Combinatorial Optimization: Assignment 2

Due date: March 31, 2025

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1. Show that the dual of the dual LP is the primal LP.

2. Consider the min-cost vertex cover problem in a graph $G = (V, E)$ where every $v \in V$ has a cost associated with it. The cost of a vertex cover S is the sum of costs of vertices in S. Show that the following algorithm computes a vertex cover whose cost is at most twice the cost of a min-cost vertex cover.

   (a) Solve the LP-relaxation of the integer program for min-cost vertex cover.

   (b) Let x^* be the optimal solution of the above LP.

   (c) Return C where $C = \{u : x^*_u \geq \frac{1}{2}\}$.

3. Consider the following rounding rule for the above problem: $C' = \{u : x^*_u > 0\}$. Show that C' is also a vertex cover whose cost is at most twice the cost of a min-cost vertex cover.

4. Show that the maximal (inclusion-wise) non-trivial faces of a non-empty polyhedron P are its facets.

5. Show that if $\mathsf{rank}(A) < n$ then $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ has no vertices.

6. Suppose $P = \{x \in \mathbb{R}^n : Ax \leq b,\, Cx \leq d\}$. Show that the set of vertices of $Q = \{x \in \mathbb{R}^n : Ax \leq b,\, Cx = d\}$ is a subset of the set of vertices of P.

7. Given two extreme points a and b of a polyhedron P, we say that they are adjacent if the line segment between them forms an edge (i.e. a face of dimension 1) of the polyhedron P. This can be rephrased by saying that a and b are adjacent on P if and only if there exists a cost function c such that a and b are the only two extreme points of P minimizing $c^T x$ over P.

   Consider the polytope P defined as the convex hull of all perfect matchings in a (not necessarily bipartite) graph G. Give a necessary and sufficient condition for two matchings M_1 and M_2 to be adjacent on this polyhedron.

   [Hint: Consider $M_1 \oplus M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1)$ and prove that your condition is necessary and sufficient.]

8. Show that two vertices u and v of a polyhedron P are adjacent if and only there is a unique way to express their midpoint $\frac{1}{2}(u + v)$ as a convex combination of vertices of P.

9. Consider the set $X = \{(\pi(1), \pi(2), \ldots, \pi(n)) : \pi \text{ is a permutation of } \{1, 2, \ldots, n\}\}$. Show that the dimension of $\mathsf{conv}(X)$ is $n - 1$. To show $\mathsf{dim}(\mathsf{conv}(X)) \geq n - 1$, show n affinely independent permutations π (and prove they are affinely independent).

10. A independent set S in a graph $G = (V, E)$ is a set of vertices such that there are no edges between any two vertices in S. If we let P denote the convex hull of all (incidence vectors of) independent sets of $G = (V, E)$, it is clear that $x_i + x_j \leq 1$ for any edge $(i, j) \in E$ is a valid inequality for P.

    (a) Give a graph G for which P is *not* equal to

    $$\begin{aligned} \{x \in \mathbb{R}^{|V|} : x_i + x_j &\leq 1 \;\; \forall\, (i,j) \in E\} \\ x_i &\geq 0 \;\; \forall\, i \in V\}. \end{aligned}$$

    (b) If the graph G is bipartite then show that P equals

    $$\begin{aligned} \{x \in \mathbb{R}^{|V|} : x_i + x_j &\leq 1 \;\; \forall\, (i,j) \in E\} \\ x_i &\geq 0 \;\; \forall\, i \in V\}. \end{aligned}$$