Min-Cost Popular Matchings in a Hospitals/Residents Instance with Complete Preferences*

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Abstract. We consider a matching problem in a hospitals/residents instance G, i.e., a many-to-one matching instance, where every vertex has a strict ranking of its neighbors and hospitals have capacities. A matching M is said to be *popular* if M does not lose an election against any matching where vertices cast votes for one matching versus another. There are efficient algorithms to find popular matchings in G but it is NP-hard to find a *min-cost* popular matching when edges have costs. When preferences are complete, there is a polynomial-time algorithm for this problem in the one-to-one setting, i.e., when every hospital has capacity 1. Interestingly, the set of popular matchings in a many-to-one instance can be richer than the set of popular matchings in the corresponding one-to-one instance (obtained by cloning vertices) and no polynomial-time algorithm is currently known for the min-cost popular matching problem with complete preferences in the many-to-one setting. We show a polynomial-time algorithm for this problem. Our algorithm includes a subroutine for computing a min-cost popular *perfect* matching—this subroutine also works for instances with incomplete preferences.

1 Introduction

We consider a matching problem in a bipartite graph $G = (R \cup H, E)$ where R is a set of residents and H is a set of hospitals. Every vertex $v \in R \cup H$ has a strict ranking of its neighbors. Every resident r seeks to be matched to a hospital while every hospital h has an integral capacity $\mathsf{cap}(h) \geq 1$ and seeks to be matched to $\mathsf{cap}(h)$ many residents. Such a graph G is called a hospitals/residents instance and this is a very well-studied model in two-sided matching markets. This model includes several real-world settings such as matching students to schools and colleges [1,3] and residents to hospitals [7,34].

Definition 1. *A matching M in $G = (R \cup H, E)$ is a subset of E such that $|M(r)| \leq 1$ for each $r \in R$ and $|M(h)| \leq \mathsf{cap}(h)$ for each $h \in H$, where $M(v) = \{u : (u, v) \in M\}$ for any $v \in R \cup H$.*

There is a function $\mathsf{cost} : E \to \mathbb{R}$, so $\mathsf{cost}(r, h)$ is the cost of matching resident r to hospital h. The cost of a matching M is the sum of costs of edges in M, i.e., $\mathsf{cost}(M) = \sum_{e \in M} \mathsf{cost}(e)$. Our goal is to define "best matchings" with respect to vertex preferences and design a polynomial-time algorithm to find a min-cost best matching.

An efficient algorithm for the min-cost best matching problem will enable us to efficiently solve a variety of best matching problems such as finding an *egalitarian* best matching or finding a best matching that contains as many of our favorite edges as possible and so on. In the setting of two-sided preferences, stable matchings are typically considered to be the best matchings. A matching M is stable if there is no edge that "blocks" M: an edge (r, h) is said to block M if (i) either r is unmatched or it prefers h to its partner in M and (ii) either h has less than $\mathsf{cap}(h)$ partners or it prefers r to its worst partner in M.

* Part of this work (Section 3) appeared in FSTTCS 2023 [27]. Section 2 and our main result (Theorem 3) are new.

Stable matchings always exist in G and the classic Gale-Shapley algorithm [14] finds one. However stability is a strong and rather restrictive notion, thus we may miss out on matchings with lower cost (say, more egalitarian matchings) by defining only stable matchings to be the best matchings. There are several applications such as assigning projects to trainees or students where stability may be relaxed for the sake of obtaining matchings with lower cost or improved utility.

Popularity. A well-studied relaxation of stability is the notion of *popularity*. Popularity is based on voting by vertices on matchings. In the one-to-one (also called the *marriage*) setting, the preferences of a vertex over its neighbors extend naturally to preferences over matchings—so every vertex orders matchings in the order of its partners in these matchings and being unmatched is its worst option. Popular matchings are (weak) *Condorcet winners* [11,29] in this voting instance where vertices are voters and matchings are candidates. Thus a popular matching M does not lose a head-to-head election against any matching N where each vertex either casts a vote for the matching in $\{M, N\}$ that it prefers or it abstains from voting if its assignment is the same in M and N.

Recall that we are in the many-to-one or hospitals/residents setting, i.e., hospitals have capacities. So we need to specify how a hospital votes over different subsets of its neighbors. Thus for a hospital h, we need to compare two subsets $M(h)$ and $N(h)$ of $\mathsf{Nbr}(h)$, where $\mathsf{Nbr}(h)$ is the set of neighbors of h. We will follow the method from [6] for this comparison. The definition in [6] of voting by a hospital h between two subsets of $\mathsf{Nbr}(h)$ is the following:

- First, all residents that are contained in both sets are removed and then a bijection from the first set to the second set is determined.[3] Every vertex is compared with its image under this bijection, thus the vote depends on the bijection that is chosen.

Voting by vertices. Let us first formally define how a resident r casts its vote between two neighbors h and h'. The function $\mathsf{vote}_r(h, h')$ is 1 if r prefers h to h', it is -1 if r prefers h' to h, and it is 0 otherwise, i.e., $h = h'$. The function $\mathsf{vote}_r(\cdot, \cdot)$ that compares two neighbors of r extends naturally to any pair of matchings M and N as $\mathsf{vote}_r(M, N) = \mathsf{vote}_r(M(r), N(r))$. Note that $M(r)$ (resp., $N(r)$) is null if r is unmatched in M (resp., N) and this is the worst state for any vertex. Every resident casts a vote in $\{0, \pm 1\}$ in the M-vs-N election.

A hospital h with capacity $\mathsf{cap}(h)$ is allowed to cast up to $\mathsf{cap}(h)$ many votes. Let S and T be two subsets of $\mathsf{Nbr}(h)$; hospital h compares these two subsets as follows:

- a bijection ψ is chosen from $S' = S \setminus T$ to $T' = T \setminus S$;
- every resident $r \in S'$ is compared with $\psi(r) \in T'$;
- the number of wins minus the number of losses is h's vote for S versus T.

The bijection ψ that is chosen will be the one that *minimizes* h's vote for S' versus T'. More formally, the vote of h for S versus T, denoted by $\mathsf{vote}_h(S, T)$, is defined as follows where $|S'| = |T'| = k$ and $\Pi[k]$ is the set of permutations on $\{1, \ldots, k\}$:

$$\mathsf{vote}_h(S, T) = \min_{\sigma \in \Pi[k]} \sum_{i=1}^{k} \mathsf{vote}_h(S'[i], T'[\sigma(i)]), \tag{1}$$

where $S'[i]$ is the i-th ranked resident in S' and $T'[\sigma(i)]$ is the $\sigma(i)$-th ranked resident in T'.

