c \beta^{\frac{2}{\alpha}} d^2 > 1, \\ \lambda \exp\left(-\lambda c \beta^{\frac{2}{\alpha}} d^2\right) & \text{o.w.} \end{cases} \tag{23}$$

Thus, even without an outage probability constraint, we see that the good put expression (23) is independent of both λ and p, similar to Theorem 4.1 whenever $\lambda c \beta^{\frac{2}{\alpha}} d^2 > 1$, since the product of the density λ and the optimal ALOHA probability p is a constant. This is a result of a underlying fundamental limit on the maximum density of successful transmissions in a wireless network, which in case of the ALOHA protocol is equal to λp.

In Fig. 2, we plot the network goodput as a function of the ALOHA access probability p. As derived, we can see that the optimal $p = .63$ for $\lambda = 10^{-3}$ with $d = 10m$, $\alpha = 3$ for $B = 2$ bits/sec/Hz transmission rate.

After analyzing the transmission capacity of wireless network with ALOHA protocol in detail, we next highlight a surprising feature of the ALOHA protocol of having both spatial and temporal correlation in interference received at any point in space. With ALOHA protocol, at each time slot, each node transmits independently of all other nodes, but the shared randomness between node locations due to PPP assumption gives rise to counter-intuitive correlations across time and space. We capture this critical phenomenon in the next section, and which will be useful for analyzing the performance of ARQ type protocols in Chapter **??**.

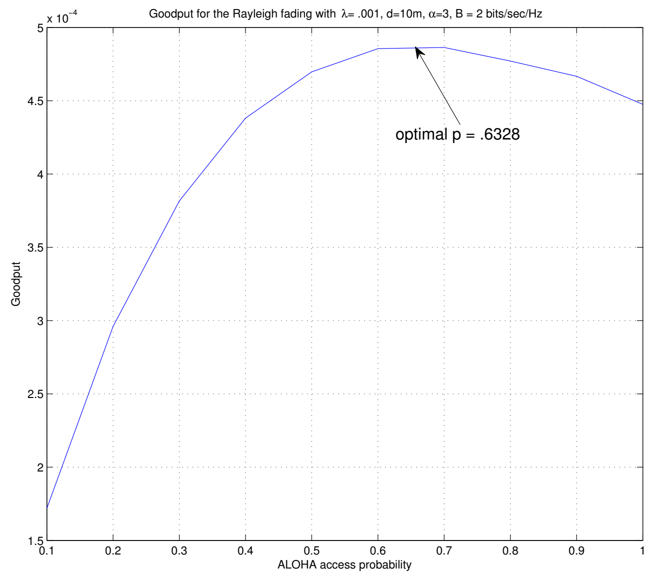

Figure 2: Network goodput G with Rayleigh fading as a function of ALOHA access probability p.

7 Correlations with ALOHA protocol

In this section, we show a counter-intuitive result from [4] that shows that interference received at any point in space in a PPP network is both temporally and spatially correlated while using the ALOHA protocol. One would assume that given that the locations of nodes in a PPP network are uniformly random in any given area, and with ALOHA protocol each node transmits independently across space and time, the interference received at different locations or time instants would be independent, however, that is shown to be incorrect as follows.

To be concrete, in a PPP distributed wireless network with the ALOHA protocol, we fix the transmitter locations drawn from a single realization of PPP, while at each time slot, each node decides to transmit with probability p independently of all other nodes. Finally, the averaging is taken with respect to the PPP. Thus, the shared randomness of transmitter locations gives rise to spatial and temporal correlation.

From (9), we know that conditioned on a transmitter being at T_0, the conditional outage probability at R_0 is

$$\mathbb{P}(\mathsf{SINR} \leq \beta) = \mathbb{P}\left(\frac{Pd^{\alpha}|h_{00}|^2}{\sum_{m:T_m\in\Phi_T} Pd_{m0}^{-\alpha}\mathbf{1}_m|h_{m0}|^2+1} \leq \beta\right).$$

In the above expression, the only stochastic quantity is the interference (shot noise)

$$I_0 = \sum_{m:T_m \in \Phi_T} P d_{m0}^{-\alpha} \mathbf{1}_m |h_{m0}|^2$$

seen at receiver R_0.

To unravel these correlations of interference, let the interference seen at location $u \in \mathbb{R}^2$ at time k be

$$I_u(k) := \sum_{x \in \Phi_T} P g(x-u) \mathbf{1}_p(x,k) |h_{xu}(k)|^2,$$

where $g(.)$ be the path-loss function that only depends on the distance. Throughout this book, we use $g(x-u) = |x-u|^{-\alpha}$. The indicator function $\mathbf{1}_p(x,k)$ means that the transmitter at $x \in \Phi_T$ transmits with probability p using the ALOHA protocol at time k.

The spatio-temporal correlation coefficient between $I_u(k)$ and $I_v(\ell)$ is

$$\begin{aligned} \mathsf{corr}_{xy}(k,l) &= \frac{\mathbb{E}\left\{\left(I_u(k) - \mathbb{E}\{I_u(k)\}\right)\left(I_v(\ell) - \mathbb{E}\{I_v(\ell)\}\right)\right\}}{Var(I_u(k))^{\frac{1}{2}} Var(I_v(\ell))^{\frac{1}{2}}} \\ &= \frac{\mathbb{E}\{I_u(k) I_v(\ell)\} - (\mathbb{E}\{I_u(k)\})^2}{\mathbb{E}\{I_u(k)^2\} - (\mathbb{E}\{I_u(k)\})^2}, \end{aligned} \tag{24}$$

since $I_u(k)$ and $I_v(\ell)$ are identically distributed. We now compute the three expectations in (24) using the Campbell's Theorem and second-order product density (correlation) of the PPP as follows.

First the expected value of $I_u(k)$, which is

$$\begin{aligned} \mathbb{E}\{I_u(k)\} &\overset{(a)}{=} \mathbb{E}\{I_o(k)\}, \\ &= \mathbb{E}\left\{\sum_{x \in \Phi_T} P g(x) \mathbf{1}_p(x,k) |h_{xo}(k)|^2\right\}, \\ &\overset{(b)}{=} \mathbb{E}\left\{\sum_{x \in \Phi(k)} P g(x) \mathbf{1}_p(x,k)\right\}, \\ &\overset{(c)}{=} p\lambda \int_{\mathbb{R}^2} g(x) dx, \end{aligned} \tag{25}$$

where (a) follows since the distribution of $I_u(k)$ is invariant to location of u and o is the origin, (b) follows since the $|h_{xo}|^2$ is Rayleigh distributed with $\mathbb{E}\{|h_{xo}|^2\} = 1$, and finally, (c) follows from the Campbell's Theorem for PPP.

