Homework 1: 5 problems: due September 25, 2006

1. A *surjection* is an onto function i.e. every element of the co-domain has a pre-image. Show that the number of surjections from $[s]$ to $[n]$ is

$$\sum_{k=0}^{s}(-1)^k\binom{n}{k}(n-k)^s.$$

Hence, conclude that the above expression is 0 iff $s < n$.

2. **(Bonferroni's inequalities.)** Let $A_1, A_2, \ldots, A_k \subseteq [n]$. For $S \subseteq [k]$, let $A_S \triangleq \cap_{s\in S} A_s$, $A_{\{\}} \triangleq [n]$. Then, for $0 \le r \le k$, show that

$$\begin{aligned} |\overline{A_1 \cup A_2 \cup \ldots \cup A_k}| &\geq \textstyle\sum_{S\subseteq[k]:|S|\le r}(-1)^{|S|}|A_S|, \quad r \text{ odd;} \\ |\overline{A_1 \cup A_2 \cup \ldots \cup A_k}| &\leq \textstyle\sum_{S\subseteq[k]:|S|\le r}(-1)^{|S|}|A_S|, \quad r \text{ even.} \end{aligned}$$

That is, show that the successive steps in the inclusion-exclusion identity alternately bound the final value from above and below.

3. Show by a counting argument that the number of k-element subsets of $\{0, 1, \ldots, n-1\}$ that contain no two consecutive numbers modulo n, is exactly

$$\frac{n}{n-k}\cdot\binom{n-k}{k}.$$

Hint: We want to colour k non-consecutive points red. Consider two cases, 1 is coloured red, and otherwise. In the first case, consider $n-k$ uncoloured points on a circle and place k red points in the spaces between them. Handle the second case similarly.

4. (a) **(Möbius inversion.)** Let $f, g : \{1, 2, \ldots, n\} \to \mathbb{C}$ be two functions. Suppose

$$f(n) = \sum_{d|n} g(d).$$

Let $\mu(\cdot)$ be the *Möbius function* on positive integers defined as follows: Let positive integer x have the prime factorisation $x = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$. Then,

$$\mu(x) = \begin{cases} 0 & \text{if } r_i \geq 2 \text{ for some } 1 \leq i \leq k \\ (-1)^k & \text{otherwise.} \end{cases}$$

Show that

$$g(n) = \sum_{d|n}\mu(n/d)f(d).$$

(b) Prove the following identity via a counting argument:

$$n = \sum_{d|n}\varphi(d).$$

Hence, derive a formula for $\varphi(\cdot)$ in terms of $\mu(\cdot)$.

5. (a) Consider the $2^n \times 2^n$ matrix $\mathcal{I}$ with rows and columns indexed by the subsets of $[n]$ defined as follows:

$$\mathcal{I}_{A,B} \triangleq \begin{cases} 1 & \text{if } A \subseteq B \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $\mathcal{I}$ is known as the *set inclusion* matrix. Find $\mathcal{I}^{-1}$ explicitly i.e. you should be able to write down $\mathcal{I}^{-1}_{A,B}$ for any $A, B \subseteq [n]$.

(b) Show that the expression for $\mathcal{I}^{-1}$ derived above gives rise to the general inclusion-exclusion formula.

(c) Using the first part of this exercise or otherwise, show that the *set disjointness* matrix $\mathcal{D}$ defined as:

$$\mathcal{D}_{A,B} \triangleq \begin{cases} 1 & \text{if } A \cap B = \{\} \\ 0 & \text{otherwise,} \end{cases}$$

is invertible. Find an explicit expression for $\mathcal{D}^{-1}$.