Homewok 3 (9 problems, due: 20 November 2006)

C3.1 Let $(S_1, S_2, \ldots, S_m)$ and $(T_1, T_2, \ldots, T_m)$ be two sequences of finite sets with the following properties: (a) $S_i \cap T_i = \emptyset$ for $i = 1, 2, \ldots, n$, and (b) $S_i \cap T_j \neq \emptyset$ whenever $i \neq j$. Using an averaging argument similar to the one used in class to prove the LYM inequality, conclude the following result of Bollobás:

$$\sum_{i=1}^{m} \binom{|S_i| + |T_i|}{|S_i|}^{-1} \leq 1.$$

Derive the LYM inequality from the above result.

C3.2 Let $G = (V_1, V_2, E)$ be a bipartitite graph. Suppose there is a non-negative integer d such that for all $S \subseteq V_1$ we have $|N(S)| \geq |S| - d$. Show that G has a matching of size at least $|V_1| - d$.

C3.3 For a graph G, a vertex cover is a set of vertices of G such that every edge has at least one end point in the set. Show that for a bipartite graph the size of the largest matching and the size of the smallest vertex cover are equal. What about non-bipartite graphs?

C3.4 In a graph, the degree of a vertex is the number of edges incident on it. A graph is r-regular if every vertex has degree exactly r. Show that every r-regular bipartite graph is a union of r perfect matchings. Conclude that any bipartite graph with maximum degree r is the union of at most r matchings. Observe that the theorem holds even if multiple edges are allowed.

C3.5 Consider the complete bipartite graph $K_{n,n}$ ($|V_1|, |V_2| = n, E = V_1 \times V_2$). Let $w : E \to \mathbb{R}$. Let S_w denote the set of all perfext matchings of $K_{n,n}$ with maximum weight (with respect to the given function w). Show the following theorem of Sundar Vishwanathan:

for every $w : E \to \mathbb{R}$ there is a $w' : E \to \{0, 1\}$ such that $S_w = S_{w'}$.

Hint: Use C3.4.

C3.6 Let M be an $n \times n$ with positive real entries. M is said to be doubly stochastic if the sum of each row and each column is 1. Show that every doubly stochastic matrix is a convex combination of permutation matrices, that is

$$M = \sum_{\pi} w_\pi M_\pi,$$

where w_π are non-negative real weights that add up to 1, and M_π is the permutation matrix corresponding to π. (This is called the Birkhoff-von Neumann theorem.)

C3.7 For a finite poset P, let $\kappa(P)$ be the length of the longest chain in P and $\chi(P)$ be the minimum number of antichains needed to cover P. Prove the 'dual of Dilworth's theorem': $\kappa(P) = \chi(P)$. Conclude that $\chi(P) \cdot \alpha(P) \geq |P|$ (where $\alpha(P)$ is the size of the largest antichain in P). Use this to prove the Erdős-Szekeres theorem: every sequence of $n^2 + 1$ integers has a monotone subsequence of length $n + 1$.

C3.8 How many different ways are there to colour the faces of a regular tetrahedron using c colours? The group of rotations of the regular tetrahedron has 12 elements. Two colourings should be reagarded as the same if one can be obtained from the other by rotation.

C3.9 How many different patterns can be formed by colouring the faces of a cube so that there are three red faces, two white faces and one blue face? Two patters should be regarded as the same if one pattern can be obtained from the other by rotation.

Please send me email (jaikumar@tifr.res.in) when you spot errors. – Jaikumar