A Characterization of Strategy-Proof Probabilistic Assignment Rules

Donthu, Roy, Sadhukhan, Sreedurga (2025)

Souvik Roy

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Outline

Motivation

- Assignment of indivisible objects with initial endowments (the “housing market”).
- Probabilistic assignments are used for fairness.
- Deterministic case: Ma (1994) — TTC $=$ unique mechanism satisfying:
  - Strategy-proofness
  - Pareto efficiency
  - Individual rationality
- A weakening of this result with Pair efficiency is due to Ekici (2024).
- Probabilistic case: with fractional endowments, in general, there is no rule satisfying SD-Strategy-proofness, SD-Pareto efficiency, and SD-Individual rationality.
- Almost nothing is known in the probabilistic setting with deterministic endowments.

Objectives

- Introduce a weaker incentive notion: **SD-top-strategy-proofness**.
- Characterize probabilistic assignment rules satisfying:
  - SD-Pareto efficiency/SD-pair efficiency
  - SD-individual rationality
  - SD-top-strategy-proofness
- Characterization on restricted domains (introduced in Sen (2011)):
  - **Free Pair at the Top (FPT)**
  - **Free Triple at the Top (FTT)**

Basic Model

- $N = \{1, \dots, n\}$ agents, $X = \{x_1, \dots, x_n\}$ objects.
- Strict preferences: $P_i \in P$.
- Initial endowment: deterministic, agent i owns x_i.
- Probabilistic assignment: bistochastic matrix $A = (A_{ij})$.
- $A_{i\bullet}$ = agent i's lottery over objects.

Stochastic Dominance (SD)

Given preferences P_i and lotteries λ, μ:

$$\lambda \succeq_{P_i} \mu \quad \text{iff} \quad \lambda(U(x, P_i)) \geq \mu(U(x, P_i)) \;\; \forall x.$$

- Standard comparison of lotteries in random assignment.
- Used to define all properties: efficiency, IR, strategy-proofness.

SD-Pareto Efficiency

A probabilistic assignment A is SD-Pareto dominated by A' if:

- $A'_{i\bullet} \succeq_{P_i} A_{i\bullet}$ for all i, and
- strict for at least one j.

A rule is SD-efficient if it never produces a dominated outcome.

SD-Pair Efficiency (New Notion)

A is SD-pair dominated by A' if:

- There exists pair (i, j) such that both strictly gain.
- Everyone else's allocation unchanged.

Much weaker than SD efficiency.

- Can allow infinitely many SD-pair-efficient outcomes.

SD-Individual Rationality

$$A_{i\bullet} \succeq_{P_i} E_{i\bullet} \quad \forall i$$

Since E is deterministic, this means:

$$A_{i\,x_i} = 1 \quad \text{if } x_i \text{ is not worse than any other object for } i.$$

SD-Top-Strategy-Proofness

Very weak incentive requirement:

$$\phi_{i,P_i(1)}(P_i, P_{-i}) \geq \phi_{i,P_i(1)}(P_i', P_{-i})$$

- Only prohibits manipulations that increase probability of most-preferred object.
- Weaker than SD-strategy-proofness.
- Natural in settings where agents focus on getting their top object.

FPT and FTT Domains

Free Pair at the Top (FPT):

- For any x, y, some preference ranks x first, y second.

Free Triple at the Top (FTT):

- For any x, y, z, some preference has $x \succ y \succ z$.
- **FPT** is weaker; **FTT** is stronger.

Top Trading Cycles (TTC)

- Build directed graph: each agent points to owner of their top object.
- Since finite, cycle must exist.
- Assign each agent in cycle the object they point to.
- Remove them; repeat on reduced market.

Main Theorem 1 (SD-Pareto Version)

Theorem

On any FPT domain: *A probabilistic assignment rule is*

- *SD-Pareto efficient*
- *SD-IR*
- *SD-top-strategy-proof*

iff it is the TTC rule.

Main Theorem 2 (SD-Pair Version)

Theorem

On any FTT domain: A probabilistic assignment rule is

- *SD-pair efficient*
- *SD-IR*
- *SD-top-strategy-proof*

iff it is the TTC rule.

Why FTT is Needed?

- Pair-efficiency is extremely weak.
- Example: infinitely many SD-pair-efficient lotteries exist even with 3 agents.
- Need stronger domain richness to ensure uniqueness.

Ex-Post Versions (Theorems 3 & 4)

Using Birkhoff–von Neumann:

$$A = \sum_k \alpha_k \Pi_k$$

Define:

- **Ex post Pareto efficiency**: all Π_k are Pareto efficient.
- **Ex post pair efficiency**: all Π_k are pair-efficient.

Result: Conclusions of Theorems 1 and 2 still hold.

Proof Ideas (Theorem 1)

- Use cycle structure of TTC.
- Show all objects of a cycle must be allocated within cycle with probability 1.
- SD-top-strategy-proofness prevents deviation from TTC cycle selection.
- SD-efficiency pushes the rule to allocate exactly like TTC.

Key Lemma (Cycle Containment)

For a cycle C:

$$\phi_{j-1,x_j} + \phi_{j,x_j} = 1 \quad \forall j$$

- Ensures no probability mass leaves the cycle.
- Comes from SD-IR and bistochasticity.

Proof Ideas (Theorem 2)

- Pair-efficiency only restricts two agents at a time.
- Need FTT richness to rule out “mixing” assignments.
- Construct improvements pairwise to show any deviation from TTC violates pair-efficiency.

Example: Infinite SD-Pair Efficient Assignments

For $n > 2$, profile:

$$P_i : x_i \succ x_{i+1} \succ x_{i+2} \succ \ldots$$

Assignments:

$$A^b : A^b_{i,x_i} = b, \;\; A^b_{i,x_{i+1}} = 1 - b.$$

- $b < 1$: SD-Pareto dominated by A^1.
- All A^b: SD-pair efficient.

Conclusions

- TTC uniquely extends to probabilistic environments.
- SD-top-strategy-proofness sufficient to characterize TTC.
- FPT/FTT domain richness key for uniqueness.
- Bridges deterministic theory (Ma 1994) and probabilistic mechanism design.

Future Directions

- Probabilistic *initial* endowments.
- Extending beyond SD comparison: cardinal utilities?
- Randomized TTC-like rules under more general constraints.

Questions?