[3] If the sets are not of equal size, then dummy vertices that are less preferred to all non-dummy vertices are added to the smaller set.

For example, let $S = \{r_1, r_3, r_4\}$ and let $T = \{r_2, r_3, r_5\}$ where $r_1 \succ r_2 \succ r_3 \succ r_4 \succ r_5$ is h's preference order. So h prefers r_1 to r_2 and so on. Then $S' = \{r_1, r_4\}$ and $T' = \{r_2, r_5\}$. The bijection $\psi : S' \to T'$ would be as follows: $\psi(r_1) = r_5$ and $\psi(r_4) = r_2$ as this is the most adversarial way of comparing S' with T'. This results in $\mathsf{vote}_h(S,T) = \mathsf{vote}_h(r_1, r_5) + \mathsf{vote}_h(r_4, r_2) = 1 - 1 = 0$.

For any pair of matchings M and N in G, let $\mathsf{vote}_h(M,N) = \mathsf{vote}_h(S,T)$, where $S = M(h)$ and $T = N(h)$. So $\mathsf{vote}_h(M,N)$ counts the number of votes by h for $M(h)$ versus $N(h)$ when the two sets $M(h) \setminus N(h)$ and $N(h) \setminus M(h)$ are compared in the order that is most adversarial or *negative* for M. The two matchings M and N are compared using $\Delta(M,N) = \sum_{v \in R \cup H} \mathsf{vote}_v(M(v), N(v))$. We will say matching M is better or more popular than matching N if $\Delta(M,N) > 0$.

Definition 2. *A matching M is popular if $\Delta(M,N) \geq 0$ for all matchings N in G.*

Popular matchings will be our "best matchings". Every stable matching in G is popular [6]. Thus popular matchings always exist in a hospitals/residents instance $G = (R \cup H, E)$ and it is easy to find one. Our problem is to compute a *min-cost* popular matching in G where $\mathsf{cost} : E \to \mathbb{R}$ is part of the input.

Recall that a one-to-one matching instance is also called a marriage instance. A polynomial-time algorithm to compute a min-cost popular matching in a marriage instance with complete preferences is known [10]. When preferences are incomplete, it is NP-hard to find a min-cost popular matching in a marriage instance [12]. In light of this hardness result, let us assume that every resident is adjacent to all hospitals—thus the underlying graph is the complete bipartite graph. No polynomial-time algorithm is currently known for the min-cost popular matching problem with complete preferences in the many-to-one setting.

A first attempt to solve this problem would be to reduce it to the min-cost popular matching problem in the marriage setting. Suppose we construct the marriage instance G' obtained from G where each hospital h is replaced by $\mathsf{cap}(h)$ many *clones* $h_1, \ldots, h_{\mathsf{cap}(h)}$; moreover, we will have $h_1 \succ_r \cdots \succ_r h_{\mathsf{cap}(h)}$ for all residents r (Section 1.4 has more details). There is a natural map p from the set of popular matchings in the marriage instance G' to the set of matchings in the hospitals/residents instance G, where for any popular matching M' in G', the many-to-one matching $M = p(M')$ is obtained by identifying all the clones of the same hospital. It can be shown that the resulting matching M is popular in G.

Thus p : {popular matchings in G'} $\to$ {popular matchings in G}. Is p surjective? If so, then we can obtain a min-cost popular matching in G by computing a min-cost popular matching in G'.

1.1 An interesting example

Consider the following example where $R = \{r, r'\}$ and $H = \{h, h'\}$ with $\mathsf{cap}(h) = 1$ and $\mathsf{cap}(h') = 2$. Vertex preferences are as follows.

$$
\begin{array}{ll}
r: h \succ h' & \qquad h: r \succ r' \\
r': h \succ h' & \qquad h': r \succ r'
\end{array}
$$

Both residents prefer h to h' and both hospitals prefer r to r'. We claim $M = \{(r,h'), (r',h)\}$ is popular. Let us compare M with $N = \{(r,h), (r',h')\}$: M gets the vote of r' and a single vote from h' while N gets the votes of r and h. So $\phi(M,N) = \phi(N,M) = 2$ and the M-vs-N election is a tie. Another matching of size 2 here is $T = \{(r,h'), (r',h')\}$: M gets the votes of r' and h in the M-vs-T election while T gets a single vote from h', so M is more popular than T. It is easy to check that M does not lose against any smaller matching, thus M is a popular matching in G.

Let us construct the marriage instance $G' = (R \cup H', E')$ corresponding to G where $E' = R \times H'$, $R = \{r, r'\}$, and $H' = \{h, h'_1, h'_2\}$. Vertex preferences in G' are as follows: note that h' gets replaced by $h'_1 \succ h'_2$ in the preference orders of all residents and the preference lists of h'_1 and h'_2 are the same as the preference list of h'.

$$\begin{array}{lll} r\colon h \succ h'_1 \succ h'_2 & \qquad h'_1\colon r \succ r' & \qquad h\colon r \succ r' \\ r'\colon h \succ h'_1 \succ h'_2 & \qquad h'_2\colon r \succ r' & \end{array}$$

The matching M in G has two realizations in G': these are $M' = \{(r, h'_1), (r', h)\}$ and $M'' = \{(r, h'_2), (r', h)\}$. We show below that neither M' nor M'' is popular in G'.

- M' is more popular than M'' since r and h'_1 prefer M' to M'' while h'_2 prefers M'' to M' and the vertices r' and h are indifferent between M' and M''.
- $N' = \{(r, h), (r', h'_2)\}$ is more popular than M' since the vertices r, h, and h'_2 prefer N' to M' while the vertices r' and h'_1 prefer M' to N'.

Hence though $M = \{(r, h'), (r', h)\}$ is popular in G, neither of its realizations is popular in G'. The instance G' has a unique stable matching $P' = \{(r, h), (r', h'_1)\}$ which is also the unique popular matching in G'.

The above example shows that $p : \{\text{popular matchings in } G'\} \to \{\text{popular matchings in } G\}$ is not surjective. Solving the min-cost popular matching problem in the above instance G' would have yielded $P = \{(r, h), (r', h')\}$ in G because $P' = \{(r, h), (r', h'_1)\}$ is the only popular matching in G'. The matching P – though popular in G – need not be a *min-cost* popular matching in G because M is also popular and edge costs can be easily set so that $\mathsf{cost}(M) < \mathsf{cost}(P)$. Thus solving the min-cost popular matching problem in G' does not solve our problem in G.