Next, we derive the expression for second moment of the interference, as follows.

$$\begin{aligned}
\mathbb{E}\{I_u(k)^2\} &= \mathbb{E}\{I_o(k)^2\}, \\
&= \mathbb{E}\left\{\left(\sum_{x\in\Phi_T} Pg(x)\mathbf{1}_p(x,k)|h_{xo}(k)|^2\right)^2\right\}, \\
&= \mathbb{E}\left\{\sum_{x\in\Phi_T} Pg^2(x)\mathbf{1}_p(x,k)|h_{xo}(k)|^4\right\} \\
&\quad +\mathbb{E}\left\{\sum_{x,y\in\Phi_T,x\neq y} Pg(x)g(y)\mathbf{1}_p(x,k)\mathbf{1}_p(y,k)|h_{xo}(k)|^2|h_{yo}(k)|^2\right\}, \\
&= p\mathbb{E}\{h^4\}\lambda\int_{\mathbb{R}^2} g^2(x)dx, \\
&\quad +p^2\left(\mathbb{E}\{h^2\}\right)^2\lambda^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g(x)g(y)dxdy, \qquad (26)
\end{aligned}$$

where the first term in the last equality follows from Campbell's Theorem (Theorem 3.7) for PPP, and the second term from the fact that $|h_{xo}|^2$ and $|h_{y0}|^2$ are independent, and second-order product density of the PPP [5] which states that

$$\mathbb{E}\left\{\sum_{x,y\in\Phi(k),x\neq y} f(x)f(y)\right\} = \lambda^2\int\int f(x)f(y)dxdy.$$

Finally, by exactly following the same procedure as above, the cross-correlation of the interference is given by

$$\begin{aligned}
\mathbb{E}\{I_u(k)I_v(\ell)\} &= p^2\lambda\int_{\mathbb{R}^2} g(x-u)g(x-v)dx, \\
&\quad +p^2\lambda^2\left(\int_{\mathbb{R}^2} g(x)dx\right)^2, \qquad (27)
\end{aligned}$$

where we have used $\mathbb{E}\{|h|^2\} = 1$.

Thus, using (25), (26), and (27), the spatio-temporal correlation of the interference from (24) is

$$\mathsf{corr}_{u,v}(k,\ell) = \frac{p\int_{\mathbb{R}^2} g(x-u)g(x-v)dx}{\mathbb{E}\{|h|^4\}\int_{\mathbb{R}^2} g^2(x)dx}. \qquad (28)$$

For the $x^{-\alpha}$ path-loss function, that we use throughout this book, we next show that the spatial correlation coefficient is zero.

Example 7.1 *For the special case of $g(x) = x^{-\alpha}$, the spatial correlation coefficient is zero. For the purposes of analysis, we let $g_\psi(x) = \frac{1}{\psi+x^\alpha}$ and let $\psi \to 0$, since otherwise $\int x^{-\alpha}dx = \infty$. From* (28)*, the spatial correlation coefficient with path-loss* *function $g_\psi(x)$ is*

$$\begin{aligned}\mathsf{corr}_{u,v}(k,\ell) &= \frac{p\int_{\mathbb{R}^2} g_\psi(x-u)g_\psi(x-v)dx}{\mathbb{E}\{|h|^4\}\int_{\mathbb{R}^2} g_\psi^2(x)dx}, \\ &= \frac{p\int_{\mathbb{R}^2} \frac{1}{\psi+(x-u)^\alpha}\frac{1}{\psi+(x-v)^\alpha}dx}{\mathbb{E}\{|h|^4\}\int_{\mathbb{R}^2} \frac{1}{\psi+x^\alpha}dx}.\end{aligned}$$

Taking the limit as $\psi \to 0$, it follows that $\lim_{\psi\to 0}\mathsf{corr}_{u,v}(k,\ell) = 0$. This result is an artifact of the path-loss model of $x^{-\alpha}$, where the nearest interferers are the dominant interferers. Thus, for two distinct receivers, most of the interference comes from small discs around them that are non-overlapping, and since the number of points of PPP lying in non-overlapping discs are independent, the result follows.

Thus, even though the interferences seen at different receivers are not independent, however, they are uncorrelated. Consequently, assuming independence of interference for ease of analysis is not too limiting.

Example 7.2 *From (28), we can get the temporal correlation coefficient as a special case of $\mathsf{corr}_{u,v}(k,\ell)$ by specializing $v = u$ as $\mathsf{corr}(k,\ell) = \frac{p}{\mathbb{E}\{|h|^4\}}$.*

Clearly from (28), the temporal correlation coefficient is non-zero, and hence repeated transmissions (SINR at different times) between a transmitter-receiver pair are not independent. Direct impact of this observation is encountered in the analysis of ARQ protocols, where repeated transmission attempts are made till the packet is successfully received. Typically, for the ease of exposition, SINRs at repeated attempts are assumed independent, which is inaccurate as shown in the next example. We next present a simple example to derive the joint success probability at a receiver from [4]. A more general exact derivation for ARQ protocols will be described in Chapter **??** under a maximum retransmissions/delay constraint.

Example 7.3 *We consider a receiver located at origin o, and for simplicity drop the AWGN contribution and define the success to be the event that*

$$\mathsf{SINR}(k) = \frac{d^{-\alpha}|h(k)|^2}{I_o(k)} > \beta.$$

Then

$$\begin{aligned}
\mathbb{P}\left(\mathsf{SINR}(k) > \beta, \mathsf{SINR}(\ell) > \beta\right) &= \mathbb{P}\left(|h(k)|^2 > d^\alpha I_o(k)\beta, |h(\ell)|^2 > d^\alpha I_o(\ell)\beta\right), \\
&\stackrel{(a)}{=} \mathbb{E}\left\{\exp^{-d^\alpha I_o(k)\beta}\exp^{-d^\alpha I_o(\ell)\beta}\right\}, \\
&= \mathbb{E}\left\{\exp^{-d^\alpha \beta \sum_{x\in\Phi} Pg(x)\mathbf{1}_p(x,k)|h_{xo}(k)|^2}\right. \\
&\quad \left.\exp^{-d^\alpha \beta \sum_{x\in\Phi} Pg(x)\mathbf{1}_p(x,\ell)|h_{xo}(\ell)|^2}\right\} \\
&\stackrel{(b)}{=} \mathbb{E}\left\{\prod_{x\in\Phi}\left(\frac{p}{1+d^\alpha x^{-\alpha}\beta} + 1 - p\right)^2\right\}, \\
&\stackrel{(c)}{=} \exp^{\left(-\lambda\int_{\mathbb{R}^2} 1-\left(\frac{p}{1+d^\alpha x^{-\alpha}\beta}+1-p\right)^2 dx\right)},
\end{aligned}$$

where (a) follows from the fact that $|h(\ell)|^2, |h(k)|^2 \sim \exp(1)$ and are independent, (b) follows by taking the expectation with respect to ALOHA (similar to (10)) and $|h_{xo}(k)|^2, |h_{xo}(\ell)|^2$ that are i.i.d. $\sim \exp(1)$, and finally (c) follows from the probability generating functional of the PPP (Theorem 3.6).

