Lecture 21: Copositive Programming

(the size of a maximum independent set)

A Copositive Program for the Independence Number:

A copositive program looks just like an SDP except that the constraint $X \succcurlyeq 0$ is replaced by $X \in COP_n$ or $X \in POS_n$.

Let us fix a graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ for $n \geq 1$. Recall that $\overline{E}$ denotes the edge set of the complementary graph $\overline{G}$. So $\overline{E} = \{(i,j) : i \neq j \text{ and } (i,j) \notin E\}$.

Theorem. The copositive program

$$\begin{aligned} \min\ & t \\ \text{s.t.}\ & y_{ij} = -1 \text{ for } (i,j) \in \overline{E} \\ & y_{ii} = t-1 \ \forall i = 1, \ldots, n \\ & Y \in COP_n \end{aligned}$$

has value $\alpha(G)$.

the independence number of G. For convenience, we'll denote it by α henceforth.

Note that replacing $Y \in COP_n$ with $Y \succcurlyeq 0$ gives us the SDP for $\theta(G)$ (the theta function of G). Thus we know that the value of the above copositive program is at most $\theta(G)$. We will now show that it is precisely $\alpha(G)$. Recall that $\theta(G) \geq \alpha(G)$.

We will prove the above theorem in two parts.

Part 1: This will show that the value of the above copositive program $\geq \alpha$. The dual of the above copositive program is the following:

$$\begin{aligned} \max\ & J_n \bullet X \\ \text{s.t.}\ & Tr(X) = 1 \\ & x_{ij} = 0 \text{ for } (i,j) \in \overline{E} \\ & X \in POS_n. \end{aligned}$$

(J_n is the $n \times n$ matrix with all entries equal to 1.)

The proof that this is the dual program is exactly the same as the proof of one of the problems in Assignment 4. So we'll skip this proof.

Claim. The above dual program is feasible and its value $\geq \alpha$.

Once we prove the above claim, we can use weak duality to conclude that the value of the primal program $\geq \alpha$.

Proof of the claim. Let $x \in \mathbb{R}^n$ be the characteristic vector of a maximum independent set $S \subseteq V$, i.e. $x_i = 1$ if $i \in S$ and $x_i = 0$ otherwise.

Consider the matrix $X = \left(\frac{x}{\sqrt{\alpha}}\right)\left(\frac{x}{\sqrt{\alpha}}\right)^T \in POS_n$.

We have $Tr(X) = \frac{1}{\alpha} \cdot \alpha = 1$

since $x_i \cdot x_i = 1 \iff i \in S$.

We also have $\underbrace{J_n \cdot X}_{\text{This is } \sum_{i,j} x_{ij}.} = \frac{1}{\alpha} \sum_{i,j} x_i x_j = \frac{\alpha^2}{\alpha} = \alpha$

(since $\sum_{i,j} x_i x_j = \sum_{i,j \in S} 1 = \alpha^2$)

Thus there is a dual feasible solution that attains the value α. Hence the value of the primal program $\geq \alpha$.

Part 2: This will show that the value of the primal program $\leq \alpha$. Let us rewrite our primal program a little differently.

Let $Y \in SYM_n$. The constraints

$$y_{ij} = -1 \quad \text{if } (i,j) \in \overline{E}$$

$$y_{ii} = t-1 \quad \text{for all } i = 1, \ldots, n$$

can be equivalently expressed as

$$Y = tI_n + Z - J_n$$

where Z is an $n \times n$ matrix with $z_{ij} = 0$ whenever $(i,j) \notin E$. That is, $z_{ij} \neq 0 \Rightarrow (i,j) \in E$.

Let A be the adjacency matrix of G. If $Y = tI_n + Z - J_n$ is a feasible solution of our primal program and if z is the largest entry of Z, then $Y' = tI_n + zA - J_n$ is also a feasible solution of the primal program. This is because $Z' = zA$ satisfies $z'_{ij} = 0$ for $(i,j) \notin E$ and $Y' = \underbrace{Y}_{\text{in } COP_n} + \underbrace{zA - Z}_{\substack{\text{a non-neg. symmetric matrix} \\ \text{(any such matrix is in } COP_n)}}$ is in COP_n.

Moreover, the value of the primal program is the same for Y and Y'. Thus we may assume without loss of generality that Y in our primal program is a matrix of the form $tI_n + zA - J_n$. Thus we have the following lemma.

<u>Lemma.</u> The copositive program

$$\begin{aligned} \min\ & t \\ \text{s.t.}\ & tI_n + zA - J_n \in COP_n \\ & t, z \in \mathbb{R} \end{aligned}$$

is feasible and has the same value as our primal program; in particular, its value $\geq \alpha$.