The following theorem is our main result.

Theorem 3. *Let $G = (R \cup H, E)$ be a hospitals/residents instance with a function $\mathsf{cost} : E \to \mathbb{R}$, where every vertex in $R \cup H$ has complete and strict preferences. A min-cost popular matching in G can be computed in polynomial time.*

1.2 Incomplete preferences

The assumption that every resident has to rank all hospitals in a strict order of preference is reasonable in some settings such as the assignment of trainees to projects where every trainee is asked to rank all projects. But there are other settings, say while matching students to schools, where the instance G need not be the complete bipartite graph because a student may not want to go to a school that is too far away from home. One of the main steps in our algorithm in Theorem 3 involves finding a min-cost popular (many-to-one) matching in an instance that admits a perfect matching, i.e., one where every vertex is fully matched.

There are several natural scenarios where a hospitals/residents instance with *incomplete* preferences admits a perfect matching, e.g., the “lab rotation” problem where each student gets assigned to a lab during training. At certain intervals, based on the preferences of students and those in charge of labs, there may be a reassignment of students to labs. Every student in our instance is adjacent to her current lab and labs that she prefers to her current lab, thus preferences need not be complete here. The set of feasible solutions is the set of perfect matchings. The instance admits a perfect matching which is the current assignment of students to labs. In such applications we seek a "best perfect matching". Though the instance admits perfect matchings, it could be the case that no perfect matching is popular [23]. A natural candidate for a best perfect matching is a popular *perfect* matching defined below.

Definition 4. *A perfect matching M is a popular perfect matching if $\Delta(M,N) \geq 0$ for all perfect matchings N in G.*

Thus a popular perfect matching M need not be popular – it may lose elections – but no perfect matching defeats M. For any hospitals/residents instance G, there is a natural function f (analogous to p) from the set of popular perfect matchings in the corresponding marriage instance G' to the set of popular perfect matchings in G. We show the following structural result.

Theorem 5. *For any popular perfect matching M in a hospitals/residents instance $G = (R \cup H, E)$ with strict preferences (these can be incomplete), there is always some realization of M that is a popular perfect matching in the corresponding marriage instance $G' = (R \cup H', E')$.*

Thus, unlike the map p (see the example in Section 1.1), the map f is surjective. In other words, the set of popular perfect matchings in G is *not* richer than the set of popular perfect matchings in G'. There is a polynomial-time algorithm to find a min-cost popular perfect matching in a marriage instance [26]. Hence Theorem 5 leads to a polynomial-time algorithm to find a min-cost popular perfect matching in a hospitals/residents instance $G = (R \cup H', E')$ with a function $\mathsf{cost} : E \to \mathbb{R}$.

Theorem 6. *Let $G = (R \cup H, E)$ be a hospitals/residents instance where vertices have strict preferences (perhaps incomplete) and there is a function $\mathsf{cost} : E \to \mathbb{R}$. If G admits a perfect matching then a min-cost popular perfect matching in G can be computed in polynomial time.*

1.3 Background

The notion of popularity was proposed by Gärdenfors [16] in 1975 in the stable marriage problem (i.e., in the one-to-one setting) where he observed that every stable matching is popular. When preferences include ties (even one-sided ties), popular matchings need not always exist and it is NP-hard to decide if one exists [4,9].

It was shown in [19] that stable matchings are min-size popular matchings in a marriage instance. Polynomial-time algorithms to find a max-size popular matching were given in [19,23]. A subclass of max-size popular matchings called *dominant* matchings was studied in [10]. A popular matching M is dominant if M is more popular than every larger matching. Every popular matching in an instance with complete preferences is a maximum matching (hence, dominant) and a characterization of dominant matchings as stable matchings in another graph yielded a polynomial-time algorithm in [10] for the min-cost popular matching in marriage instances with complete preferences. When preferences are incomplete, even in the special case of marriage instances that admit perfect matchings, it is NP-hard to find a min-cost popular matching [12].

It was shown in [23] that popular *maximum* matchings, i.e., maximum matchings that are popular within the set of maximum matchings, always exist in a marriage instance. It follows from [23,26] that a maximum matching M in a marriage instance G is a popular maximum matching if and only if M can be realized as a stable matching in an auxiliary instance. Thus, in contrast to the NP-hardness of the min-cost popular matching problem in instances with incomplete preferences, there is a polynomial-time algorithm for the min-cost popular maximum matching problem in all marriage instances. We refer to [8] for a survey on results in popular matchings in the marriage setting.

The stable matching problem has been extensively studied in the hospitals/residents setting and also in the many-to-many setting [2,5,13,17,18,20,21,35,37]. In contrast to min-cost popular matchings, a min-cost stable matching in a hospitals/residents instance (and also in a many-to-many instance) with incomplete preferences can be computed in polynomial time [13,18,36,38]. The notion of popularity was extended from the marriage setting to the many-to-many setting in [6] and [33], independently. A polynomial-time algorithm to compute a max-size popular matching in the many-to-many setting was given in [6]. It was also shown in [6] that every stable matching in a many-to-many instance is popular; so though a rather strong definition of popularity was adopted here, it was shown that popular matchings always exist. The definition of popularity considered in [33] is weaker than the one in [6]; in order to compare a pair of matchings M_0 and M_1, every hospital h uses the bijection that compares the top neighbor in $M_0(h) \setminus M_1(h)$ with the top neighbor in $M_1(h) \setminus M_0(h)$, and so on, i.e., the permutation σ in Eq. (1) is the identity permutation.

Popular matchings where vertices have capacity lower bounds have been studied in the hospitals/residents setting and in the many-to-many setting [30,31,32], where it is only matchings that satisfy these lower bounds that are feasible. It was shown in these works that popular feasible matchings always exist and can be computed in polynomial time. Note that it is not always the case that the tractability of a matching problem in the one-to-one setting translates to its tractability in the many-to-one setting. It was very recently shown that it is NP-hard to find a matching that maximizes *Nash social welfare* (i.e., the geometric mean of edge utilities) in a many-to-one instance [22]. Note that the maximum weight matching algorithm solves this problem in the one-to-one setting.[4]

1.4 Our techniques

For any hospitals/residents instance $G = (R \cup H, E)$ with strict preferences, there is a corresponding marriage instance G', i.e., each vertex in G' has capacity 1. The vertex set of G' is $R \cup H'$ where $H' = \cup_{h \in H}\{h_1, \ldots, h_{\mathsf{cap}(h)}\}$. The set of neighbors of each h_i in G' is exactly the same as the set of h's neighbors in G. So $G' = (R \cup H', E')$ is a marriage instance where $E' = \{(r, h_i) : (r, h) \in E$ and $1 \le i \le \mathsf{cap}(h)\}$. Every vertex in G' has a strict preference order over its neighbors as described below.