Solving for this integral we get

$$\frac{\mathbb{P}\left(\mathsf{SINR}(k) > \beta, \mathsf{SINR}(\ell) > \beta\right)}{\mathbb{P}\left(\mathsf{SINR}(k) > \beta\right)^2} = \exp^{\left(2\lambda p^2\beta^{2/\alpha}d^2\pi^2\frac{\alpha-2}{\alpha^2}csc\left(\frac{2\pi}{\alpha}\right)\right)} > 1.$$

Thus, link success probabilities are positively correlated. Hence, if the transmission between a transmitter and receiver is successful at a given instant, it is more likely to be successful again. Thus, the analysis of ARQ type strategies, where a packet is repeatedly sent until it is successfully received is complicated, since the probability of success or failure in successive slots is not independent. Most works assume the independence and make an inaccurate prediction on the performance of ARQ protocols. In Chapter **??**, *we will illustrate the exact performance of a ARQ protocol in a wireless network.*

Until now in this chapter we have concentrated on analyzing the performance of wireless network when each transmitter uses an ALOHA protocol. However, clearly, one can choose the set of simultaneously active transmitters better than the ALOHA protocol by considering received SINRs, neighbor distance etc. to reduce interference and improve the transmission capacity. In the next section, we present two such protocols and analyze their performance.

8 Transmission Capacity with Scheduling in Ad Hoc Networks

In a PPP network, the most significant contributor of interference at each receiver is its nearest interferer. The expected interference from the nearest interferer is of the same order as the expectation of the sum of the interference from all other interferers. With the ALOHA protocol, each node transmits independently and there is no restriction on the set of simultaneously transmitting nodes. Thus, the nearest interferer is active with a fixed probability, and limits the performance of the ALOHA protocol. To improve the outage probability at any receiver, thus there is a case for inhibiting transmission from some of the nearest interferers. This is turn, however, reduces the number of simultaneous transmissions (spatial capacity) in the network. Thus, an efficient scheduling strategy (transmitter inhibition strategy) has to find which transmitters to turn off such that the improvement in the outage probability compensates for the spatial capacity loss.

One such strategy is based on *guard zones*, where any transmitter within a distance of d_{gz} from a receiver is not allowed to transmit [6]. Other class of strategies includes various versions of CSMA, where each trasnmitter measures the channel and decides to transmit depending on a function of its measurement. Intuitively, both these strategies, should improve the transmission capacity with respect to ALOHA, however, exactly quantifying the improvement analytically is complicated. The challenge is that with guard zone strategy or CSMA, the set of active transmitters is correlated (not PPP anymore) and the distribution of interference seen at any receiver does not have a closed form expression or known Laplace transform or probability generating functionals. To overcome this difficulty, typically, approximations are made on the interference distributions to get some analytical tractability. In this section, we first discuss the guard zone based strategy and then follow it up with analyzing two variants of CSMA.

8.1 Guard zone strategy

Consider a PPP network Φ with density λ nodes per unit area as described in Section 2, where each transmitter has a corresponding receiver at distance d. With a guard zone, only those transmitters that are not within a distance of d_{gz} from any receiver are allowed to transmit, see Fig. 3. For the typical transmitter-receiver pair (T_0, R_0), where R_0 is at the origin, let the active set of interferers be denoted by $\Phi_{gz} = \{T_m : T_m \in \Phi \backslash \mathbf{B}(0, d_{gz})\}$. Then, the outage probability at receiver R_0 is

$$P_{out}^{gz}(B) = \mathbb{P}\left(\frac{Pd^{-\alpha}|h_{00}|^2}{\sum_{m:T_m\in\Phi_{gz}} Pd_{m0}^{-\alpha}|h_{m0}|^2 + 1} \leq \beta\right). \tag{29}$$

Then with an outage probability constraint of ϵ, the maximum density of successful transmissions is

$$\lambda_{gz}^1 = \sup\{\lambda : P_{out}^{gz}(B) \leq \epsilon\}.$$

Since the distribution of interference $I_{gz} = \sum_{m:T_m\in\Phi_{gz}} Pd_{m0}^{-\alpha}|h_{m0}|^2$ received at any receiver is not known, deriving exact expression for the outage probability is not possible. To facilitate the analysis, I_{gz} is assumed to follow a Gaussian distribution. Similar to the computation of mean and variance in (17), the exact mean m_{gz} and variance var_{gz} of I_{gz} can be computed using the Campbell's Theorem (Theorem 3.7) (proof is left as an excercise) as

$$\mathsf{m}_{gz} = \frac{4\pi d^{\alpha} d_{gz}^{2-\alpha}}{\alpha^2 - 4}\lambda, \tag{30}$$

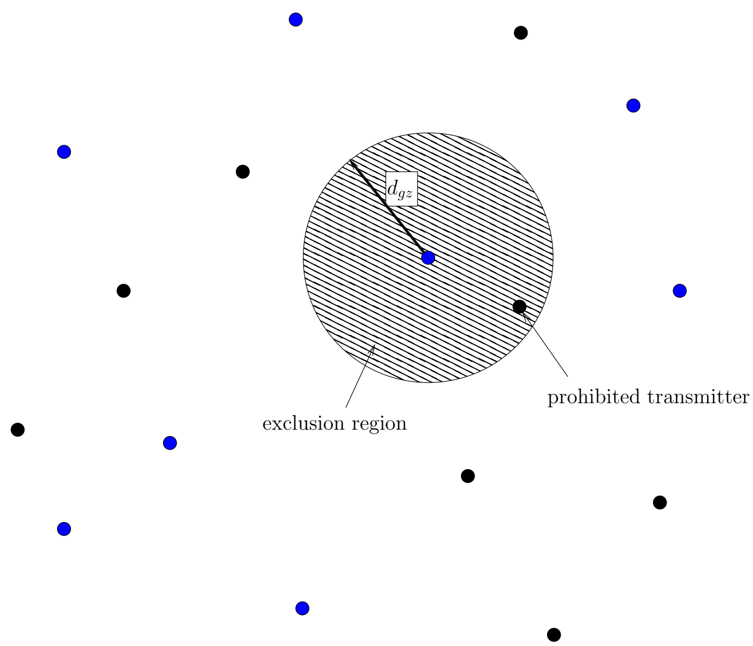

Figure 3: Black dots are transmitters and blue dots are receivers. Only those transmitters are allowed to transmit that lie outside of discs of radius d_{gz} centered at all the receivers.

