Graph Partitioning

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Outline

1. Introduction
   - Graph Partitioning Problems
   - Partitioning into Connected Parts

2. Results
   - $k$-Partitionable Graphs
   - Basic Properties
   - Proof for Near-Triangulations
   - Bounded Degree Graphs
Graph Partitioning

- Partition the vertices and/or edges of a graph.
- Partition must satisfy specified properties.
- Does there exist a partition with specified properties?
- Optimize a specified cost function associated with possible partitions.
- Variety of graph partitioning problems.
Graph Coloring

- Partition the vertex set.
- No two vertices in the same part should be adjacent.
- Number of parts is at most $k$.
- Does there exist such a partition?
- Minimize the number of parts.
- NP-Hard in general.
Min and Max Cut

- Partition the vertex set.
- Number of parts is 2.
- Minimize (or maximize) number of edges with an end vertex in each part.
- Min-cut can be solved in polynomial-time.
- Max-cut is NP-Hard.


Arborocity

- Partition the edges.
- Each part should be acyclic.
- Minimize the number of parts.
- Solvable in polynomial-time.
Connected Partitions

- Partition the vertices.
- Number of parts and size of each part specified.
- Each part should induce a connected subgraph of the graph.
- Does there exist such a partition?
- NP-Hard in general, even if number of parts is 2.
- Generalization of perfect matchings.
Formal Definition

- **Input**
  - A graph $G$ with $n$ vertices.
  - Positive integers $n_1, n_2, \ldots, n_k$ such that $\sum_{1 \leq i \leq k} n_i = n$.

- **Output**
  - A partition $V_1, V_2, \ldots, V_k$ of $V(G)$ such that $|V_i| = n_i$ and $V_i$ induces a connected subgraph of $G$, if it exists.
  - We call such a partition a $k$-partition of $G$. 
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Motivation
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Theorem (Györi and Lovász)

A graph $G$ with $n$ vertices is $k$-connected iff for any subset \{\(v_1, v_2, \ldots, v_k\)\} of $k$ vertices, and any positive integers $n_1, n_2, \ldots, n_k$ such that $\sum_{1 \leq i \leq k} n_i = n$, there exists a partition of $V(G)$ into $k$ parts $V_1, V_2, \ldots, V_k$ such that $v_i \in V_i$, $|V_i| = n_i$ and $V_i$ induces a connected subgraph of $G$ for all $1 \leq i \leq k$. 
Definition

A graph $G$ with $n$ vertices is said to be $k$-partitionable if for all positive integers $n_1, n_2, \ldots, n_k$ such that $\sum_{1 \leq i \leq k} n_i = n$, there exists a partition of $V(G)$ into $k$ parts $V_1, V_2, \ldots, V_k$ such that $|V_i| = n_i$ and $V_i$ induces a connected subgraph of $G$, for $1 \leq i \leq k$.

Definition

A graph $G$ is said to be decomposable if it is $k$-partitionable for all $k \geq 1$. 
Algorithmic Complexity

- NP-Hard to find a $k$-partition of an arbitrary graph, for all $k \geq 2$.
- No polynomial-time algorithm known to find a $k$-partition for a $k$-connected graph for $k \geq 4$. The partition always exists by the Györi-Lovász Theorem.
- NP-Hard to recognize $k$-partitionable and decomposable graphs, for $k \geq 2$.
- Not clear whether recognizing $k$-partitionable and decomposable graphs is in NP, for arbitrary $k$. 
Sufficient Conditions for $k$-Partitionability

- $k$-connected graphs are $k$-partitionable for all $k \geq 1$. (Györi-Lovász Theorem).
- $k$-connected graphs are not $(k + 1)$-partitionable in general.
- Complete bipartite graph $K_{k,k+2}$ has no perfect matching.
- Does $k$-connectivity with some additional property imply higher partitionability?
Planar Graphs

- $K_{1,3}$ is a planar 1-connected graph that is not 2-partitionable.
- $K_{2,4}$ is a planar 2-connected graph that is not 3-partitionable.
- Planar 4-connected graphs are Hamiltonian (Tutte’s Theorem), which implies they are decomposable.
- What happens for 3-connected planar graphs? ($K_{3,5}$ is not planar).
- Conjecture: Planar 3-connected graphs are 6-partitionable.
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Plane Triangulations

Definition

A plane triangulation is a planar simple graph in which every face is a triangle. Equivalently, it is a maximal planar graph with at least 3 vertices.