- For $h \in H$ and $1 \le i \le \mathsf{cap}(h)$: the preference order of h_i in G' is exactly the same as h's preference order in G.
- For $r \in R$: the preference order of r in G' is the same as r's preference order in G where every neighbor h in G gets replaced by all its clones in the order $h_1 \succ h_2 \succ \cdots \succ h_{\mathsf{cap}(h)}$. So if r's preference order in G is $h \succ h'$ then r's preference order in G' is:

$$h_1 \succ h_2 \succ \cdots \succ h_{\mathsf{cap}(h)} \succ h'_1 \succ h'_2 \succ \cdots \succ h'_{\mathsf{cap}(h')}.$$

As seen in the example in Section 1.1, every popular matching in a hospitals/residents instance G with complete preferences need *not* have a realization as a popular matching in the corresponding marriage instance G'.

[4] For any edge (u, v), the weight of this edge is log of the product of utilities that u and v have for each other.

– When $|R| > \sum_{h\in H}\mathsf{cap}(h)$, a simple observation shows that if M is a popular matching in G then M has to be stable in G. Thus the min-cost stable matching algorithm in G solves the min-cost popular matching problem in G.
– The non-trivial cases are when $|R| \le \sum_{h\in H}\mathsf{cap}(h)$.

The case when $|R| < \sum_{h\in H}\mathsf{cap}(h)$ is quite different from the case when $|R| = \sum_{h\in H}\mathsf{cap}(h)$.

Combinatorial arguments. When $|R| < \sum_{h\in H}\mathsf{cap}(h)$, as seen in the example in Section 1.1, the set of popular matchings in the many-to-one instance G can be richer than the set of popular matchings in the corresponding one-to-one instance G'. We show that any popular matching M in G has to be *almost stable* in G. That is, blocking edges (if any) can be incident to partners of one particular hospital. This allows us to construct a new marriage instance $\widetilde{G}$ and use combinatorial arguments to show that a matching M is popular in G if and only if M has a realization as a stable matching $\widetilde{M}$ in $\widetilde{G}$. Thus when $|R| < \sum_{h\in H}\mathsf{cap}(h)$, the min-cost stable matching algorithm in $\widetilde{G}$ solves the min-cost popular matching problem in G. This result is shown in Section 2.

When $|R| = \sum_{h\in H}\mathsf{cap}(h)$, the graph G admits a perfect (many-to-one) matching and popular matchings can be *highly unstable*. In other words, a popular matching can admit many blocking edges that are incident to residents matched to different hospitals $h, h', h'', \ldots$ in G. Interestingly, in contrast to the case $|R| < \sum_{h\in H}\mathsf{cap}(h)$, we will be able to show that each popular matching in G has a *realization* as a popular matching in the marriage instance G'.

For any matching M in G, note that there is a natural realization of M in G' defined as follows:

$$M' = \cup_{h\in H}\{(r_i, h_i) : 1 \le i \le \mathsf{cap}(h) \text{ and } r_i \text{ is the } i\text{-th most preferred partner of } h \text{ in } M\}.$$

Though M is a popular matching in the instance G, the above realization M' need not be popular in the marriage instance G'. Consider the following instance where $R = \{p, q, r\}, H = \{h, h'\}$ and $\mathsf{cap}(h) = 2$ while $\mathsf{cap}(h') = 1$.

$$\begin{array}{ll} p\colon h \succ h' & \qquad h\colon p \succ q \succ r \\ q\colon h \succ h' & \qquad h'\colon p \succ q \succ r \\ r\colon h \succ h' & \end{array}$$

It can be checked that $M = \{(p, h), (q, h'), (r, h)\}$ is a popular matching in G. But the natural realization $M' = \{(p, h_1), (q, h'), (r, h_2)\}$ is not popular in G' since $N' = \{(p, h'), (q, h_2), (r, h_1)\}$ is more popular than M'; the four vertices q, r, h_2, h' prefer N' to M' while the two vertices p and h_1 prefer M' to N'.

Stable matching problems in a hospitals/residents instance G can be easily translated into stable matching problems in the corresponding marriage instance G' since a matching M is stable in G if and only if its natural realization M' is stable in G'. However popular matchings are more complex than stable matchings and M can be popular in G though M' is not popular in G'. In the above example, note that there is another realization $M'' = \{(p, h_2), (q, h'), (r, h_1)\}$ of M that is a popular matching in G'.

The LP method. We use LP duality to show that for any popular matching M in a hospitals/residents instance $G = (R \cup H, E)$ with $|R| = \sum_{h\in H}\mathsf{cap}(h)$, there is always some realization M'' of M such that M'' is a popular matching in the marriage instance $G' = (R \cup H', E')$. Furthermore, by replacing popular matching with *popular perfect* matching, this result extends to instances with incomplete preferences.

It is known that every popular perfect matching in a marriage instance G' has a *dual certificate* $\vec{\alpha}$ that certifies its optimality [26] (see Lemma 16 in Section 3.1). Every popular perfect matching M in a hospitals/residents instance G has a weaker dual certificate $\vec{\gamma}$ that certifies its optimality. We show how to transform the weaker certificate $\vec{\gamma}$ into a stronger certificate $\vec{\alpha}$. This allows us to obtain an appropriate realization M'' of M such that M'' is a popular perfect matching in G' with $\vec{\alpha}$ as its dual certificate. This result is given in Section 3.

Thus we obtain the following characterization of popular matchings in a hospitals/residents instance $G = (R \cup H, E)$ with complete preferences. A matching M in $G = (R \cup H, E)$ is popular if and only if:

1. $|R| > \sum_{h \in H} \mathsf{cap}(h)$ and the natural realization M' is stable in G'.
2. $|R| = \sum_{h \in H} \mathsf{cap}(h)$ and M has a realization as a popular matching M'' in G'.
3. $|R| < \sum_{h \in H} \mathsf{cap}(h)$ and M has a realization as a stable matching $\widetilde{M}$ in the new instance $\widetilde{G}$.

2 A characterization of popular matchings

Our input is a many-to-one matching instance $G = (R \cup H, E)$ with strict and complete preferences, every hospital $h \in H$ has a capacity $\mathsf{cap}(h) \geq 1$. In order to characterize popular matchings in G, we will use the one-to-one (or marriage) instance $G' = (R \cup H', E')$ corresponding to $G = (R \cup H, E)$. In the instance G', every $h \in H$ is replaced by the clones $h_1, \ldots, h_{\mathsf{cap}(h)}$ and every $(r, h) \in E$ has $\mathsf{cap}(h)$ many copies (r, h_i) for $1 \leq i \leq \mathsf{cap}(h)$. See Section 1.4 for details on vertex preferences.

