Locally Commutative Categories

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Abstract. It is known that a finite category can have all its base monoids in a variety $\mathbf{V}$ (i.e. be locally $\mathbf{V}$, denoted $\ell\mathbf{V}$) without itself dividing a monoid in $\mathbf{V}$ (i.e. be globally $\mathbf{V}$, denoted $\mathbf{gV}$). This is in particular the case when $\mathbf{V}$=$\mathbf{Com}$, the variety of commutative monoids. Our main result provides a combinatorial characterization of locally commutative categories. This is the first such theorem dealing with a variety for which local differs from global. As a consequence, we show that $\ell\mathbf{Com} \subset \mathbf{gV}$ for every variety $\mathbf{V}$ that strictly contains the commutative monoids.

1 Introduction

In algebraic theory of automata, a language $L \subseteq A^*$ is said to be recognized by the finite monoid M if there exist a morphism $\phi : A^* \to M$ and a subset $F \subseteq M$ such that $L = \phi^{-1}(F)$. It is well-known that languages that can be so recognized are precisely the regular languages and that for each regular language there is a unique minimal monoid, called the *syntactic monoid of* L and denoted $M(L)$, that recognizes it. One expects that combinatorial properties of L would be reflected in the algebraic structure of $M(L)$: this intuition is completely valid and a driving theme of the field is to prove theorems of the following form:

"A language L belongs to the combinatorially-defined class $\mathcal{V}$ iff the syntactic monoid $M(L)$ belongs to the algebraically-defined class $\mathbf{V}$."

For technical, but unavoidable, reasons, one sometimes has to deal with subsets of A^+ (instead of A^*) and semigroups (instead of monoids). Most often, "algebraically-defined" means that $\mathbf{V}$ is an M-variety, that is a class of finite monoids which is closed under division (i.e. morphic image and submonoid) and direct product. The notion of S-variety is similarly defined for finite semigroups. Books such as [1, 2, 4] offer a comprehensive treatment of this theory. One interesting by-product of results of the above form is that when membership in $\mathbf{V}$ is decidable, one gets a decision procedure to test if L is in $\mathcal{V}$, since the monoid $M(L)$ can be effectively computed from any of the common representations used for regular languages (automaton, regular expression, grammar, logical formula). Two classical theorems of that nature are the correspondence between star-free languages and aperiodic monoids [5] and the correspondence between piecewise-testable languages and $\mathcal{J}$-trivial monoids [6].

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Consider the situation where two automata are connected in series: for the second machine it is no longer the case that the space of inputs it can receive forms a free monoid, since the input sequence is mediated through the first machine and some combinations may never arise. Technically, the right point of view needed for analyzing the computations of the second automaton, is to view the machine as operating over a free category rather than over a free monoid. In order to understand the all-important case of serial connection of automata and its algebraic incarnation i.e the wreath product of monoids, it is essential to generalize the above setting to the level of categories, e.g. deciding if a monoid M divides a wreath product of the form $S \circ T$ amounts to decide if a certain category, constructible from M and T, divides S. In this framework, one considers languages as sets of finite-length paths in a directed multigraph (instead of finite-length sequences over a set) and such languages may be recognized by finite categories (instead of finite monoids). The notion of the syntactic category of a language appears naturally and so does the notion of a C-variety, i.e. a class of finite categories closed under division and direct product. Thus the manipulation and understanding of finite categories are essential ingredients in manipulation and understanding of regular languages as observed and formalised in the seminal work of [10].

Given a C-variety $\mathbf{W}$, it is easily seen that the monoids in $\mathbf{W}$ form an M-variety. It is thus natural to consider the following question: for a fixed M-variety $\mathbf{V}$, what are the C-varieties $\mathbf{W}$ for which the monoids in $\mathbf{W}$ are precisely those of $\mathbf{V}$? Two natural examples emerge readily: the variety $\mathbf{gV} = \{C : C$ divides M for some $M \in \mathbf{V}\}$, and the variety $\ell\mathbf{V} = \{C :$ every base monoid of C is in $\mathbf{V}$ $\}$ are respectively the smallest and the largest C-variety with that property. It turns out that a combinatorial description of the languages recognized by monoids in $\mathbf{V}$ immediately implies a combinatorial description of the languages recognized by categories in $\mathbf{gV}$; similarly, an algebraic description of the monoids in $\mathbf{V}$ implies an algebraic description of the categories in $\ell\mathbf{V}$. Our understanding is thus complete whenever $\mathbf{gV} = \ell\mathbf{V}$; this happens in a number of interesting cases, e.g. for every non-trivial variety of groups, for semilattices, for aperiodic monoids. But there are also cases where $\mathbf{gV} \not\subseteq \ell\mathbf{V}$, e.g. for the trivial variety, for commutative monoids [9], for $\mathcal{J}$-trivial monoids [3]; apart from the case of the trivial variety, it becomes quite a challenge to find an algebraic description of $\mathbf{gV}$ or a combinatorial description of the languages recognized by members of $\ell\mathbf{V}$.

