Lecture 23 Linear Programming

We will now complete the proof of strong duality theorem.

What is left to show: our assignment of values to y_i's and λ_j's – we showed that this makes the value of Game 2 $= \sum_j c_j x_j^*$. We still have to show all y_i's and λ_j's are ≥ 0.

We used β_t-values in this assignment of values to y_i's and λ_j's. Recall that the rows of $X_=$ are $\underbrace{\vec{N_1}, \ldots, \vec{N_k}}_{\text{normals of hyperplanes whose intersection is } (x_1^*, \ldots, x_k^*)}$.

Since $\vec{N_1}, \ldots, \vec{N_k}$ are linearly independent, we said $(c_1, \ldots, c_k)$ can be expressed as a linear combination of $\vec{N_1}, \ldots, \vec{N_k}$. Thus

$$[c_1 \cdots c_k] = [\beta_1 \cdots \beta_k] \begin{bmatrix} \vec{N_1} \\ \vdots \\ \vec{N_k} \end{bmatrix}$$

In fact, something stronger is true.

Claim. $(c_1, \ldots, c_k)$ is a conic combination of $\vec{N_1}, \ldots, \vec{N_k}$.

That is, $[c_1 \ldots c_k]$ can be expressed as a non-negative linear combination of $\vec{N_1}, \ldots, \vec{N_k}$. Thus all y_i's and λ_j's are ≥ 0.

Why is the above claim true? We will prove this using a classical lemma from optimization. This is called Farkas' lemma.

Farkas' lemma. Exactly one of the following two statements is true:

(1) $(c_1, \ldots, c_k)$ is a conic combination of $\vec{N_1}, \ldots, \vec{N_k}$.

(2) $\exists$ a hyperplane $h_1 x_1 + \cdots + h_k x_k = 0$ such that

$$[c_1 \ldots c_k] \begin{bmatrix} h_1 \\ \vdots \\ h_k \end{bmatrix} < 0 \quad \text{and} \quad \begin{bmatrix} \vec{N_1} \\ \vdots \\ \vec{N_k} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_k \end{bmatrix} \geq 0.$$

Farkas' lemma essentially says if $(c_1, \dots, c_k)$ is not in the cone of $\vec{N_1}, \dots, \vec{N_k}$ then there is a separating hyperplane. Let us take this for granted and complete our proof.

Assuming Farkas' lemma, if $(c_1, \dots, c_k)$ is not a conic combination of $\vec{N_1}, \dots, \vec{N_k}$ then there exists such a hyperplane $h_1 x_1 + \dots + h_k x_k = 0$.

Consider $(x_1^* + \varepsilon h_1, \dots, x_k^* + \varepsilon h_k)$ for a suitable $\varepsilon > 0$.

$$\sum_j c_j (x_j^* + \varepsilon h_j) = \sum_j c_j x_j^* + \varepsilon \sum_j c_j h_j$$

$$< \sum_j c_j x_j^* \quad \left(\text{since } [c_1 \dots c_k] \begin{bmatrix} h_1 \\ \vdots \\ h_k \end{bmatrix} < 0\right)$$

this will contradict the optimality of $(x_1^*, \dots, x_k^*)$ if we show $(x_1^* + \varepsilon h_1, \dots, x_k^* + \varepsilon h_k)$ to be primal feasible.

That is what we will show now. We will show there is a suitably small ε such that the above point is primal feasible.

$$\begin{bmatrix} X_= \\ \\ X_> \end{bmatrix} \begin{bmatrix} x_1^* + \varepsilon h_1 \\ \vdots \\ x_k^* + \varepsilon h_k \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_t \\ 0 \\ \vdots \\ 0 \\ \left.\right\} \text{some slack} \end{bmatrix} + \begin{bmatrix} \ge 0 \\ \vdots \\ \ge 0 \\ * \\ \vdots \\ * \end{bmatrix} \begin{matrix} \left.\right\} k \text{ of them} \\ \\ \\ \end{matrix}$$

The $*$ values in the rightmost column are the dot products of $\vec{N}_{k+1}, \dots, \vec{N}_{n+k}$ with $(h_1, \dots, h_k)$ mult. with ε — these values can be negative. However $(x_1^*, \dots, x_k^*)$ satisfies these constraints with slack.

[illegible] us now write the dual [illegible]. The variables are y_u for $u \in V$ and z_e for $e \in E$.

So if we choose ε small enough, then the slack compensates for this negative value and so $(x_1^* + \varepsilon h_1, \ldots, x_k^* + \varepsilon h_k)$ is primal feasible.

Since the point $(x_1^* + \varepsilon h_1, \ldots, x_k^* + \varepsilon h_k)$ achieves a better objective function value than $(x_1^*, \ldots, x_k^*)$ - which contradicts the optimality of $(x_1^*, \ldots, x_k^*)$ - we can conclude that option (1) of Farkas' lemma holds. Thus $(c_1, \ldots, c_k)$ is a conic combination of $\vec{N_1}, \ldots, \vec{N_k}$. ▨

Let us use LP-duality to show a new proof of Max-flow min-cut theorem.

Max-flow as an LP

Our variables will be x_e for $e \in E$.
Let us also add a new arc from t to s so that our flow becomes a circulation. That is, flow conservation will be obeyed at all vertices, incl. s and t. Set $c_{(t,s)} = \infty$.