Let M be a popular matching in G and let M' be any realization of M in G'. The subgraph $G'_{M'} = (R \cup H', E'_{M'})$ of G' from [6] will be very useful to us. The edge set $E'_{M'}$ of this subgraph is defined as follows:

$$E'_{M'} = M' \cup \{(r, h_i) : (r, h) \in E \setminus M \text{ and } 1 \leq i \leq \mathsf{cap}(h)\}.$$

So for any edge $(r, h) \notin M$, all the $\mathsf{cap}(h)$ many copies (r, h_i) where $1 \leq i \leq \mathsf{cap}(h)$ are in $E'_{M'}$ while for each edge $(r, h) \in M$, exactly *one* edge, which is the edge in M' (say, (r, h_i)) is in $E'_{M'}$. Thus $E'_{M'} \subseteq E'$ and every vertex in $G'_{M'}$ inherits its preference order from G'.

The following edge weight function $\mathsf{wt}_{M'}$ defined in $G'_{M'}$ will be useful to us. For any $e \in E'_{M'}$:

$$\text{let } \mathsf{wt}_{M'}(e) = \begin{cases} 2 & \text{if } e \text{ blocks } M'; \\ -2 & \text{if the endpoints of } e \text{ prefer their partners in } M' \text{ to each other;} \\ 0 & \text{otherwise.} \end{cases}$$

So $\mathsf{wt}_{M'}(e)$ is the sum of votes of the endpoints of edge e for each other versus their respective assignments in M' (recall that each vote is in $\{0, \pm 1\}$). Note that each endpoint of an edge in M' casts a vote of 0 for the other endpoint; other than these pairs in M', it is only pairs (r, h_i) where $(r, h) \notin M$ that are allowed to cast votes for each other—that is why the edge set of interest to us has been restricted to $E'_{M'}$ (instead of the entire set E').

The following theorem is reminiscent of [28, Theorem 2.1] that characterized popular b-matchings (i.e., vertices have capacities) using an auxiliary graph. Due to complete preferences, note that a necessary condition for a matching M to be popular in our instance G is that M is a maximum matching in G.

Theorem 7. *Let M be any maximum matching in G. Then M is popular if and only if there is a realization M' of M that satisfies the following two properties in $G'_{M'}$.*

1. *There is no alternating cycle C with respect to M' in $G'_{M'}$ such that $\mathsf{wt}_{M'}(C) > 0$.*
2. *There is no alternating path ρ with an unmatched vertex as an endpoint such that (i) $\mathsf{wt}_M(\rho) > 0$ and (ii) the two endpoints of ρ are not clones of each other.*

Proof. We will first prove these two properties are sufficient. Then we will show they are necessary.

Direction "⇐". Since preferences are complete, it suffices to compare M against maximum matchings to prove its popularity. Let M' be a realization of M that satisfies properties 1 and 2. A realization N' of any maximum matching N can be obtained in $G'_{M'}$ as follows.

- For every edge $(r, h) \in N \cap M$: the edge (r, h_i) is in N' where $(r, h_i) \in M'$.
- For every $(r, h) \in N \setminus M$: in the evaluation of $\mathsf{vote}_h(M, N)$ while comparing the set $M(h) \setminus N(h)$ with the set $N(h) \setminus M(h)$, let r' be the resident that h compares r with. So the matching M' contains the edge (r', h_i) for some $i \in \{1, \dots, \mathsf{cap}(h)\}$; the edge (r, h_i) will be included in N'.[5]

Since M and N are maximum matchings, $|M'| = |N'|$; so the symmetric difference $M' \oplus N'$ is a collection of alternating cycles and even length alternating paths. Recall that $\Delta(M, N) = \sum_{v \in R \cup H} \mathsf{vote}_v(M(v), N(v))$. From the construction of the matching N', we have:

$$\sum_{v \in R \cup H} \mathsf{vote}_v(M(v), N(v)) = -\sum_C \mathsf{wt}_{M'}(C) - \sum_\rho (\mathsf{wt}_{M'}(\rho) - 1),$$

where the first sum on the right is over all alternating cycles C and the second sum is over all even length alternating paths ρ in $M' \oplus N'$. Note that we are subtracting 1 from $\mathsf{wt}_{M'}(\rho)$ to count the vote of the endpoint of ρ matched in M' but left unmatched in N' (this vote is in favor of M).

For each alternating cycle $C \in M' \oplus N'$, we have $\mathsf{wt}_{M'}(C) \leq 0$ by property 1. It follows from our construction of N' that the endpoints of any even length alternating path $\rho \in M' \oplus N'$ (so one endpoint of ρ is unmatched in M') are *not* clones of each other.[6] Thus $\mathsf{wt}_{M'}(\rho) \leq 0$ by property 2. Hence $\Delta(M, N) \geq 0$. So $\Delta(M, N) \geq 0$ for any maximum matching N and thus for all matchings in G. Hence we can conclude that M is a popular matching in G.

Direction "⇒". Suppose the matching M is popular in G. We will show any realization M' of M satisfies properties 1 and 2 in the corresponding instance $G'_{M'}$. Suppose M' does not satisfy property 1. Let C be an alternating cycle with respect to M' in $G'_{M'}$ such that $\mathsf{wt}_{M'}(C) > 0$.

- Let $N' = M' \oplus C$. We have $\mathsf{wt}_{M'}(N') = \mathsf{wt}_{M'}(C) > 0$. By identifying all clones of the same hospital, the matching N' in $G'_{M'}$ becomes a matching N in G.
  - For any hospital h with no clone in C, the two sets $M(h)$ and $N(h)$ are the same. For each hospital h with one or more clones in C, the cycle C defines a bijection between $M(h) \setminus N(h)$ and $N(h) \setminus M(h)$. Since $\mathsf{wt}_{M'}(C) > 0$, this means for each h in C, there is a way of comparing elements in $M(h) \setminus N(h)$ with those in $N(h) \setminus M(h)$ such that summed over all vertices in C, the votes in favor of N outnumber the votes in favor of M.

[5] If $r' = \mathsf{null}$ then h_i is any clone of h that is left unmatched in M.