The main result of this paper is to provide a combinatorial description of the languages recognized by members of $\ell\mathbf{Com}$, the C-variety of locally commutative categories. This is the first instance of such result for a non-trivial variety $\mathbf{V}$ where $\mathbf{gV} \neq \ell\mathbf{V}$. We give our description via congruences of finite-index and some novel ideas have to be introduced. We also show that $\ell\mathbf{Com}$ is contained in $\mathbf{gV}$ for every M-variety $\mathbf{V}$ that strictly contains all commutative monoids. We then use known techniques to derive results about the S-variety $\mathbf{LCom} = \{S : eSe \in \mathbf{Com}$ for every $e = e^2\}$. The paper is organized as follows: section 2 presents the basic notions that are needed, section 3 proves the main theorem about locally commutative categories and section 4 describes the consequences of that result.

2 Basic notions

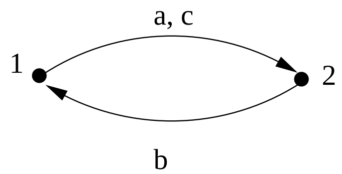
A directed multigraph $G = (V, A, \alpha, \omega)$ consists of a set V of vertices, a set A of directed edges and two mappings $\alpha, \omega : A \to V$, which assigns to each edge a the start vertex $\alpha(a)$ and the end vertex $\omega(a)$ of that edge. Two edges a, b are consecutive iff $\omega(a) = \alpha(b)$. A path of length $n > 0$ is a sequence of n consecutive edges; we extend the mappings α and ω to paths in the natural way. For each vertex v we allow an empty path 1_v of length 0 for which $\alpha(1_v) = \omega(1_v) = v$. The length of a path x will be denoted by $|x|$, and the number of occurrences of an edge a in x by $|x|_a$. Two paths x, y are coterminal, denoted $x \sim y$ if $\alpha(x) = \alpha(y)$ and $\omega(x) = \omega(y)$. A loop is a path x such that $\alpha(x) = \omega(x)$ and a loop edge is a loop that consists of a single edge; we denote by $\overline{x}$ the path obtained from x by removing its loop edges. For a path x and a vertex v, let $x[v]$ stand for the subsequence of x consisting of all edges of the path that are incident on vertex v; note that $x[v]$ is not itself a path, and that when x is a loop $\overline{x}[v]$ has even length for each v.

An equivalence β on the set G^* of all paths in G is a graph congruence iff $x \,\beta\, y$ implies $x \sim y$ and $x_1 \,\beta\, y_1, x_2 \,\beta\, y_2, \omega(x_1) = \alpha(x_2)$ imply $x_1 x_2 \,\beta\, y_1 y_2$. The set of congruence classes, G^*/β, then forms a category. For each path x, we denote the corresponding congruence class containing x by $[x]_\beta$. We note that for every vertex v, the set $\{[x]_\beta : x \text{ is a loop on } v\}$ forms a monoid; we call these the base monoids of G^*/β. We refer the reader to [10] for the technical definition of division of categories and we define a C-variety to be a class of finite categories which is closed under division and direct product.

Monoids can be identified with 1-vertex categories in an obvious way. If we restrict a C-variety to its 1-vertex members, we then get an M-variety. In general, there may exist several C-varieties which coincide on the monoids they contain. For a given M-variety $\mathbf{V}$ it is always the case that $\mathbf{gV} = \{C : C \text{ divides } M \text{ for some } M \in \mathbf{V}\}$ is the smallest C-variety having $\mathbf{V}$ as its restriction to monoids: similarly $\ell\mathbf{V} = \{C : \text{every base monoid of } C \text{ is in } \mathbf{V}\ \}$ is always the largest C-variety with this property. Thus the C-variety corresponding to $\mathbf{V}$ is unique iff $\mathbf{gV} = \ell\mathbf{V}$. Although this holds in several instances, here are two examples where this is not the case.

Example 1. Let $\mathbf{V} = \mathbf{1}$ be the M-variety consisting of the 1-element monoid only. Then for every graph G, $G^*/\beta \in \mathbf{g1}$ iff β and $\sim$ coincide. On the other hand, let B be the subset of those edges of G for which the start and the end vertices belong to different strongly connected components. Define $x \,\gamma\, y$ iff $x \sim y$ and, for each $b \in B$, $x = x_0 b x_1$ iff $y = y_0 b y_1$. Clearly G^*/β is in $\ell\mathbf{1}$ but not in $\mathbf{g1}$ if B is non-empty. In fact, it is an exercise to show that $G^*/\beta \in \ell\mathbf{1}$ iff $\gamma \subseteq \beta$. An interesting consequence of this observation is that $\ell\mathbf{1} \subset \mathbf{gV}$ whenever $\mathbf{1} \subset \mathbf{V}$. Indeed an edge b of B can appear in a path zero or one time only: if M is a non-trivial monoid, i.e. M contains an element $m \neq 1$, it can be used to distinguish paths in which b occurs from paths in which it does not, by mapping b to m and every other edge of the graph to 1. Taking a direct product of $|B|$ copies of M insures that we can recover the equivalence class (in γ) of a path from its value in $M^{|B|}$.