Theorem

Plane triangulations are 6-partitionable.

The proof also gives a polynomial-time algorithm to find a 6-partition.
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**Definition**

A plane near-triangulation is a planar simple graph in which all internal (bounded) faces are triangles and the outer face is a simple cycle.

**Theorem**

Plane near-triangulations are 4-partitionable.
Lemma

Let $u, v, w$ be the vertices on the boundary of some face of a plane triangulation with at least 4 vertices. There exists a vertex $x \notin \{v, w\}$ such that contracting edge $ux$ gives a plane triangulation.
Lemma

Let $u$ be any vertex in a plane triangulation with at least 4 vertices. There are at least two edges $uv, uw$ incident with $u$ such that contracting $uv$ or $uw$ gives a plane triangulation.
Contractible Edges
Contractible Edges
Contractible Edges
Lemma

Let $u$ be a vertex in the external cycle of a chordless near-triangulation $G$ with at least 4 vertices. Then at least one of the following holds:

(i) There exists an internal vertex $x$ adjacent to $u$ such that contracting the edge $ux$ gives a chordless near-triangulation.

(ii) Contracting any external edge incident with $u$ gives a chordless near-triangulation.
Contractible Edges
3-Partitioning Near-Triangulations

**Lemma**

Let $G$ be a plane near-triangulation with $n$ vertices and let $u, v$ be two adjacent vertices in the outer face of $G$. Then for any 3 positive integers $n_1, n_2, n_3$ such that $n_1 + n_2 + n_3 = n$, there exists a partition of $V(G)$ into 3 parts $V_1, V_2, V_3$ such that $u \in V_1$, $v \in V_2$, $|V_i| = n_i$ and $G[V_i]$ is connected, for $1 \leq i \leq 3$. 
3-Partitioning Near-Triangulations
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Lemma

Let \( u, v, w \) be vertices on the boundary of some face of a plane triangulation \( G \) with \( n \) vertices. Then for all positive integers \( n_1, n_2, n_3, n_4 \) such that \( n_1 + n_2 + n_3 + n_4 = n \), there exists a partition of \( V(G) \) into parts \( V_1, V_2, V_3, V_4 \), such that \( u \in V_1 \), \( v \in V_2 \), \( w \in V_3 \), \( |V_i| = n_i \) and \( V_i \) induces a connected subgraph of \( G \) for \( 1 \leq i \leq 4 \).
Lemma

Let $G$ be a plane triangulation with $n$ vertices and let $u, v, w$ be the vertices on the boundary of some face in $G$. Then for all positive integers $n_1, n_2, n_3, n_4$ such that $n_1 + n_2 + n_3 + n_4 = n - 1$, there exists a partition of $V(G) - v$ or $V(G) - w$ into parts $V_1, V_2, V_3, V_4$, such that $u \in V_1$, $|V_i| = n_i$ and $V_i$ induces a connected subgraph of $G$ for $1 \leq i \leq 5$. 
5-Partitioning a Plane Triangulation

Lemma

Let $u$ be any vertex in a plane triangulation $G$ with $n$ vertices. Then for all positive integers $n_1, n_2, n_3, n_4, n_5$ such that $n_1 + n_2 + n_3 + n_4 + n_5 = n$, there exists a partition of $V(G)$ into parts $V_1, V_2, V_3, V_4, V_5$ such that $u \in V_1$, $|V_i| = n_i$ and $V_i$ induces a connected subgraph of $G$ for $1 \leq i \leq 5$. 
6-Partitioning a Plane Triangulation
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Graph Partitioning
6-Partitioning a Plane Triangulation
Near-Triangulations are not 5-partitionable
Triangulations are not 7-partitionable
Planar 3-connected graphs are 6-partitionable.
Partitioning 2-connected Graphs

Theorem

Every 2-connected graph with maximum degree at most 3 is 4-partitionable.

Theorem

Every 2-connected claw-free ($K_{1,3}$-free) graph is 4-partitionable.
Counterexamples
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Is every $k$-connected graph with maximum degree at most $k + 1$ $2k$-partitionable?

Is every $k$-connected $k$-regular graph decomposable, that is, $l$-partitionable for all $l \geq 1$. 


Thank You