[6] For any hospital h, if $|M(h)| \leq |N(h)|$ then all clones of h matched in M' are matched in N' as well and similarly, if $|M(h)| > |N(h)|$ then any clone of h left unmatched in M' is left unmatched in N' as well.

- Adding up the votes of all vertices in C for M versus N (as per the bijection defined by C), the total number of votes for M is less than the total number of votes for N. Hence it follows from the definition of the function vote (see Eq.(1)) that $\sum_{v \in C} \mathsf{vote}_v(M, N) < 0$. So $\Delta(M, N) < 0$ and this contradicts the fact that M is a popular matching in G.

Suppose M does not satisfy property 2. Let ρ be an even length alternating path with respect to M' in $G'_{M'}$ such that (i) $\mathsf{wt}_{M'}(\rho) > 0$ and (ii) the endpoints of ρ are not clones of each other.

- Let $N' = M' \oplus \rho$. We have $\mathsf{wt}_{M'}(N') = \mathsf{wt}_{M'}(\rho) > 0$. By identifying all the clones of the same hospital, the matching N' in $G'_{M'}$ becomes a matching N in G.
  - As before, for any hospital h with no clone in ρ, the two sets $M(h)$ and $N(h)$ are the same. For each hospital h with one or more clones in ρ, the path ρ defines a bijection between $M(h) \setminus N(h)$ and $N(h) \setminus M(h)$. Observe that we are crucially using the fact that the endpoints of ρ are not clones of the same vertex—if they were clones h_i and h_j of the same hospital h, then in the M' vs N' comparison in $G'_{M'}$, we would have h_i comparing $r = M'(h_i)$ against null along with h_j comparing null against $r' = N'(h_j)$ whereas the correct $M(h)$ vs $N(h)$ comparison in G would have weeded out null from the comparison.
- As argued above, since $\mathsf{wt}_{M'}(\rho) > 0$ (so $\mathsf{wt}_{M'}(\rho) \geq 2$), for each h in ρ, there is a way of comparing elements in $M(h) \setminus N(h)$ with those in $N(h) \setminus M(h)$ such that summed over all vertices in ρ and taking into account the vote of the endpoint of ρ left unmatched in N', the votes in favor of N outnumber the votes in favor of M by at least 1. Thus, adding up the votes of all vertices in ρ for M versus N (as per the bijection defined by ρ), the total number of votes for M is less than the total number of votes for N. Hence it follows from the definition of vote (see Eq.(1)) that $\sum_{v \in \rho} \mathsf{vote}_v(M, N) \leq -1$. So $\Delta(M, N) < 0$, a contradiction to M's popularity in G.

Thus properties 1 and 2 characterize popular matchings in G. □

We will use the above characterization of popular matchings to show the following simple lemma.

Lemma 8. *If $|R| > \sum_{h \in H} \mathsf{cap}(h)$ then every popular matching in G has to be stable.*

Proof. Suppose G has a popular matching M that is not stable. We assume M is a maximum matching, so M fully matches all hospitals. Since $|R| > \sum_{h \in H} \mathsf{cap}(h)$, there is at least one resident (call it s) left unmatched in M. Because M is not stable, there is a pair (r, h) that blocks M. So:

1. hospital h prefers resident r to its least preferred partner r' in M and
2. either resident r is unmatched in M or it prefers hospital h to its partner h' in M.

Let M' be a realization of M that obeys properties 1 and 2 (given in Theorem 7) in $G'_{M'}$. Let $(r', h_i) \in M'$ where r' is the least preferred partner of h in M. Thus $\mathsf{wt}_{M'}(r, h_i) = 2$. If r is unmatched in M then the length 2 alternating path $\rho = r - h_i \sim r'$ with respect to M' in $G'_{M'}$ has weight $2 + 0 = 2$ (note that $h_i \sim r'$ denotes $(r', h_i) \in M'$). This contradicts the assumption that M' obeys property 2 (given in Theorem 7) in $G'_{M'}$. Thus r is matched in M. Let $(r, h'_j) \in M'$.

Consider the length 4 alternating path $\rho' = s - h'_j \sim r - h_i \sim r'$ with respect to M' in $G'_{M'}$. Since s prefers being matched to h'_j compared to being left unmatched, $\mathsf{wt}_{M'}(s, h'_j) \geq 0$. So the path ρ' has weight at least $0 + 0 + 2 + 0 = 2$. This again contradicts the assumption that M' obeys property 2 (given in Theorem 7) in $G'_{M'}$. Hence it has to be the case that M is stable. □

Thus when $|R| > \sum_{h \in H} \mathsf{cap}(h)$, the min-cost stable matching algorithm solves the min-cost popular matching problem in G. Let us now consider the case $|R| < \sum_{h \in H} \mathsf{cap}(h)$. This is a non-trivial and technical case as we can have examples such as the one in Section 1.1 where the instance G admits a popular matching that cannot be realized as a popular matching in G'.

The case $|R| < \sum_{h \in H} \mathsf{cap}(h)$. Since preferences are complete, every popular matching matches all residents; moreover, there is at least one hospital that is not fully matched here. The proofs of Lemma 9 and Lemma 10 are given in the appendix.

Lemma 9. *Let M and N be two popular matchings in G. For every hospital h, $|M(h)| = |N(h)|$.*

Lemma 10. *If there are two different hospitals (say, h' and h'') that are not fully matched in a popular matching M, then M is stable.*

Suppose two different hospitals h' and h'' are not matched up to capacity in a stable matching in G. Since all popular matchings match every hospital to the same capacity (by Lemma 9), this means neither h' nor h'' is fully matched in any popular matching in G. This implies every popular matching in G has to be stable (by Lemma 10). Thus in this case, the min-cost stable matching algorithm solves the min-cost popular matching problem in G.

In the rest of this section we will solve the min-cost popular matching problem in G when all but one hospital are fully matched in a stable matching (say, S) in G.

All but one hospital h are fully matched in S. We will construct a multigraph $\widetilde{G} = (R \cup H', \widetilde{E})$ such that the min-cost popular matching problem in G reduces to the min-cost stable matching problem in $\widetilde{G}$. The vertex set of $\widetilde{G}$ is the same as the vertex set $R \cup H'$ of G'. The edge set of $\widetilde{E}$ is as follows. Let $h \in H$ be the unique hospital that is not fully matched in the stable matching S in G. Let $|S(h)| = k < \mathsf{cap}(h)$.

– Every edge e in G' incident to $h_1, \ldots, h_k$ has two copies e_+ and e_- in $\widetilde{G}$.
– Every other edge $e' \in G'$ has only one copy e'_+ in $\widetilde{G}$.

