Homework 1 (7 problems) due August 25, 2008

1. Let p be prime. Show that for $k = 2, 3, \dots, p-1$, $\binom{p}{k}$ is divisible by p.

2. In how many ways can one write $k \geq 2n$ as a sum of n integers (the order is important), each at least 2?

3. In how many ways can one pick a k element subset out of $[n]$ ($n \geq 2k-1$) without consecutive elements?

4. Fix $1 \leq k \leq n$. How many integer sequences $1 \leq a_1 < a_2 < \cdots < a_k \leq n$ satisfy

$$a_i \equiv i \pmod 2?$$

5. How big is the largest family of subsets of $[n]$ such that every pair of sets has a non-empty intersection?

6. Let n be a positive integer and let $k_1, k_2, \dots, k_n$ be non-negative integers such that $k_1 + 2k_2 + \cdots + nk_n = n$. How many partitions of $[n]$ are there with k_i i-element blocks ($i = 1, 2, \dots, n,$)?

7. How many compositions does the integer $n \geq 1$ have? Show that the number of composition of n with an even number of even parts is exactly 2^{n-2}.