Example 2. Let $\mathbf{V} = \mathbf{Com}$, the variety of all commutative finite monoids. On any graph G, define $x \, \gamma_{t,q} \, y$ iff $x \sim y$ and for each $a \in A$ either ($|x|_a < t$ and $|x|_a = |y|_a$) or ($|x|_a \geq t, |y|_a \geq t$ and $|x|_a \equiv_q |y|_a$, where $\equiv_q$ denotes modulo q equality). It can be shown that $G^*/\beta \in \mathbf{gCom}$ iff $\gamma_{t,q} \subseteq \beta$ for some $t \geq 0, q \geq 1$. On the other hand, consider the following graph G:



define $x \, \beta \, y$ iff $x \sim y$ and ($|x| \leq 3$ and $x = y$) or ($|x| > 3$ and $x \sim y$). Then $G^*/\beta \in \ell\mathbf{Com}$ but not in $\mathbf{gCom}$.

This example is in some sense generic as [9] proves that a category C is in $\mathbf{gCom}$ iff it satisfies $xyz = zyx$ whenever x and z are coterminal; this result is combinatorially quite delicate to obtain. By definition, a category C is in $\ell\mathbf{Com}$ iff $xy = yx$ for every two loops x, y on the same vertex. The above example shows that knowing the number of occurrences of each edge in a path is not enough information to characterize the value of the path in a locally commutative category. Our present paper will provide a combinatorial description of the information that is missing in order to do so.

3 Combinatorial characterization of locally commutative categories

3.1 Free locally commutative categories

Let G be a graph and define on G^* the congruence $x \, \gamma_\infty \, y$ iff $x \sim y$ and $|x|_a = |y|_a$ for every edge a. Let also θ_∞ be the coarsest congruence satisfying the equation $xyz \, \theta_\infty \, zyx$ whenever $x \sim z$. It was shown in [9] that $\gamma_\infty = \theta_\infty$.

The free locally commutative congruence on G^*, which we denote θ^ℓ_∞, is the coarsest congruence satisfying $xy \, \theta^\ell_\infty \, yx$ whenever x and y are loops on the same vertex. Obviously, θ^ℓ_∞ refines $\theta_\infty = \gamma_\infty$. We also observe that $x \, \theta^\ell_\infty \, y$ iff $|x|_a = |y|_a$ for every loop edge a and $\overline{x} \, \theta^\ell_\infty \, \overline{y}$, i.e. the presence of loop edges cannot affect the congruence relation provided they are in equal number in both paths. There is another combinatorial property that is preserved by commutation of loops; let v be a vertex such that $|xy|_a = 0$ for each loop edge a on v and such that $|xy[v]|_a \leq 1$ for each a; then the subsequence $xy[v]$ is an even permutation of the subsequence $yx[v]$. We now proceed to show that these combinatorial properties, the last one suitably modified, characterize θ^ℓ_∞.

In general, it is not the case that every edge appears at most once in a path. Suppose $|x|_a = k$; we make the k occurrences of a in x formally distinct by labelling them, in the order they appear, as $a_{\lambda(1)}, \dots, a_{\lambda(k)}$, where λ is a permutation of $\{1, \dots, k\}$. A *labelling* $\Lambda(x)$ of a path x is the result of applying this process to each edge. Thus the edges forming $\Lambda(x)$ can be viewed as being distinct. We will write $I(x)$ when the labelling is based on identity permutations for each edge, i.e. for each a, if $|x|_a = k$, the occurrences of a in x are renamed $a_1, \dots, a_k$ in that order.

We define on G^* $x \, \gamma_\infty^\ell \, y$ iff $x \, \gamma_\infty \, y$ and there exists a labelling Λ for $\overline{y}$ such that for every vertex v the sequence $\Lambda(\overline{y})[v]$ is an even permutation of the sequence $I(\overline{x})[v]$. It can be checked that γ_∞^ℓ is a congruence relation.

We state a useful property of γ_∞^ℓ

Proposition 1. *Let $x = x_1 \rho x_2$ and $y = y_1 \rho y_2$ be two paths in a graph such that $x \, \gamma_\infty^\ell \, y$ and ρ is a loop on some vertex v such that for each edge a in ρ we have $|x|_a = |y|_a = 1$. Then $x_1 x_2 \, \gamma_\infty^\ell \, y_1 y_2$.*

Proof. Clearly $x_1 x_2 \, \gamma_\infty \, y_1 y_2$. For the second property that we need to prove, we can assume that x and y do not contain any loop edge, since this property is dealing with $\overline{x}$ and $\overline{y}$. From the definition of γ_∞^ℓ there exists a labelling function Λ such that for each vertex v, $\Lambda(y)[v]$ is an even permuation of $I(x)[v]$. But every edge that appears in ρ is unique and so Λ and I must have labelled ρ exactly the same way. Also for every vertex v, $\Lambda(\rho)[v]$ and $I(\rho)[v]$ have the same length, which is even since ρ is a loop. This implies that $\Lambda(y_1 y_2)[v]$ is an even permutation of $I(x_1 x_2)[v]$ for every vertex v, as required. □