Primal LP

$$\max \; x_{ts}$$

subject to

$$\sum_{e:\, e \text{ entering } u} x_e - \sum_{e:\, e \text{ leaving } u} x_e \le 0 \quad \forall u \in V$$

$$x_e \le c_e \quad \forall e \in E$$

$$x_e \geq 0 \quad \forall e \in E$$

Observe that setting $\sum_{e:\, e \text{ ent. } u} x_e - \sum_{e:\, e \text{ leaving } u} x_e \le 0$ will imply $\sum_{e:\, e \text{ ent. } u} x_e - \sum_{e:\, e \text{ leaving } u} x_e = 0$. (Why?)

Let us now write the dual LP. The variables are y_u for $u \in V$ and z_e for $e \in E$.

Dual LP

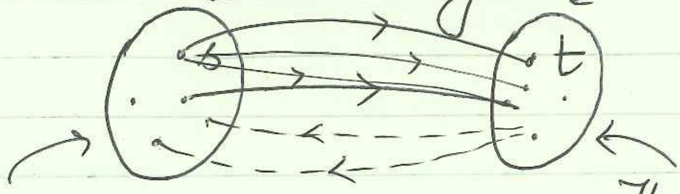
$$\min \sum_e c_e z_e$$

subject to

$$y_s - y_t \geq 1$$
$$y_v - y_u + z_{(u,v)} \geq 0 \quad \forall (u,v) \in E$$
$$y_u \geq 0, \quad z_e \geq 0 \quad \forall u \in V \ \forall e \in E$$

In order to understand the above LP better, let us add another restriction to it: constrain z_e, y_u for all edges e and vertices u to be 0/1 values. So $y_s = 1$ and $y_t = 0$.

For any edge (u,v) with $y_u = 1$ and $y_v = 0$, we are forced to set $z_{(u,v)} = 1$. Since we want to ~~maximize~~ minimize $\sum_e c_e z_e$, we will set the remaining z_e values to 0.



These are vertices whose y-value is 1

These are vertices whose y-value is 0.

So the goal of the integer program is to come up with a partition of V into S and $V-S$ such that $s \in S$, $t \in V-S$ and the sum of capacities of edges in $S \times (V-S)$ is minimized. Thus the integer program computes an s-t min-cut.

What about the dual LP? It computes a min fractional s-t cut. It assigns distance labels to arcs such that on any s-t path the sum of distance labels is ≥ 1.

Let $\underset{u_0}{\overset{}{s}} \text{---} u_1 \text{---} \cdots u_{i-1} \overset{d_i}{\text{---}} u_i \text{---} \cdots \underset{u_k}{t}$ be any s-t path.

$d_i \geq u_{i-1} - u_i$ for each i.

So $\sum_{i=1}^{k} d_i \geq u_0 - u_k \geq 1$.

The constraint matrix of this LP has a special property - it is totally unimodular.

This means any square submatrix of the constraint matrix has determinant in $\{0, \pm 1\}$. This implies the feasible region is an integral polytope, i.e, the vertices or extreme points of this polytope have integral coordinates. This means the objective function value and the optimal solution of the dual LP and the integer program are the same. That is, adding the integrality constraints does not change the optimal value. Thus the dual LP computes an s-t min-cut. Since the primal LP computes a max flow, LP-duality implies max-flow = min-cut.

Total Unimodularity

Let us look at the constraint matrix of the max-flow LP. This matrix is as follows:

$$
\begin{array}{c}
 \\ u_1 \\ \vdots \\ \vdots \\ \vdots \\ u_n \\ e_1 \\ \vdots \\ e_m
\end{array}
\begin{array}{c}
\begin{array}{ccc} e_1 & \dots & e_m \end{array} \\
\left[\begin{array}{ccc}
1 & & -1 \\
0 & & 0 \\
0 & & 1 \\
-1 & & 0 \\
0 & & 0 \\
0 & & 0 \\
\hline
 & I & \\
 & & \\
 & &
\end{array}\right]
\end{array}
$$

$\rightarrow$ every column in the upper part of this matrix has one +1 and one -1 in it, all other entries are 0.

$\rightarrow$ $\boxed{\text{Identity matrix}}$

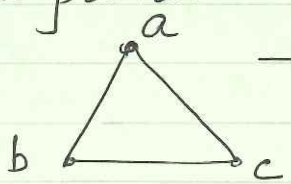
Take any $s \times s$ submatrix of the above matrix. To compute its determinant, keep expanding along any row/column that has a single 1 or a single -1 till there is an all-0s row/column or every row/column has ≥ 2 non-zero entries. (otherwise we have a 1x1 matrix and determinant = val. of this entry)

Claim. The determinant of such a matrix (as underlined above) is 0. [Please do this as an exercise.]

Let us now prove a result in graph theory. Before we state this result, let us define a vertex cover: this is a subset of the vertex set such that every edge has at least 1 endpoint in this set.

Kőnig - Egerváry theorem. In any bipartite graph, size of a maximum matching = size of a minimum vertex cover.

It is important that the graph is bipartite. Consider



$\rightarrow$ here size of maximum matching is 1 and the size of minimum v.c. is 2.

In any graph, it is easy to see that size of maximum matching $\leq$ size of minimum vertex cover.

What Kőnig-Egerváry theorem says that the above constraint is tight for bipartite graphs. We will prove this using LP-duality.

Consider the following LP in a bipartite graph $G = (A \cup B, E)$.

$$\max \sum_{e \in E} x_e$$

s.t.

$$\sum_{e \in \delta(u)} x_e \leq 1 \quad \forall u \in A \cup B.$$

$$x_e \geq 0 \quad \forall e \in E$$

Here $\delta(u)$ = set of edges incident to vertex u.

If we add the constraints $x_e \in \mathbb{Z}$ then this computes a maximum matching in G. The claim is the above constraint matrix is again totally unimodular. So adding integrality constraints is redundant since the optimal solution is integral.