We need to show its value $\leq \alpha$. For that, we'll use the following theorem.

<u>The Motzkin-Straus Theorem.</u> For any graph G,

$$\frac{1}{\alpha} = \min \left\{ x^T (A + I_n) x : x \geq 0 \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}.$$

Let us take the Motzkin - Straus theorem for granted and complete the proof of Part 2. Then we'll prove the Motzkin - Straus theorem.

Completing the proof of Part 2. We'll construct a feasible solution of value α for the modified primal program. If $x \geq 0$ and $\sum_{i=1}^{n} x_i = 1$, the Motzkin - Straus theorem implies:

$$x^T(\alpha(A + I_n))\, x \geq 1 = x^T J_n x.$$

This means that

$$x^T(\alpha I_n + \alpha A - J_n)\, x \geq 0 \text{ if } x \geq 0, \sum_{i=1}^{n} x_i = 1.$$

Note that if any non-zero $x \geq 0$ satisfies $x^T(\alpha I_n + \alpha A - J_n)\, x \geq 0$ then we can rescale x so that $\sum_{i=1}^{n} x_i = 1$.

Thus we have $x^T(\alpha I_n + \alpha A - J_n)\, x \geq 0$ for all $x \geq 0$.

Hence $(\alpha I_n + \alpha A - J_n)$ is in COP_n and is a feasible solution to the rewritten copositive program with $t = z = \alpha$.

Thus the optimal value of our primal program $\leq \alpha$. ∎

Proof of the Motzkin - Straus theorem.

Let $f(x) = x^T(A + I_n)\, x$ and let m^* denote the minimum of $f(x)$ over the unit simplex.

The inequality $m^* \leq 1/\alpha$ is easy to show.
Let $x = \frac{1}{\alpha} \cdot I_G$ where I_G is the characteristic vector of a maximum independent set in G.

$$f(x) = \frac{1}{\alpha^2}\left(\underbrace{\sum_{i \in I_G} x_i^2}_{\alpha} + \underbrace{2 \sum_{(i,j)\in E} x_i x_j}_{0} \right) = \frac{1}{\alpha}.$$

Thus $m^* \leq 1/\alpha$.

To show $m^* \geq 1/\alpha$, let us start from a minimizer x^* of f over the unit simplex and transform it into another minimizer y^* such that $J = \{ i : y_i^* > 0 \}$ is an independent set. As done above, we get

$$m^* = f(x^*) = f(y^*) = \underbrace{2 \sum_{(i,j)\in E} y_i^* y_j^*}_{=0} + \sum_{i \in J} (y_i^*)^2.$$

<u>Claim.</u> $\sum_{i\in J} (y_i^*)^2$ is minimized subject to $\sum_{i \in J} y_i^* = 1$ when all the y_i^* values are equal to $\frac{1}{|J|}$.

We'll take this claim for granted and finish the proof of the Motzkin-Straus theorem. Then we'll prove this claim.

The above claim implies that $m^* = |J| \cdot \frac{1}{|J|^2}$

$$= \frac{1}{|J|} \geq \frac{1}{\alpha}.$$

It remains to construct y^*, given a minimizer x^*.
We define $F = \{ (i,j) \in E : x_i^* x_j^* > 0 \}$.

If $F = \phi$ then we set $y^* = x^*$ and are done. Otherwise we choose $(i,j) \in F$ and define z as:

$$z_k = \begin{cases} x_k^* + \varepsilon & \text{if } k = i \\ x_k^* - \varepsilon & \text{if } k = j \\ x_k^* & \text{otherwise,} \end{cases}$$

where ε is some real number (positive or negative).

It is easy to check that $f(z) = f(x^*) + l(\varepsilon)$, where l is a linear function in ε – note that the ε^2 term gets cancelled.

We have $x_i^* > 0$, $x_j^* > 0$ (since $x_i^* x_j^* > 0$). Moreover, $x_i^* < 1$, $x_j^* < 1$ (since $x_i^* + x_j^* \leq 1$). Thus by choosing ε appropriately, z is in the unit simplex.

Observe that $l(\varepsilon) = 0$ — otherwise by choosing either $\varepsilon > 0$ or $\varepsilon < 0$, we will get $f(z) < f(x^*)$, a contradiction to x^* being a minimizer of f.

Hence $f(z) = f(x^*)$ and by taking $\varepsilon = -x_i^*$, we can make $z_i = 0$. Replacing x^* with z yields a new minimizer for which the set F of nonzero edges has become strictly smaller. We can thus repeat the above process till $F = \phi$. Thus we obtain y^*. ▨

(The earlier claim will be proved in the notes of Lecture 22.)