$$\mathsf{var}_{gz} = \frac{\pi d^{2\alpha} d_{gz}^{2(1-\alpha)}}{\alpha^2 - 1}\lambda. \tag{31}$$

Thus, using the Gaussian distribution approximation on I_{gz} with mean m_{gz} and variance var_{gz}, we have that

$$\begin{aligned} P_{out}^{gz}(B) &= \mathbb{P}\left(\frac{Pd^{-\alpha}|h_{00}|^2}{\sum_{m:T_m\in\Phi_{gz}} Pd_{m0}^{-\alpha}|h_{m0}|^2+1} \le \beta\right), \\ &\stackrel{(a)}{=} \exp^{\left\{-\frac{d^\alpha \beta}{P}\right\}} \exp^{\{-d^\alpha \beta I_{gz}\}}, \\ &\stackrel{(b)}{=} \exp^{\left\{-\frac{d^\alpha \beta}{P}\right\}} \exp^{\left\{-d^\alpha \beta \mathsf{m}_{gz} + \frac{d^{2\alpha}\beta^2 \mathsf{var}_{gz}}{2}\right\}}, \end{aligned} \tag{32}$$

where (a) follows since $|h_{00}|^2 \sim \exp(1)$, and (b) follows from using the moment generating function of the Gaussian distribution. Thus, (32) reveals how the outage probability decreases with increasing the guard zone distance d_{gz}, and we can get λ_g^1 by equating it to the outage probability constraint ϵ.

The probability for any receiver to not have any transmitter in a disc of radius of d_{gz} around it is equal to the void probability of transmitter PPP Φ in a disc of radius d_{gz}, which is given by $\exp^{-\lambda\pi d_{gz}^2}$. Thus, with the inhibition criterion of the guard zone based policy, the density of the active transmitter process is $p_a\lambda$, where $p_a = \exp^{-\lambda\pi d_{gz}^2}$. Hence the operational density of transmitters is $\lambda_{gz}^2 = \lambda\exp^{-\lambda\pi d_{gz}^2}$. Note that this separate analysis of λ_{gz}^1 and λ_{gz}^2 is not completely accurate since they depend on each other, however, acts as a reasonable approximation.

There is inherent tension between inhibition radius d_{gz} and the transmission capacity; increasing d_{gz} decreases the interference and the outage probability (increases λ_g^1), but at the same time decreases the number of active transmitters (decreases λ_g^2). Note λ_g^1 is the maximum density that can be supported given an outage probability constraint of ϵ, and considering only the interference coming from outside the disc of radius d_{gz}, under the guard-zone based strategy. Hence we want to find the best d_{gz} and λ such that λ_g^2 (density corresponding to inhibition) is equal to λ_g^1 (corresponding to the outage probability constraint). Equivalently, one can also write the optimization problem as

$$\lambda^\star(d_{gz}) = \max_{d_{gz},\lambda}\left[\min\{\lambda_{gz}^1, \lambda_{gz}^2\}\right]. \tag{33}$$

Corresponding transmission capacity is $\lambda^\star(d_{gz})(1-\epsilon)B$ bits/sec/Hz/m^2.

Problem (33) is a non-linear optimization problem which can solved using numerical computations. In Fig. 4, we plot the transmission capacity as a function of d_{gz} for outage probability constraint of 10% $(\epsilon = .1)$. We can see that that transmission capacity increases with d_{gz} for $d_{gz} \le d_{gz}^\star$ and decreases thereafter, since the decrease in the number of concurrent transmissions for $d_{gz} > d_{gz}^\star$ overtakes the improvement in the outage probability. We can also see that there is significant improvement by employing a guard-zone based inhibition policy over uninhibited transmissions ($d_{gz} = 0$).

An alternative to guard-zone based scheduling strategy, is the CSMA protocol, where each node monitors the channels and follows a contention resolution strategy. We discuss two versions of CSMA protocols in the next section for wireless networks and analyze their performance.

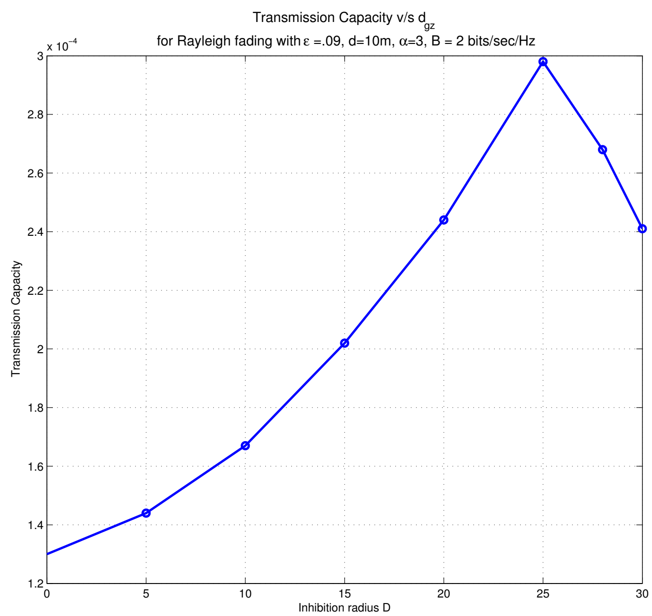

Figure 4: Transmission Capacity with Rayleigh fading as a function of guard zone d_{gz}.

8.2 CSMA

8.2.1 Channel Gain Based

We consider the same signal model as in Section 2, where the location of transmitters T_n is assumed to follows a PPP Φ with density λ. Each transmitter T_n is defined to be *qualified* to transmit if its channel gain with its associated receiver R_n, $|h_{nn}|^2$, exceeds a threshold τ_h. Thus, this protocol requires channel feedback from each R_n to T_n. Let

$$\Phi_Q = \{T_n \in \Phi : |h_{nn}|^2 \geq \tau_h\}$$

denote the set of qualified nodes or contenders. Note that Φ_Q is a randomly thinned version of Φ_T, since $|h_{nn}|^2$ are i.i.d., and therefore Φ_Q is also a homogenous PPP with density λp_{τ_h}, where $p_{\tau_h} = \mathbb{P}(|h_{nn}|^2 \geq \tau_h)$.

