Algorithms: Assignment sheet 4

Due date: December 22, 2025

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1. Show that any language in the class NP can be decided by an algorithm with running time $2^{O(n^k)}$ for some constant k.

2. Recall that the complexity class co-NP is the set of those languages L such that $\overline{L} = \Sigma^* - L$ is in NP. Show that if $NP \neq$ co-NP, then $P \neq NP$.

3. Recall the set cover problem: we are given a universe U along with a collection $\mathcal{C}$ of subsets of U such that $\bigcup_{S \in \mathcal{C}} S = U$; moreover, each set S has a non-negative cost $c(S)$. The problem is to find a min-cost subcollection $\mathcal{C}' \subseteq \mathcal{C}$ whose union is U.

   Consider the following LP-relaxation of the problem.

$$\begin{aligned} \text{minimize} \quad & \sum_{S \in \mathcal{C}} c(S)\, x_S \\ \text{subject to} \quad & \sum_{S : u \in S} x_S \;\geq\; 1 \quad \text{for } u \in U \\ & x_S \;\geq\; 0 \quad \text{for } S \in \mathcal{C} \end{aligned}$$

   Let $(x_S^* : S \in \mathcal{C})$ be an optimal solution to the above LP. Suppose the collection $\mathcal{C}$ satisfies the following additional condition: *every element of U appears in at most f sets in $\mathcal{C}$*. Consider the following rounding algorithm: if $x_S^* \geq 1/f$ then include S in our subcollection.

   Show that such a subcollection is indeed a valid set cover of size at most $f \cdot \mathsf{OPT}$, where OPT is the cost of the min-cost set cover.

4. Consider the following rounding rule for the above problem: if $x_S^* > 0$, then include S in our subcollection. Show that such a subcollection is also a valid set cover of cost at most $f \cdot \mathsf{OPT}$.

   (*Hint:* Write the dual and use complementary slackness.)

5. Consider any bipartite graph $G = (A \cup B, E)$. By König-Egerváry theorem, we know that the size of a minimum vertex cover in G is the same as the size of a maximum matching. So we can determine the cardinality of a minimum vertex cover in G in polynomial time by running a maximum matching algorithm. Suppose we want to *find* a minimum vertex cover C, i.e., the set C and not just the value of $|C|$. Show a polynomial time algorithm for this problem.

6. We wish to solve the minimum vertex cover problem on input instances that consist of a graph G, together with a valid vertex colouring of G with 3 colours. Give a 4/3-approximation algorithm for the minimum cardinality vertex cover problem on such instances.

   (*Hint:* You can use the fact that we can compute a minimum cardinality vertex cover in a bipartite graph in polynomial time.)

7. The following problem is called the *multiway cut* problem and it is NP-hard for $k \geq 3$: Given a set of terminals $S = \{s_1, \ldots, s_k\} \subseteq V$, a multiway cut is a set of edges whose removal

disconnects the terminals from each other. The multiway cut problem asks for the minimum weight such set of edges. (Note that the case $k = 2$ is the minimum s-t cut problem.)

Design a simple $2 - 2/k$ factor approximation algorithm for this problem.

(*Hint:* First compute a minimum weight cut for each s_i that disconnects s_i from the rest of the terminals.)

8. Consider the following multiprocessor scheduling problem. The input consists of n jobs: $J_1, \ldots, J_n$. Job J_i needs run-time p_i, for each $1 \le i \le n$. The jobs are to be scheduled on m identical processors so as to minimize $\max_{1 \le i \le m} S_i$, where S_i = sum of the run-times of jobs assigned to processor i. This is an NP-hard problem.

   Let us design an approximation algorithm for this problem. Let our algorithm consider the n jobs one-by-one and assign job J_k, for $1 \le k \le n$, to a processor which at that point is the least loaded processor (i.e., it has the least value of S_i, taking into account only the assignments of $J_1, ..., J_{k-1}$). Show that this algorithm has an approximation ratio of $2 - 1/m$.

9. Given a graph $G = (V, E)$ with an even number of vertices and non-negative edge costs, the *balanced* max-cut problem asks for a partition $(A, V \setminus A)$ such that (i) $|A| = |V \setminus A|$ and (ii) the cost of edges crossing the cut $(A, V \setminus A)$ is maximized. This problem is NP-hard.

   Show a randomized polynomial time algorithm for this problem to return a partition $(X, V \setminus X)$ such that $|X| = |V \setminus X|$ and the expected cost of edges crossing $(X, V \setminus X)$ is at least $\mathsf{opt}/2$, where opt is the sum of edge costs in the optimal solution to the balanced max-cut problem.

10. Show that the following problem cannot be approximated within any polynomial time computable factor α, unless $P = NP$.

    Given a graph $G = (V, E)$ with positive weights on its edges, and a positive integer k, find a subset S of vertices of cardinality k such that the total weight of edges in the subgraph induced by S is minimized.