An immediate corollary follows

Corollary 1. *If two paths x and y satisfy $x \, \gamma_\infty^\ell \, y$ and there are n loops $\rho_1, \dots, \rho_n$ appearing in x and y where for each edge a in a loop ρ_i we have $|x|_a = |y|_a = 1$ then the paths obtained by deleting these loops from x and y (say x' and y'), satisfy $x' \, \gamma_\infty^\ell \, y'$*

Proof. This follows by repeatedly applying proposition 1 once for every loop ρ_i. □

Lemma 1. *For two paths x and y, $x \, \gamma_\infty^\ell \, y$ iff $x \, \theta_\infty^\ell \, y$.*

Proof. The implication from right to left is easy and left to the reader.

Now for the other direction we assume $x \, \gamma_\infty^\ell \, y$. Since every loop edge appears the same number of times in the two paths, its suffices to show $\overline{x} \, \theta_\infty^\ell \, \overline{y}$, so we now suppose that x and y have no loop edges. Because of the labelling involved in the definition of γ_∞^ℓ, we can think of x and y as having at most one occurrence of any edge.

We will prove our claim by induction on the length of the paths. For the base case of $|x| = 1$ the lemma is trivially true. Also note that if x and y are two coterminal paths that start with the same edge a and $x = ax'$ and $y = ay'$, then $x \, \gamma_\infty^\ell \, y$ implies $x' \, \gamma_\infty^\ell \, y'$, since the occurrence of a is unique. Thus from the inductive hypothesis we obtain $x' \, \theta_\infty^\ell \, y'$ and this proves $x \, \theta_\infty^\ell \, y$.

Assume next that x and y start with different edges. Let $x = ax_0bx_1, y = by_0ay_1, v = \alpha(x) = \alpha(y)$; If v appears in y_1, i.e. $y_1 = y_{10}y_{11}$ with $\omega(y_{10}) = v$, then we can commute by_0 and ay_{10} and we are back at the previous case. A similar argument holds if x_1 contains v. Otherwise the vertex v, which is the common end vertex of x_0 and y_0, must appear at least once more in those two subpaths because $x[v]$ is an even permutation of $y[v]$. This implies the presence of an edge c in x_0 with start vertex v. This edge also appears in y, hence must appear in y_0. We can thus use loop commuting to bring the c as first edge in each path, and so $x\,\theta^\ell_\infty\, cx'\,\gamma^\ell_\infty\, cy'\,\theta^\ell_\infty\, y$ for some x' and y' (note that this follows from the already proven fact that θ^ℓ_∞ refines γ^ℓ_∞). Now we are back to the case handled before. $\square$.

The lemma above combinatorially captures the algebraic congruence θ^ℓ_∞ and so provides a tool for describing the language recognized by free locally commutative categories. But it is impossible to work directly with the congruence γ^ℓ_∞ for the case of finite categories since we have to deal with paths that are equivalent even though their lengths are different and so the concept of even permutations does not work anymore. This motivates us to find another way of characterising θ^ℓ_∞.

Consider the following special case. Let $x = ax_1yx_2zx_3a$ be a path where a is an edge which is coterminal with the subpaths y and z. One verifies that $x\,\theta^\ell_\infty zx_3yx_2ax_1a\ \theta^\ell_\infty\, zx_3yx_1ax_2a\,\theta^\ell_\infty\, zx_1ax_3yx_2a\,\theta^\ell_\infty\, ax_3zx_1yx_2a\,\theta^\ell_\infty\, ax_1yx_3zx_2a$ $\theta^\ell_\infty\, ax_1zx_2yx_3a$. Thus we are able to interchange in x the coterminal subpaths y and z by using commutation of loops, because x contains an edge twice which is coterminal with these subpaths. The equivalence between exchange of coterminal paths and commutation of loops holds under a more general condition that we formalize below.

For a path x, define Γ^*_x as the reflexive and transitive closure of the relation Γ_x defined on the vertices by $v_1\Gamma_x v_2$ whenever there is an edge a such that $|x|_a \geq 2$ and $\alpha(a) = v_1, \omega(a) = v_2$ or $\alpha(a) = v_2, \omega(a) = v_1$.

