# Combinatorial Optimization: Assignment 1

*Due date: Feb 21, 2025*

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1. An edge cover of a graph $G = (V, E)$ is a subset $R$ of $E$ such that every vertex of $V$ is incident to at least one edge in $R$. Let $G$ be a bipartite graph with no isolated vertex. Show that the size of a minimum edge cover $R^*$ of $G$ is equal to $|V|$ minus the size of the maximum matching $M^*$ of $G$. Give an efficient algorithm to find a minimum edge cover of $G$. Does the equation $|R^*| = |V| - |M^*|$ also hold for non-bipartite graphs?

2. Consider a bipartite graph $G = (A \cup B, E)$. Assume that for some vertex sets $A_1 \subseteq A$ and $B_1 \subseteq B$, there exists a matching $M_A$ that matches all vertices in $A_1$ and a matching $M_B$ that matches all vertices in $B_1$ . Prove that there exists a matching that matches all vertices in $A_1 \cup B_1$.

3. Consider a bipartite graph $G = (A \cup B, E)$ where each vertex has degree $k$. Prove that such a graph always has a perfect matching in two different ways: (i) by using Hall's theorem and (ii) by using the linear programming formulation that inspired the primal-dual algorithm for the assignment problem.

   Using the above fact (that a perfect matching always exists in a $k$-regular bipartite graph), show that the edges of a $k$-regular bipartite graph $G$ can be partitioned into $k$ matchings. That is, the edge chromatic number of $G$ is precisely $k$.

4. Prove that the running time of the primal-dual algorithm for the assignment problem is $O(n^3)$, where $n$ is the number of vertices on each side of the bipartite graph. Recall that our input graph here was $K_{n,n}$.

5. Show that the running time of the maximum matching algorithm in a bipartite graph $G = (A \cup B, E)$ can be improved to $O(m\sqrt{n})$, where $|E| = m$ and $|A \cup B| = n$. In the $O(mn)$ algorithm, we were building a Hungarian forest $F_A$ with unmatched vertices of $A$ at level 0 and searching for an augmenting path.

   Now find a maximal number of vertex-disjoint shortest length augmenting paths in $F_A$ in each iteration and augment $M$ along all of them. Show that one iteration can be implemented in $O(m)$ time (we assume $G$ is connected, so $m \geq n - 1$).

   Show that the shortest length augmenting path with respect to $M$ grows by at least 2 in each iteration. So in $\lfloor \frac{\sqrt{n}}{2} \rfloor$ iterations, the shortest length augmenting path with respect to $M$ becomes at least $\sqrt{n} - 1$. Thus conclude that there are at most $2\sqrt{n}$ iterations.

6. Our algorithm for the assignment problem allows us to conclude that there is always an *integral* solution to the linear program that inspired the primal-dual algorithm. Reprove this result in the following way.

   Take a possibly non-integral optimum solution $x^*$ . If there are many optimum solutions, then take one with as few non-integral values $x^*_{ij}$ as possible. Show that if some coordinate of $x^*$ is non-integral, then there exists a cycle $C$ with all edges $e = (i, j) \in C$ having a non-integral value $x^*_{ij}$. Now show how to derive another optimum solution with fewer non-integral values, leading to a contradiction.

7. *Dulmage-Mendelsohn decomposition.* Let $M$ be a maximum matching in a bipartite graph $G = (A \cup B, E)$. Partition $A \cup B$ into sets $\mathcal{E}_M, \mathcal{O}_M$, and $\mathcal{U}_M$ as follows:

   - let $\mathcal{E}_M$ be the set of vertices reachable by even length alternating paths with respect to $M$ from a vertex unmatched in $M$.
   - let $\mathcal{O}_M$ be the set of vertices reachable by odd length alternating paths with respect to $M$ from a vertex unmatched in $M$.
   - let $\mathcal{U}_M = (A \cup B) - \mathcal{E}_M \cup \mathcal{O}_M$.

   Show that $\mathcal{E}_M \cap \mathcal{O}_M = \emptyset$. Let $\mathcal{E}_{M'}, \mathcal{O}_{M'}$, and $\mathcal{U}_{M'}$ be sets analogously defined with respect to another maximum matching $M'$ in $G = (A \cup B, E)$. Prove that $\mathcal{E}_M = \mathcal{E}_{M'}$, $\mathcal{O}_M = \mathcal{O}_{M'}$, and $\mathcal{U}_M = \mathcal{U}_{M'}$.

8. Let $G$ be any bipartite graph and let $\mathcal{E}$ denote the set $\mathcal{E}_M$ of the previous exercise (where $M$ is any maximum matching in $G$) and let $\mathcal{O}$ denote the set $\mathcal{O}_M$ of the previous exercise. Let $C$ be any minimum vertex cover of $G$. Show that $\mathcal{O} \subseteq C$ and $C \cap \mathcal{E} = \emptyset$.

9. Given a graph $G = (V, E)$, an *inessential* vertex is a vertex $u$ such that there exists a maximum matching of $G$ not matching $u$. Let $\mathcal{E}$ be the set of inessential vertices in $G$. Let $\mathcal{O}$ denote the set of vertices not in $\mathcal{E}$ but adjacent to at least one vertex in $\mathcal{E}$. Let $\mathcal{U} = V - (\mathcal{E} \cup \mathcal{O})$. The triple $\langle \mathcal{E}, \mathcal{O}, \mathcal{U} \rangle$ is called the *Edmonds-Gallai* partition of V . Prove that the size of a maximum matching in $G$ equals $\frac{1}{2}(|V| + |\mathcal{O}| - o(G - \mathcal{O}))$, where $o(G - \mathcal{O})$ is the number of connected components of odd size of $G - \mathcal{O}$.

10. The previous exercise shows that $\mathcal{O}$ is a minimizer $U$ in the Tutte-Berge theorem. Can there be several such minimizers $U$? Either give an example with several sets $U$ achieving the minimum, or prove that the set $U$ has to be the set $\mathcal{O}$ (which means the set $U$ is unique).