We define that two transmitters T_n and T_m contend with each other if the received interference power they see from each other, $|h_{nm}|^2 d^{-\alpha}$, is greater than the CSMA threshold τ_c, $|h_{nm}|^2 d^{-\alpha} > \tau_c$. For a transmitter T_n, its contention neighborhood is the set of nodes that contend with it,

$$\Phi_{CN}(n) = \{T_m \in \Phi_Q : |h_{mn}|^2 |T_m - T_n|^{-\alpha} \geq \tau_c\}.$$

The *inhibition* module of the CSMA protocol allows only one of the transmitters from $\Phi_{CN}(n)$ to transmit at any time to suppress interference.

To decide which node of $\Phi_{CN}(n)$ gets to transmit in a decentralized manner, each node of $\Phi_{CN}(n)$ is equipped with a timer value clk_n that is a uniformly distributed random variable between $[0,1]$. Thus, the node with the minimum timer value transmits in each slot, and if any node in Φ_N hears a transmission from node $T_{n^\star}$, it does not transmit in that entire slot and resets its timer value. For each slot, the node $n^\star \in \Phi_{CN}(n)$ transmits at time $\mathsf{clk}_{n^\star}$, where $n^\star = \arg\min_{n:T_n\in\Phi_{CN}(n)} \mathsf{clk}_n$.

Remark 8.1 *There are two modules in this CSMA protocol, the first that finds qualified nodes that have sufficient channel gains to their respective receivers, and the second that chooses one node among the set of qualified nodes to minimize interference. Allowing only qualified nodes to contend increases the chance of success, however, limits the number of simultaneously spatial transmissions, thus the choice of τ_h, (parameter controlling the qualification) is critical. Similar tradeoff exists as a function of τ_c that controls the size of the neighborhood, since only one node in each neighborhood is allowed to transmit.*

Let $\mathcal{E}_n = \mathbf{1}_{\{|h_{00}|^2\ge\tau_h, \mathsf{clk}_n\le\min_{T_m\in\Phi_{CN}(n)}\mathsf{clk}_n\}}$ represent the event that transmitter T_n is qualified, and has the least timer in its neighborhood and gets to transmit. Then a typical transmitter T_0 located at the origin gets to transmit with the above CSMA protocol if $\mathcal{E}_0 = 1$. Thus, the probability that a typical transmitter T_0 transmits is $p_{csma} = \mathbb{E}\{\mathcal{E}_0\}$. Note that events $\{|h_{00}|^2 \ge \tau_h\}$ and $\{\mathsf{clk}_n \le \min_{T_m\in\Phi_{CN}(n)}\mathsf{clk}_n\}$ are independent.

Next, we show that the cardinality of set Φ_N^0 (the neighborhood set of T_0 located at origin), $\#(\Phi_N^0)$, is Poisson distributed with mean $p_{\tau_h}\bar{N}$, where

$$\bar{N} = \lambda\pi\mathbb{E}\left\{\left(\frac{\tau_c}{|h_{m0}|^2}\right)^{\frac{2}{\alpha}}\right\}.$$

By definition, the set Φ_N^0 consists of all nodes of Φ_Q that lie in a disc $\mathbf{B}\left(o,\left(\frac{\tau_c}{|h_{m0}|^2}\right)^{\frac{1}{\alpha}}\right)$. Since Φ_Q is a PPP with density λp_{τ_h}, the number of nodes of Φ_Q lying in $\mathbf{B}\left(o,\left(\frac{\tau_c}{|h_{m0}|^2}\right)^{\frac{1}{\alpha}}\right)$ are Poisson distributed with mean $\lambda p_{\tau_h}\pi\mathbb{E}\left\{\left(\frac{\tau_c}{|h_{m0}|^2}\right)^{\frac{2}{\alpha}}\right\}$.

Thus,

$$\begin{aligned}
p_{csma} &= p_{\tau_h}\mathbb{E}\{\mathsf{clk}_0 \le \min_{T_m\in\Phi_N^0}\mathsf{clk}_n\}\},\\
&\stackrel{(a)}{=} p_{\tau_h}\mathbb{E}\left\{\frac{1}{1+\#(\Phi_N^0)}\right\},\\
&\stackrel{(b)}{=} p_{\tau_h}\frac{1-\exp^{-p_{\tau_h}\bar{N}}}{p_{\tau_h}\bar{N}},\\
&= \frac{1-\exp^{-p_{\tau_h}\bar{N}}}{\bar{N}},
\end{aligned}$$

where (a) follows since T_0 has the least timer among its $\#(\Phi_N^0)$ neighbors and (b) follows since $\#(\Phi_{CN}(n))$ is Poisson distributed with mean $p_{\tau_h}\bar{N}$.

Next, we compute the outage probability for the typical transmitter-receiver pair (T_0, R_0). The set of active users $\Phi_a = \{T_m \in \Phi_Q : \mathcal{E}_m = 1\}$ and the interference received at R_0 from active users is

$$I_0^a = \sum_{T_m \in \Phi_a \setminus \{T_0\}} d_{m0}^{-\alpha} |h_{m0}|^2.$$

Hence, the outage probability is given by

$$P_{out}(B) = \mathbb{P}\left(|h_{00}|^2 < \beta d^\alpha I_0^a \;\middle|\; |h_{00}|^2 > \tau_h\right),$$

Deriving the outage probability with this CSMA protocol is challenging since the set of active transmitters is no longer a homogenous PPP, and there are correlations among the active node locations.

To facilitate the analysis, following [7], we will approximate I_0^a by the interference from another non-homogenous PPP Φ_{h} with density $p_{\tau_h}\lambda \mathsf{h}(x, p_{\tau_h}\lambda)$, as a function of $x > 0$. The function $\mathsf{h}(x, p_{\tau_h}\lambda)$ is the conditional probability that T_0 at origin is active and in addition there is another active transmitter T_m with $\mathcal{E}_m = 1$ at a distance x from the origin, and where the density of qualified nodes Φ_Q is $p_{\tau_h}\lambda$. The difference between a homogenous and non-homogenous PPP is that in the non-homogenous case, the density is not constant, and depends on the location $x \in \mathbb{R}^2$.