So every *negative subscript* edge e_- in $\widetilde{G}$ has one of $h_1, \ldots, h_k$ as an endpoint. As done in [25], it will be convenient to think of the two copies e_+ and e_- of any edge e as colored by different colors (the colors are $+$ and $-$): so e_+ is the copy of e colored $+$ and e_- is the copy of e colored $-$. Note that all hospitals other than $h_1, \ldots, h_k$ have only color $+$ edges incident to them in $\widetilde{G}$. Vertex preferences in $\widetilde{G}$ on incident edges are as follows:

∗ *Every resident r prefers any color $-$ edge to any color $+$ edge.* Among edges colored by the same color, it is as per r's original preference order in G'; so r prefers e_- to e'_- (and similarly, e_+ to e'_+) if r prefers e to e'.
∗ *Every hospital prefers any color $+$ edge to any color $-$ edge.* Among edges colored by the same color, it is as per the hospital's original preference order in G'.

Recall the interesting example $G = (R \cup H, E)$ from Section 1.1. The corresponding instance $\widetilde{G}$ has vertex set $R \cup H'$ where $R = \{r, r'\}$ and $H' = \{h, h'_1, h'_2\}$. Its edge set consists of the 8 edges $(s,h)_+, (s,h'_1)_+, (s,h'_1)_-$, and $(s,h'_2)_+$ where $s \in \{r, r'\}$. Note that the instance $\widetilde{G}$ has two stable matchings: $\{(r,h)_+, (r',h'_1)_+\}$ and $\{(r,h'_1)_-, (r',h)_+\}$. Ignoring the $\pm$ subscripts and replacing all clones by the original hospital, we get the two popular matchings $\{(r,h), (r',h')\}$ and $\{(r,h'), (r',h)\}$ in G. The following theorem shows this is always the case in any hospitals/residents instance $G = (R \cup H, E)$ where vertices have complete preferences and $|R| < \sum_{h \in H} \mathsf{cap}(h)$.

Theorem 11. *A matching M is popular in G if and only if there is a realization $\widetilde{M}$ of M such that $\widetilde{M}$ is a stable matching in $\widetilde{G}$.*

Proof. We will first prove the sufficient condition and then the necessary condition for popularity.

Direction "$\Leftarrow$". Let $\widetilde{M}$ be a stable matching in $\widetilde{G}$. We claim $\widetilde{M}$ matches all vertices in $R \cup H'$ except $h_{k+1}, \ldots, h_{\mathsf{cap}(h)}$. This is because for any stable matching S' in G', the matching $\widetilde{S} = \{e_+ : e \in S'\}$ is stable in $\widetilde{G}$. Since the stable matching S fully matches all hospitals in G except h (which is matched to $k < \mathsf{cap}(h)$ many partners in S), it follows that all vertices in $R \cup H'$ except $h_{k+1}, \ldots, h_{\mathsf{cap}(h)}$ are matched in S'. Thus all vertices in $R \cup H'$ except $h_{k+1}, \ldots, h_{\mathsf{cap}(h)}$ are matched in $\widetilde{S}$. All stable matchings in $\widetilde{G}$ match the same subset of vertices [15].[7] Thus the stable matching $\widetilde{M}$ matches all vertices in $R \cup H'$ except $h_{k+1}, \ldots, h_{\mathsf{cap}(h)}$.

Consider the set M' of edges obtained by ignoring the $\pm$ edge colors in $\widetilde{M}$. Every edge in M' is an edge in E' (with no $\pm$ subscripts), thus M' is a matching in G'. We will show that M' obeys properties 1 and 2 (as given in Theorem 7) in $G'_{M'}$. By Theorem 7, this will imply the popularity of M.

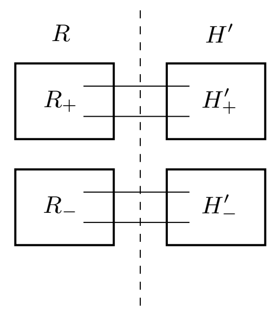

Fig. 1. $R = R_+ \cup R_-$ and $H' = H'_+ \cup H'_-$. Any blocking edge to M' in the instance G' will be in $R_- \times H'_+$.

We will partition the sets R and H as follows (see Fig. 1): initialize $R_- = H'_- = \emptyset$. For every edge $e = (r, h_i) \in M'$, if $e_- \in \widetilde{M}$ then add the vertices r and h_i to to R_- and H'_-, respectively. Let $R_+ = R \setminus R_-$ and $H'_+ = H' \setminus H'_-$. So R_+ (resp., R_-) is the set of residents matched along color $+$ (resp., color $-$) edges in $\widetilde{M}$ and $H'_- \subseteq \{h_1, \ldots, h_k\}$. Observe that $M' \subseteq (R_+ \times H'_+) \cup (R_- \times H'_-)$.

Lemma 12. *Every edge $e \in R_+ \times H'_-$ satisfies $\mathsf{wt}_{M'}(e) = -2$ and every edge $e \in (R_+ \times H'_+) \cup (R_- \times H'_-)$ satisfies $\mathsf{wt}_{M'}(e) \leq 0$.*

The proof of Lemma 12 is given in the appendix. We are now ready to prove M's popularity.

1. Consider any alternating cycle C with respect to M' in $G'_{M'}$. Since every edge in M' is either in $R_+ \times H'_+$ or in $R_- \times H'_-$, the number of edges in the cycle C that belong to the set $R_- \times H'_+$ is exactly equal to the number of edges in C that belong to the set $R_+ \times H'_-$. Every edge $e \in R_+ \times H'_-$ satisfies $\mathsf{wt}_{M'}(e) = -2$ (by Lemma 12) and by the definition of $\mathsf{wt}_{M'}$, we have $\mathsf{wt}_{M'}(e) \leq 2$ for any $e \in R_- \times H'_+$. Because $\mathsf{wt}_{M'}(e) \leq 0$ for any $e \in (R_+ \times H'_+) \cup (R_- \times H'_-)$ (by Lemma 12), we have $\mathsf{wt}_{M'}(C) \leq 0$. Thus M' obeys property 1.
2. Consider any even length alternating path ρ with respect to M' in $G'_{M'}$ with an unmatched vertex as an endpoint. Since it is only $h_{k+1}, \ldots, h_{\mathsf{cap}(h)}$ that are unmatched in M' (recall that these vertices are in H'_+), it is a clone of h that is one endpoint of ρ. If every vertex of ρ is in

[7] This fact was proved in [15] for simple graphs. Note that our graph $\widetilde{G}$ is not simple, it is a multigraph. However this fact holds in $\widetilde{G}$ as well because $\widetilde{G}$ can be easily converted into a simple graph by using some dummy vertices.