Lemma 2. *For any path $x = x_1x_2x_3x_4x_5$ in G, if $\alpha(x_2)\,\Gamma^*_x\,\omega(x_2)$ and $x_2 \sim x_4$ then $x_1x_2x_3x_4x_5\,\theta^\ell_\infty\, x_1x_4x_3x_2x_5$.*

Proof. Let $x = x_1x_2x_3x_4x_5$, $y = x_1x_4x_3x_2x_5$, $v_a = \alpha(x_2) = \alpha(x_4)$, $v_b = \omega(x_2) = \omega(x_4)$. If $v_a = v_b$, the result is immediate. Otherwise we prove the lemma by showing that the hypothesis implies $x\,\gamma^\ell_\infty\, y$. Clearly $|x|_a = |y|_a$ for each a. Consider now $\overline{x}$ and $\overline{y}$, or equivalently assume that x and y contain no loop edges. We have to show that there exists a labelling Λ which will make $\Lambda(y)[v]$ an even permutation of $I(x)[v]$ for every vertex v. Since y is obtained by interchanging subpaths of x, we get naturally from $I(x)$ a first labelling Λ for y. For each vertex $v \neq v_a, v_b$, we have that $x_2[v]$ and $x_4[v]$ have even length. Since $\Lambda(y)[v]$ is obtained from $I(x)[v]$ by interchanging two blocks of even length, it must be an even permutation. The problem is that $x_2[v_a]$ and $x_4[v_a]$ have odd length, hence the permutation $\Lambda(y)[v_a]$ is odd, and the same for v_b. Since $v_a\,\Gamma^*_x\,v_b$ there exists some $n > 0$ such that $v_a = v_0\,\Gamma_x\,v_1\,\Gamma_x\,v_2 \ldots \Gamma_x\,v_{n-1}\,\Gamma_x\,v_n = v_b$. Using the definition of Γ_x let e_i be the edge connecting v_{i-1} and v_i for $i > 0$. Each

e_i is directed and its direction is arbitrary. Also there are at least two occurrences of e_i in both x and y. Let us create a new labelling Λ' that switches the labels (as given by Λ) of two arbitrarily chosen instances of e_i for each i. For all other edge occurrences, Λ' is the same as Λ. For each of $v_1, \ldots, v_{n-1}$, $\Lambda'(y)[v_i]$ differs from $\Lambda(y)[v_i]$ by two transpositions, hence it remains even. For v_0 and for v_n, the difference between Λ and Λ' is one transposition, hence these become even as well. □

An edge e in a path x is called a *special edge for* x iff $\alpha(e)$ and $\omega(e)$ are not related by Γ_x^*. A maximal subpath in a path x that is completely contained inside an equivalence class of Γ_x^* is called a *component of* x. So special edges always connect components that are over different equivalence classes of Γ_x^*. Note that a component could consist of just the identity path in which case two special edges would be adjacent to each other. Clearly every special edge occurs exactly once in a path x. Every path x in G^* is thus now uniquely decomposed as $x_0 e_1 x_1 \ldots e_n x_n$, where the e_i's are the special edges for x and the x_i's are its components. The lemma above then gives the following result

Corollary 2. *If a path x has no special edges then for any path y, $x\,\theta_\infty^\ell\,y$ iff $x\,\theta_\infty\,y$ iff $x\,\gamma_\infty\,y$.*

In order to take into account the presence of special edges, we define, for each path x, a *reduced graph* $G_x = (V_x, A_x, \alpha_x, \omega_x)$ where $V_x = V/\Gamma_x^*$, A_x is the set of special edges for x, and α_x, ω_x are defined in the obvious way. The path x induces a path $Red(x)$ in the graph G_x by taking $Red(x)$ to be the sequence of special edges in the order they appear in x. Note that $Red(x)$ is a permutation of A_x and that $x\,\gamma_\infty\,y$ implies that $\Gamma_x^* = \Gamma_y^*$, hence that the graphs G_x and G_y are identical; furthermore we then have that $Red(x) \sim Red(y)$ in this graph.

We now define a congruence on G^* by $x\,\delta_\infty^\ell\,y$ iff $x\,\gamma_\infty\,y$ and $Red(x)\,\gamma_\infty^\ell\,Red(y)$.

Lemma 3. *For two paths x and y in G if $x\,\delta_\infty^\ell\,y$ and $Red(x) = Red(y)$ then $x\,\theta_\infty^\ell\,y$.*

Proof. Let $x = x_0 e_1 x_1 \ldots e_n x_n$, $y = y_0 e_1 y_1 \ldots e_n y_n$. Observe that this forces $x_i \sim y_i$ for each i. Fix an equivalence class C in V/Γ_x^* and let $0 \le i_0 < i_1 < \ldots, i_t \le n$ be the indices for which x_{i_j} is a component of x over C; the same sequence of indices gives the components of y that are over C. For each j replace the subpath of x between $x_{i_{j-1}}$ and x_{i_j} by a "meta-edge" E_j that goes from $\omega(x_{i_{j-1}})$ to $\alpha(x_{i_j})$. Do the same for y. Consider the paths $X = x_{i_0} E_1 x_{i_1} \ldots E_t x_{i_t}$ and $Y = y_{i_0} E_1 y_{i_1} \ldots E_t y_{i_t}$. We have that $X\,\gamma_\infty\,Y$ and these two paths now have no special edges since the two endpoints of each E_j are in C. By corollary 2, X can be transformed into Y by commuting loops. The corresponding sequence of operations will transform x into a path $x' = x'_0 e_1 x'_1 \ldots e_n x'_n$ where $x'_i = y_i$ for $i \in \{i_0, \ldots, i_t\}$ and $x'_i = x_i$ otherwise. Doing this for each class of V/Γ_x^* in turn will transform x into y. □

We are now in a position to prove the equivalence of δ_∞^ℓ and θ_∞^ℓ.