By using the non-homogenous PPP Φ_{h}, we are trying to model the inhibition induced by the CSMA among transmitters of the PPP Φ. Note that $\mathsf{h}(x, p_{\tau_h}\lambda) \to 0$ as $x \to 0$. Thus, the density of PPP Φ_h goes to zero for short distance x, and correspondingly there is large scale inhibition induced by the CSMA protocol at distances close to origin where T_0 is located, allowing only very few nodes to transmit at the same time as T_0. On the other hand, as $x \to \infty$, $\mathsf{h}(x, p_{\tau_h}\lambda) \to \mathbb{P}\left(\mathsf{clk}_m \le \min_{T_n \in \Phi_N^m} \mathsf{clk}_n\right)$ for some $T_m \in \Phi_Q$, since as x increases, the effect of conditioning (having an active transmitter T_0 at the origin) over the event that there is an active transmitter at distance x from the origin diminishes. Eventually at $x = \infty$, having an active transmitter T_0 at the origin has no effect on having an active transmitter at distance x from the origin, leading to $\mathsf{h}(x, p_{\tau_h}\lambda)$ being equal to the unconditional probability of having an active node among the qualified nodes, which is equal to $\mathbb{P}\left(\mathsf{clk}_m \le \min_{T_n \in \Phi_N^m} \mathsf{clk}_n\right)$ for some $T_m \in \Phi_Q$. Thus, at large distances x, the density of process Φ_{h} is equal to λp_{csma} having no inhibition effect from the transmitter located at the origin.

Thus, as a function of distance x from the origin, PPP Φ_{h} essentially models the inhibiting nature of the CSMA with respect to the typical node at the origin, having increasing number of active transmitters with increasing x. We illustrate the PPP Φ_{h} pictorially in Fig. 5.

The utility of this new process Φ_{h} is that its Laplace transform is known to be

$$\mathcal{L}_{\Phi_{\mathsf{h}} \setminus \{T_0\}}(s) = \exp^{\left(-p_{\tau_h}\lambda \int_0^\infty \int_0^{2\pi} \frac{\mathsf{h}(x, p_{\tau_h}\lambda) x d\theta dx}{1 + f(x,d,\theta)/s}\right)}, \tag{34}$$

where $f(x, d, \theta) = (x^2 + d^2 - 2xr\cos(\theta))^{-\alpha/2}$, and d is the distance between each transmitter-receiver pair [7]. This can also be derived from first principles similar to Laplace transform of a homogenous PPP.

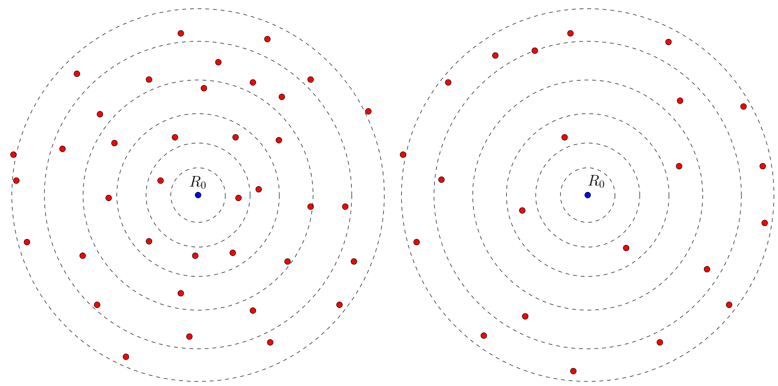

Figure 5: A pictorial description of PPP Φ_{h} in comparison to original process Φ, where the density increases with increasing distance from the origin.

Using (34), by approximating I_0^a with interference from nodes in Φ_{h}, we can write

$$P_{out}(B) = 1 - \mathcal{L}_{\Phi_{\mathsf{h}} \backslash \{T_0\}}(2i\pi r^{-\alpha} ts) \frac{\frac{1}{1-2i\pi s} \exp^{2i\pi s \tau_h} - 1}{2i\pi s}, \tag{35}$$

using the Plancerel-Parseval Theorem, [8]. The details of this derivation are intentionally deleted because of being laborious and too technical.

Thus, using (35), we can numerically compute the outage probability and consequently the transmission capacity, that is the density of successful transmissions multiplied with rate of transmission $2^\beta - 1$. In this inhibition based CSMA protocol, the two key parameters are τ_h and τ_c, that control the number of qualified transmitters, and the size of the neighborhood.

In Fig. 6, we plot the goodput $(1 - P_{out}(B))\lambda R$ as a function of density λ for fixed neighborhood contention threshold $\tau_c = 1$, and different values of transmitter channel access threshold τ_h. We see that for small values of λ, no CSMA (ALOHA with access probability 1) is better than CSMA, since interference is very low and inhibition provided by CSMA is unnecessary. On the other hand, as we increase the density, the role of CSMA transmitter channel access threshold τ_h becomes more prominent, since with sufficient interference it is important to control how many transmitters are allowed to transmit. We see a similar performance comparison in Fig. 7, where we plot the goodput as a function of density λ for fixed transmitter channel access threshold τ_h and different values of neighborhood contention threshold $\tau_c = 1$.

Remark 8.2 *Recently, a more detailed analytical analysis of CSMA protocol with just the neighborhood contention model, i.e., with $\tau_c = 0$ (no qualification criteria) has been done in [9] for small densities (λ) regime, to show that the transmission capacity scales as $\Theta\left(\epsilon^{\frac{2}{\alpha\psi}}\right)$, for $\epsilon \to 0$, where $\psi \geq 1$ depends on the fading coefficient distribution. For Rayleigh fading, $\psi = 1$.*

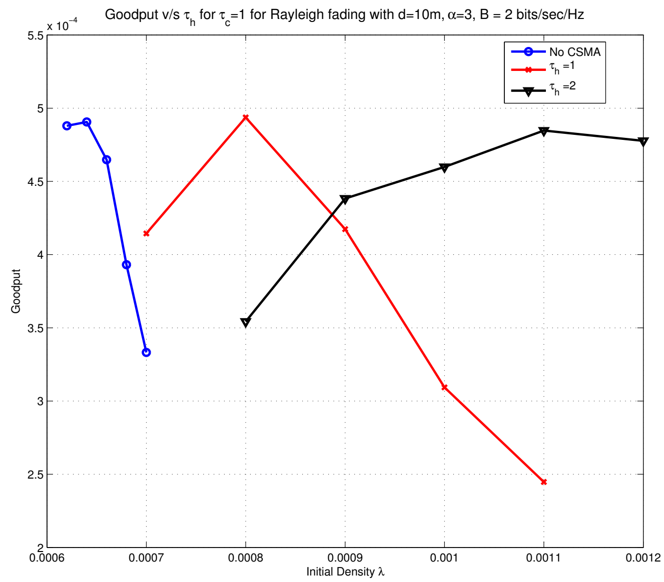

Figure 6: Network goodput with Rayleigh fading as a function of CSMA transmitter channel access threshold of τ_h for neighborhood contention threshold of $\tau_c = 1$.

Next, we present an alternate SINR based CSMA protocol, where each node monitors the SINR to its corresponding receiver, and transmits only if the SINR is larger than a threshold. In all prior sections in this chapter, we have assumed that each node's data queue is backlogged, i.e. it always has packet to transmit. This is only an abstraction, and a more general data arrival process model is considered with the SINR based CSMA protocol in the next section.

