Combinatorial Optimization: Assignment sheet 3

Due date: May 13, 2025

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1. Let $M = (E, \mathcal{I})$ be a matroid. Let k be any integer and define
$$\mathcal{I}_k = \{X \in \mathcal{I}_k : |X| \leq k\}.$$
Show that $M_k = (E, \mathcal{I}_k)$ is a matroid. (This is known as a truncated matroid.)

2. For any set $S \subseteq E$, show that $\mathsf{span}(\mathsf{span}(S)) = \mathsf{span}(S)$.

3. Consider the following collection of sets associated with matchings in a general (not necessarily bipartite) graph $G = (V, E)$. Let the ground set be the vertex set V. Let $\mathcal{I} = \{S \subseteq V : S$ is matched in one or more matchings in $G\}$. In this definition, the matching does not need to cover precisely S; other vertices can be matched as well.

   Prove that $(V, \mathcal{I})$ is a matroid. This is called the *matching matroid* of G.

4. A family $\mathcal{F}$ of sets is said to be *laminar* if for any 2 sets $A, B \in \mathcal{F}$, we have either (i) $A \subseteq B$, (ii) $B \subseteq A$, or (iii) $A \cap B = \emptyset$. Suppose we have a laminar family of subsets of E and an integer $k(A)$ for every set $A \in \mathcal{F}$. Show that $(E, \mathcal{I})$ defines a *laminar* matroid where
$$\mathcal{I} = \{X \subseteq E : |X \cap A| \leq k(A) \ \forall A \in \mathcal{F}\}.$$

5. What is the rank function of a laminar matroid (defined in Exercise 4)?

6. Deduce König-Egerváry theorem about the maximum size of a matching in a bipartite graph from the min-max relation for the largest common maximum independent set in 2 matroids.

7. Let $G = (V, E)$ be an undirected graph. For any $X \subseteq V$, let $\delta(X)$ denote the size of the cut $(X, V \setminus X)$, i.e., the number of edges with one endpoint in X and another endpoint in $V \setminus X$. If A and B are any two subsets of V, show that $\delta(A) + \delta(B) \geq \delta(A \cap B) + \delta(A \cup B)$.

8. Prove that following statement on min cuts in G: if $(S, V \setminus S)$ is a minimum s-t cut in G and $u, v \in S$, then there exists a minimum u-v cut $(U, V \setminus U)$ such that $U \subset S$ or $V \setminus U \subset S$.

9. Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be two matroids on the ground set E. Consider the family F of sets that are independent in both matroids, i.e., $F = \mathcal{I}_1 \cap \mathcal{I}_2$. Let X be a maximum cardinality set in F and let Y be any *maximal* set in F. Show that $|Y| \geq |X|/2$.

10. Assume that G has 2 edge-disjoint spanning trees. Give a winning strategy for player 2 in the spanning tree game.