Lemma 4. *For any two paths x and y in G $x\,\delta^{\ell}_{\infty}\,y$ iff $x\,\theta^{\ell}_{\infty}\,y$.*

Proof. The implication from right to left is easy and left to the reader. We prove the second implication. Suppose $x = x_0 e_1 x_1 \dots e_n x_n$; We fix in each equivalence class C of V/Γ^*_x a vertex v_C, and for each special edge e_i going from vertex v in C to a vertex v' in C', we augment the graph G by introducing four new edges: e_i^C going from v to v_C, f_i^C going from v_C to v, g_i^C going from v' to $v_{C'}$ and h_i^C going from $v_{C'}$ to v'. We create from x a new path x' in the augmented graph by the following process: if e_j is a special edge for x going from vertex v in C to a vertex v' in C', we replace e_j by $e_j^C f_j^C e_j g_j^{C'} h_j^{C'}$. If any loop edges have been added we remove them. We create y' from y similarly. $Red(x')\,\gamma^{\ell}_{\infty}\,Red(y')$ comes trivially from the fact that $x\,\delta^{\ell}_{\infty}\,y$ (since $Red(x) = Red(x')$ and $Red(y) = Red(y')$). Hence also $Red(x')\,\theta^{\ell}_{\infty}\,Red(y')$ by lemma 1. By construction, if there is a loop on vertex C appearing in $Red(x')$ in the reduced graph, there is a corresponding loop on vertex v_C appearing in x' in the augmented graph. Thus, corresponding to the sequence of loop commutations that transforms $Red(x')$ to $Red(y')$ in the reduced graph, there is a sequence of loop transformations, in the augmented graph, that transforms x' into a path (say w) in which the special edges appear in the same order as those of y'. Hence using lemma 3 it follows $x'\,\theta^{\ell}_{\infty}\,w\,\theta^{\ell}_{\infty}\,y'$. So $x'\,\gamma^{\ell}_{\infty}\,y'$ and by recalling that we obtained x' (y') from x (y) by adding a certain number of loops around every vertex v_C we apply proposition 1 and corollary 1 to get $x\,\gamma^{\ell}_{\infty}\,y$. Hence $x\,\theta^{\ell}_{\infty}\,y$. □

Thus δ^{ℓ}_{∞} provides an alternative characterisation of locally commutative free categories. We will see in the next section that this characterisation can be naturally adapted to the case of finite categories.

3.2 Locally commutative finite categories

We recall from [9] that the algebraic description of finite globally commutative categories is given by a path congruence $\theta_{t,q}$ generated by equations: $xyz\,\theta_{t,q}\,zyx$ for $x \sim z$ and $x^t\,\theta_{t,q}\,x^{t+q}$ where x is a loop.

The corresponding combinatorial congruence $\gamma_{r,q}$ is induced by relations: for $x \sim y$ we say $x\,\gamma_{r,q}\,y$ iff for all edge $a \in A$, either ($|x|_a$, $|y|_a < r$ and $|x|_a = |y|_a$) or ($|x|_a$, $|y|_a \geq r$ and $|x|_a \equiv_q |y|_a$)

We summarize the main result from [9] in the lemma below:

Lemma 5. *For every $t \geq 0$ and graph G there exists s such that for two paths x and y, $x\,\gamma_{s,q}\,y$ implies $x\,\theta_{t,q}\,y$.*

Observe that $\theta_{t,q}$ can be thought of as a rewriting system. If a path y can be obtained from path x using only loop commuting ($uw \to wu$) and loop replication ($u^t \to u^{t+q}$) <u>without</u> using loop deletion ($u^{t+q} \to u^t$), then we write $x \leq_{t,q} y$. It is a trivial observation that $x \leq_{t,q} y$ implies $x\,\theta_{t,q}\,y$. Clearly $x \leq_{t,q} y$ implies for all $a \in A$, $|x|_a \leq |y|_a$.

We now state a result that follows from the argumentation given in [8] for proving his lemma B.3.10 in Appendix B.

Proposition 2. *For paths x and y and for $t > 0$ if $\forall a \in A(G)$, $|x|_a \leq |y|_a$ and $x\,\gamma_{t+1,q}\,y$ then $x \leq_{t,q} y$.*

Proof. This follows from the argument used in the proof of lemma B.3.10 given in [8] (and is left as an exercise for the reader). □

As an extension of ideas from free locally commutative categories we introduce $\theta^{\ell}_{t,q}$ to be the finite index path congruence generated by the conditions: $xy\,\theta^{\ell}_{t,q}\,yx$ where x and y are loops and $x^t\,\theta^{\ell}_{t,q}\,x^{t+q}$ where x is a loop. Analogous to the global case we write $x \leq^{\ell}_{t,q} y$ when $x\,\theta^{\ell}_{t,q}\,y$ and y can be obtained from x by just loop commuting and loop replication.