8.2.2 SINR-based

In this section, we consider a SINR based CSMA protocol, and consider that packets arrive at each node according to a 1-dimensional PPP. In all earlier sections, we have assumed that each node always has a packet to transmit which is only an abstraction. To analyze the SINR based CSMA protocol with random packet arrivals, we consider a slightly different network model compared to Section 2, that has been introduced in [10]. We consider an area A, and model the packet arrival process as a one-dimensional PPP with arrival rate $(A/L)\lambda$, where L is the fixed packet duratio. Each packet after arrival is assigned to a transmitter location that is uniformly distributed in area A, and the receiver corresponding to a particular transmitter is located at a fixed distance d away with a random orientation, as shown in Fig. 8. For $A \to \infty$, this process corresponds to a 2-dimensional PPP of transmitter locations with density λ (Section 2), where each transmitter has packet arrival rate of $\frac{1}{L}$. Note that the performance of

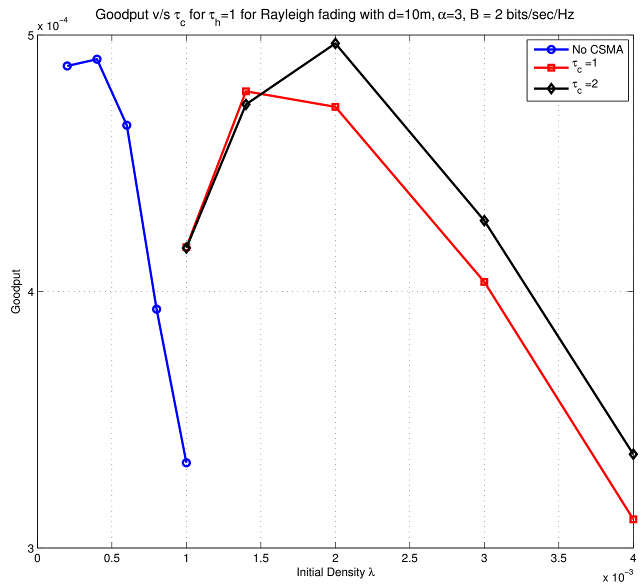

Figure 7: Network goodput with Rayleigh fading as a function of CSMA neighborhood contention threshold τ_c with channel access threshold of $\tau_h = 1$.

ALOHA protocol with data packet arrival process follows similar to what follows next and omitted for brevity and can be found in [10].

Remark 8.3 *If we use the model of Section 2, then we would first fix the transmitter locations according to a 2-dimensional PPP, and then packet traffic is generated, and each transmitting node receives packet according to 1-dimensional PPP over time. The packets are then transmitted to the respective receivers that are located at a fixed distance* d. *Thus, with the model of Section 2, one has to average over the spatial (to fix locations) and temporal (packet arrivals) statistics, which is rather challenging. Instead, by slightly altering the model, as described above, there is a single process, that defines both the spatial location and temporal packet arrival process.*

For simplicity, we ignore the AWGN contribution. Hence the SIR between transmitter T_n and its receiver R_n at time t is given by

$$\mathsf{SIR}_n(t) := \frac{d^{-\alpha}|h_{nn}|^2}{\sum_{T_s \in \Phi \setminus \{T_0\}} \mathbf{1}_{T_s}(t) d_{mn}^{-\alpha}|h_{mn}|^2}, \tag{36}$$

where $\mathbf{1}_{T_m}(t) = 1$, if the transmitter T_s is not in back-off, and 0 otherwise. With CSMA, transmitter T_s sends its packet at time t if the channel is sensed *idle* at time t, which corresponds to $\mathsf{SIR}_n(t) > \beta$. Otherwise, the transmitter backs off and makes a retransmission attempt after a random amount of time. If T_n transmits the packet, the

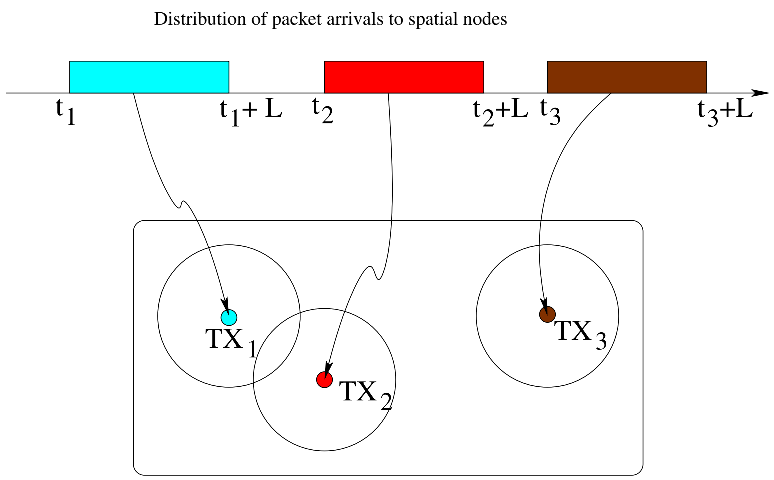

Figure 8: Spatial model for CSMA with packet arrivals.

packet transmission can still fail if SIR_n falls below β during the packet transmission time L. Thus, the outage probability

$$P_{out} = P_b + (1 - P_b) P_{\text{fail|no back-off}},$$

where P_b is the back off probability, and $P_{\text{fail|no back-off}}$ is the probability that the transmission fails during transmission. Hence, the transmission capacity with CSMA is defined as

$$C = \lambda(1 - P_{out}) B \text{ bits/sec/Hz/m}^2.$$

Remark 8.4 *CSMA introduces correlation among different transmitter's back-off events, and hence the number of simultaneously active transmitters no longer follow a PPP. Nevertheless, for analytical tractability, as an approximation, we assume that the transmitter back-off events are independent, and simultaneously active transmitter locations are still PPP distributed. The simulation results show that this assumption is reasonable [10].*

In the following Theorem, we derive the back-off probability for any transmitter with the SINR-based CSMA.

Theorem 8.5 *The back-off probability follows a recursive relationship*

$$P_b = 1 - \exp\left(-\lambda(1 - P_b) c \beta^{\frac{2}{\alpha}} d^2\right),$$

which can be solved using Lambert's function $W_0(.)$.

Proof: Under the independent back-off assumption, the set of active transmitters at time 0 is a PPP with density $\sum_{i=-L}^{0} \frac{\lambda}{L}(1 - P_b)$ by counting for all active transmitters during the packet length of L time slots. Thus, the density of active transmitters at time 0 is

$$\lambda_a = \lambda(1 - P_b).$$

Hence from Theorem 4.1, we get the recursive relation $P_b = \mathbb{P}(\mathsf{SIR}_n(0) < \beta) = 1 - \exp\left(-\lambda(1 - P_b)c\beta^{\frac{2}{\alpha}}d^2\right)$. $\square$

Next, we derive an explicit expression for the packet failure probability $P_{out}(B)$ with the CSMA protocol.