We also extend our combinatorial characterisation from the last section to $\delta^{\ell}_{t,q}$ meaning for two paths x and y $x\,\delta^{\ell}_{t,q}\,y$ iff $x\,\gamma_{t,q}\,y$ and $Red(x)\,\gamma^{\ell}_{\infty}\,Red(y)$, where γ^{ℓ}_{∞} gets defined on the reduced graph G_x. Note that this congruence only depends on the permutation of reduced paths which are of fixed length.

Using the definition of $\theta^{\ell}_{t,q}$ and the lemma 2 we can conclude the following:

Corollary 3. *For paths with no special edges, $\theta_{t,q}$ and $\theta^{\ell}_{t,q}$ are equivalent.*

This corollary along with lemma 5 gives us the intuition to expect the following result

Lemma 6. *If $x\,\delta^{\ell}_{s,q}\,y$ and $Red(x) = Red(y)$ then $x\,\theta^{\ell}_{t,q}\,y$ where s and t are related according to lemma 5.*

Proof. We direct the attention of the reader to the proof of lemma 3. Employing exactly the same technique as in that proof, fixing an equivalence class C in V/Γ^*_x we add "meta-edges" connecting two successive components of that class and obtain paths X and Y respectively from x and y. In our case here, $X\,\gamma_{s,q}\,Y$. Therefore using lemma 5 it follows $X\,\theta_{t,q}\,Y$ and since X and Y have no special edges from corollary 3 X can be transformed into Y by transformations preserving $\theta^{\ell}_{t,q}$. We apply the same operations on x to get a new path x' and then repeat the procedure with x' for each class of V/Γ^*_x to finally get y. □

We can now combine the lemma above and proposition 2 to obtain the following corollary

Corollary 4. *If $x\,\delta^{\ell}_{t+1,q}\,y$ and $Red(x) = Red(y)$ with $|x|_a \leq |y|_a$, then $x \leq^{\ell}_{t,q} y$.*

Lemma 7. *For every $t \geq 2$ and $q \geq 1$, there exists $R \geq t+1$ such that $x\,\delta^{\ell}_{R,q}\,y$ implies that there exists a path ρ satisfying $x\,\delta^{\ell}_{t+1,q}\,\rho$, where $\rho\,\theta^{\ell}_{t,q}\,y$ and for all edges $a \in A$, $|x|_a \leq |\rho|_a$.*

Proof. We will use lemmas 3.3 and 3.8 from [9] to prove this. Specifically let $R = m(G, t+1)(|E|+1) + 1$ where $m(G, t+1) = |V| + (t+1)(2^{|E|} - 1) + 2$ as defined in [9]. So for each edge a such that $|x|_a > |y|_a$ we have $|y|_a \geq R$ and since y can have at most $(|E|+1)$ components there is at least one component that has at least $m(G, t+1)$ occurences of a. We can now straight away apply the argument used to prove lemma 3.8 in [9] and obtain the result of the present lemma. □

Lemma 8. *If for two paths x and y, $|x|_a \leq |y|_a$ for all $a \in A$ and $x\,\delta^{\ell}_{t+1,q}\,y$, then $x\,\theta^{\ell}_{t,q}\,y$ for $t \geq 2$ and $q \geq 1$.*

Proof. We ask the reader to recall the technique used to prove lemma 4. We mimic the steps in that proof to augment the graph G by introducing four new edges for each special edge e_i and then modify the paths x and y to x' and y' respectively as prescribed there. (Note: we are using the same notation as in that proof.) Also let A' represent the set of edges of the augmented graph. The same argumentation of the earlier proof carries over to establish the existence of a path w such that $Red(w) = Red(y')$ and $x'\,\theta^{\ell}_{\infty}\,w\,\delta^{\ell}_{t+1,q}\,y'$. From corollary 4 it follows that $w \ \leq^{\ell}_{t,q} \ y'$ and hence $x' \ \leq^{\ell}_{t,q} \ y'$. This implies that there exists a series of loop commuting and loop duplicating transformations to obtain y' from x'. Let the loops that got duplicated, be called $\rho_1, \ldots, \rho_n$ and let them be around vertices $v_1, \ldots, v_n$ in G respectively. Also let n_i be the number of times ρ_i was duplicated. It is a trivial observation that every vertex v_i occurs somewhere in the path x and every loop ρ_i contains edges strictly from the unaugmented original graph G (since for each edge $a \in A' \backslash A$ we have $|x'|_a = |y'|_a$). Also no loop ρ_i contains any special edges as their count is one in both x' and y'. Hence every loop ρ_i could be added n_i times to path x to obtain a path u in G such that $u\,\delta^{\ell}_{\infty}\,y$ since $Red(x) = Red(u)$. This implies $x\,\delta^{\ell}_{t+1,q}\,y$ and hence from corollary 4 we have $x\,\theta^{\ell}_{t,q}\,u$. Now applying lemma 4 to u and y we get $u\,\theta^{\ell}_{\infty}\,y$ and hence $x\,\theta^{\ell}_{t,q}\,y$. $\square$