Theorem 8.6 $P_{\textit{fail|no back-off}} = 1 - \frac{\sum_{\ell=0}^{L+1}(-1)^{\ell}\binom{L+1}{\ell}e^{-\frac{\lambda}{T}\left(\int_{\mathbb{R}^2} 1-\left(\frac{(1-P_b)}{1+d^{\alpha}\beta x^{-\alpha}}+1-(1-P_b)\right)^{\ell}dx\right)}}{1-P_b}$.

Proof: Note that $P_{\text{fail|no back-off}}$ is the probability that at any time t, $\mathsf{SIR}_n(t) < \beta$ for $0 < t \le L$ given that $\mathsf{SIR}_n(0) > \beta$. Hence,

$$\begin{aligned} 1 - \mathbb{P}_{\text{fail|no back-off}} &= \mathbb{P}(\mathsf{SIR}_n(1) > \beta, \dots, \mathsf{SIR}_n(L) > \beta | \mathsf{SIR}_0 > \beta), \\ &= \frac{\mathbb{P}(\mathsf{SIR}_0 > \beta, \mathsf{SIR}_n(1) > \beta, \dots, \mathsf{SIR}_n(L) > \beta)}{\mathbb{P}(\mathsf{SIR}_0 > \beta)}, \end{aligned}$$

and the desired expression for the joint probability in the numerator follows from Proposition **??**, using the probability generating functional of the PPP (Theorem 3.6), similar to Example 7.3. The transmitters that become active at any time t between time 0 and L is a PPP with density $\frac{\lambda}{L}(1 - P_b)$.

$\square$

Hence using $P_{out} = P_b + (1 - P_b)P_{\text{fail|no back-off}}$, we get the transmission capacity $C = \lambda(1 - P_{out})R$ for CSMA by combining Theorem 8.5 and 8.6. Finding the closed form expression for $P_{\text{fail|no back-off}}$ derived in Theorem 8.6 is quite challenging. An upper bound on the $P_{\text{fail|no back-off}}$, however, can be found using the FKG inequality as follows.

Definition 8.7 *Let $(\Upsilon, \mathcal{F}, \mathcal{P})$ be the probability space. Let $A \in \mathcal{F}$, and $\mathbf{1}_A$ be the indicator function of A. Event $A \in \mathcal{F}$ is called increasing if $\mathbf{1}_A(\omega) \le \mathbf{1}_A(\omega')$, whenever $\omega \le \omega', \omega, \omega' \in \Upsilon$ for some partial ordering on ω. The event A is called decreasing if its complement A^c is increasing.*

Lemma 8.8 *(FKG Inequality [11] If both $A, B \in \mathcal{F}$ are increasing or decreasing events then $\mathbb{P}(AB) \ge \mathbb{P}(A)\mathbb{P}(B)$.*

Lemma 8.9 *For CSMA $P_{out} \le 1 - (1 - P_b)^{L+1}$.*

Proof: Clearly, $\mathsf{SIR}_n(t)$ is a decreasing function of the number of interferers, since larger the number of interferers, less is the SIR. Therefore the success event $\{\mathsf{SIR}_n(t) > \beta\}$ is a decreasing event. Hence, from the FKG inequality,

$$\mathbb{P}(\mathsf{SIR}_n(0) > \beta, \mathsf{SIR}_n(1) > \beta, \dots, \mathsf{SIR}_n(L) > \beta) \ge \mathbb{P}(\mathsf{SIR}_0 > \beta)^{L+1},$$

since $\mathsf{SIR}_n(t)$ is identically distributed for any t. Hence, $\mathbb{P}_{\text{fail|no back-off}} \le 1 - (1 - P_b)^L$, and $P_{out} \le 1 - (1 - P_b)^{L+1}$.

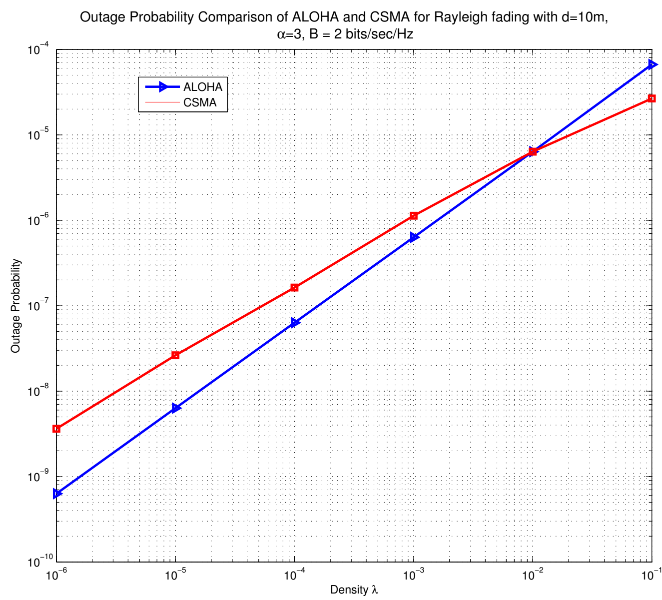

Figure 9: Outage probability comparison of ALOHA and SINR-based CSMA with Rayleigh fading.

Consequently, we get a lower bound on the transmission capacity with CSMA as

$$C \geq \lambda(1 - P_b)^{L+1} R.$$

□

Even though we have obtained closed form expression for the outage probability and consequently the transmission capacity, it is not easy to directly compare the SINR based CSMA protocol and the ALOHA protocol. We hence turn to numerical simulation for comparing their performance.

In Fig. 9, we plot the outage probabilities of ALOHA and SINR-based CSMA. For low densities λ, we see that the performance of ALOHA is better than CSMA, because of un-necessary back-offs initiated by CSMA that are not required. However, as the density λ increases, the back-off mechanism of CSMA kicks in and reduces the interference and consequently outperforms the ALOHA protocol.

9 Reference Notes

The notion of transmission capacity was introduced in [1], where upper and lower bounds for the path-loss model were presented. The exact transmission capacity expression presented in Section 4.1 for the Rayleigh fading model, and the optimal ALOHA probability that maximizes the goodput is derived from [2]. The spatial and temporal correlations with the ALOHA model presented in Section 7 can be found in [4]. Transmission capacity analysis with scheduling using guard-zone is derived from [6], while the case of scheduling with CSMA can be found in [7, 10].

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Index