We now state the main result of this paper

Theorem 1. *β is a ℓ**Com**-congruence iff there exist $R \geq 2$, $q \geq 1$ such that $\delta^{\ell}_{R,q} \subseteq \beta$.*

Proof. The direction from right to left is trivial and left as an exercise for the reader (it can be verified that $\delta^{\ell}_{R,q}$ is a ℓ**Com**-congruence). For $t \geq 2$ we choose $R = m(G, t+1)(|E|+1)+1$ according to lemma 7. Then $x\,\delta^{\ell}_{R,q}\,y$ implies there exists a path z with $|x|_a \leq |z|_a$ for each edge $a \in A$ and $x\,\delta^{\ell}_{t+1,q}\,z\,\theta^{\ell}_{t,q}\,y$. Using lemma 8 we get $x\,\theta^{\ell}_{t,q}\,y$. We recall that for the cases $t = 0$ and $t = 1$, [9] proves ℓ**Com** coincides with **gCom**. $\square$

4 Consequences of the main result

In this section, we sketch some consequences of the combinatorial description obtained above. When an M-variety **V** is such that the C-varieties **gV** and ℓ**V** differ, then ℓ**V** cannot be equal to **gW** for any M-variety **W**. How big should **W** be to insure $\ell\mathbf{V} \subset \mathbf{gW}$? In example 1 of section 2, we observed that for the trivial M-variety we have $\ell\mathbf{1} \subset \mathbf{gW}$ for every non-trivial **W**. We now argue that a similar phenomenon occurs for **Com**.

Theorem 2. $\ell\mathbf{Com} \subset \mathbf{gW}$ *for every M-variety* **W** *that strictly contains* **Com**.

Proof. Our main result shows that in every locally commutative category, the value of a path is determined by the number of occurrences of each edge (threshold t, modulo q for some $t \geq 0, q \geq 1$) and the ordering of the so-called "special" edges. The first condition can be determined by using for each edge a cyclic counter of appropriate cardinality. For the second condition, let M be any non-commutative monoid, i.e. M contains two elements m and m' such that $mm' \neq m'm$. Fix two edges of the graph, a and b, map a to m, b to m' and every other edge to 1. If a path x contains at most one occurrence of each of a and b, which is necessarily the case when these two edges are special for x, the value of the path in M is in $\{1, m, m', mm', m'm\}$. In particular if both edges occur once, the order in which they appear can be recovered from the value in the monoid. If the graph has k edges, we can use the direct product of k cyclic counters to count occurrences of each edge, and $O(k^2)$ copies of M, one for each pair of edges. The value of the counters will determine the first condition and also which edges are special for a given path; we can then look up the appropriate copies of M to know in which order the special edges have appeared, hence recover the $\delta^{\ell}_{t,q}$-value of the path. □.

Next, we transfer the last theorem to the S-variety $\mathbf{LCom} = \{S : eSe \in \mathbf{Com}$ for all $e = e^2\}$. For any semigroup S, consider the graph $G = (V, A, \alpha, \omega)$, where V is the set of idempotents of S, $A = V \times A \times V, \alpha(e, s, f) = e, \omega(e, s, f) = f$. Define the congruence β on G^* by identifying coterminal paths that multiply out to the same element in S. This construction trivially insures that $S \in \mathbf{LCom}$ iff $G^*/\beta \in \ell\mathbf{V}$. It follows from work of [7] that $S \in \mathbf{V} * \mathbf{D}$, where $\mathbf{D} = \{S : Se = e$ for all $e = e^2\}$ and $*$ denotes wreath product of varieties, iff $G^*/\beta \in \mathbf{gV}$. We thus get the following

Theorem 3. $\mathbf{LCom} \subset \mathbf{V} * \mathbf{D}$ *for every M-variety* $\mathbf{V}$ *that strictly includes the commutative monoids.*

5 Conclusion

In this paper we have proved a combinatorial description for the languages that can be recognized by finite locally commutative categories. This is the first result of that kind for a non-trivial M-variety for which the induced global and local C-varieties are different. We derived as a consequence the upper bound that for each M-variety $\mathbf{V}$ properly including the commutative monoids, the inclusion $\ell\mathbf{V} \subset \mathbf{gV}$ holds, which is similar to the situation for the trivial M-variety. It is easily checked that all these results can be proved, mutatis mutandis, for the C-variety of locally aperiodic commutative monoids. There is another famous case of an M-variety for which the induced global and local C-varieties are different, namely the variety $\mathbf{J}$ of $\mathcal{J}$-trivial monoids. However, Jorge Almeida has pointed out to us that there exists a C-variety $\mathbf{gV}$ where V is a M-variety of aperiodic monoids that strictly contains $\mathbf{J}$, such that $\mathbf{gV}$ does not contain $\ell\mathbf{J}$. It would be interesting to find an upper bound for $\ell\mathbf{J}$ in terms of globally defined C-varieties.

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