$R_+ \cup H'_+$ then $\mathsf{wt}_{M'}(\rho) \le 0$ since every edge $e \in R_+ \times H'_+$ satisfies $\mathsf{wt}_{M'}(e) \le 0$ (by Lemma 12). Suppose ρ traverses an edge $e \in R_- \times H'_+$. It may be the case that $\mathsf{wt}_{M'}(e) = 2$. However if any edge in $R_- \times H'_-$ is the last edge in ρ then the endpoints of the path ρ are clones of h. In order to get an even length alternating path ρ with one endpoint unmatched and the other endpoint *not* a clone of h, the path ρ needs to traverse an edge $e \in R_+ \times H'_-$ and $\mathsf{wt}_{M'}(e) = -2$ (by Lemma 12).

By the same reasoning as given for case 1, it follows that for any even length alternating path ρ with one unmatched endpoint (so this is a clone of h) and the other endpoint not a clone of h, the number of edges from $R_+ \times H'_-$ equals the number of edges from $R_- \times H'_+$. Since $\mathsf{wt}_{M'}(e) = -2$ for every $e \in R_+ \times H'_-$ and $\mathsf{wt}_{M'}(e) \le 0$ for all edges $e \in (R_+ \times H'_+) \cup (R_- \times H'_-)$, we can conclude that $\mathsf{wt}_{M'}(\rho) \le 0$.

Thus M' satisfies properties 1 and 2 (as given in Theorem 7) in $G'_{M'}$. Hence it follows from Theorem 7 that M is popular in G.

Direction "⇒". Let M be a popular matching in G. Since the stable matching S in G fully matches all hospitals in $H \setminus \{h\}$ and $|S(h)| = k$, it follows from Lemma 9 that M also fully matches all hospitals in $H \setminus \{h\}$ and $|M(h)| = k$. We know from Theorem 7 that there is a realization M' of M that obeys properties 1 and 2 in $G'_{M'}$.

- Let $r_1 \succ_h \cdots \succ_h r_k$ be the partners of h in M.
- For each $h' \in H \setminus \{h\}$, let $s_1 \succ_{h'} \cdots \succ_{h'} s_{\mathsf{cap}(h')}$ be the partners of h' in M.

Case 1. *Suppose there is no edge in $G'_{M'}$ that blocks M'.* Then we can construct a stable matching $\widetilde{M}$ in $\widetilde{G}$ as follows: $\widetilde{M} = \widetilde{M}_0 \cup \widetilde{M}_1$ where $\widetilde{M}_0$ and $\widetilde{M}_1$ are defined as follows.

$$\widetilde{M}_0 = \cup_{h' \neq h} \{(s_i, h'_i)_+ \text{ where } (s_i, h') \in M \text{ and } 1 \le i \le \mathsf{cap}(h')\}. \tag{2}$$

$$\widetilde{M}_1 = \{(r_i, h_i)_+ \text{ where } (r_i, h) \in M \text{ and } 1 \le i \le k\}. \tag{3}$$

Recall that $r_1 \succ_h \cdots \succ_h r_k$ are the partners of h in M. So all edges in $\widetilde{M}$ are color $+$ edges and the only vertices left unmatched in $\widetilde{M}$ are $h_{k+1}, \ldots, h_{\mathsf{cap}(h)}$. It is easy to show that $\widetilde{M}$ is stable in $\widetilde{G}$.

The difference between the matchings $\widetilde{M}$ and M' is with respect to what clone of a hospital h' gets matched to which resident in $M(h')$. For any $h' \in H$ in the matching $\widetilde{M}$: the i-th best clone of h' (which is h'_i) gets the i-th best partner of h' in M (which is s_i or r_i). Thus for any $(s_i, h') \in M$, the edge $(s_i, h'_j)_+$ does not block $\widetilde{M}$ for any $1 \le j \le \mathsf{cap}(h')$.

For every edge $(r'_i, h') \notin M$, the edge $(r'_i, h'_j) \in G'_{M'}$ for all $1 \le j \le \mathsf{cap}(h')$. Since no edge in $G'_{M'}$ blocks M', it follows that $(r'_i, h'_j)_+$ does not block $\widetilde{M}$ in $\widetilde{G}$ for any $h' \in H'$ and $1 \le j \le \mathsf{cap}(h')$. Thus no color $+$ edge blocks $\widetilde{M}$. It is only $h_1, \ldots, h_k$ that have color $-$ edges incident to them; recall that each of them prefers any color $+$ edge to any color $-$ edge and each of these hospitals is matched along a color $+$ edge in $\widetilde{M}$. Thus no color $-$ edge blocks $\widetilde{M}$.

Case 2. *Suppose there exist one or more edges in $G'_{M'}$ that block M'.* Our first claim is that if resident r is an endpoint of an edge that blocks M', then $(r, h_i) \in M'$ for some $i \in \{1, \ldots, \mathsf{cap}(h)\}$. That is, r has to be matched in M' to a clone of h, where h is the unique hospital not fully matched in M. Suppose not. Let (r, h'_j) be a blocking edge to M' where $(r, h''_\ell) \in M'$ and $h'' \neq h$. If $h' = h$ then we have a length 2 alternating path $\rho = h_t - r \sim h''_\ell$ in $G'_{M'}$ where h_t is a clone of h left unmatched in M' such that $\mathsf{wt}_{M'}(\rho) = 2 + 0 = 2$. Since $h'' \neq h$, the alternating path ρ violates property 2, which is a contradiction to the fact that M' obeys properties 1 and 2 in $G'_{M'}$.

So let us assume that $h' \neq h$. Since all hospitals in $H \setminus \{h\}$ are fully matched in M, there is an edge $(s, h'_j) \in M'$. Then we have a length 4 alternating path $\rho' = h_t - s \sim h'_j - r \sim h''_\ell$ in $G'_{M'}$ such that $\mathsf{wt}_{M'}(\rho') \geq 0 + 0 + 2 + 0 = 2$. Since $h'' \neq h$, the alternating path ρ' violates property 2; this contradicts the fact that M' obeys properties 1 and 2 in $G'_{M'}$.

Thus we have shown that every resident with a blocking edge incident to it in $G'_{M'}$ has to be matched in M' to a clone of h. Let r be the *worst* partner of h in M with a blocking edge (in $G'_{M'}$) with respect to M' incident to it. The following lemma (proof in the appendix) will be useful.

Lemma 13. *If $(r, h_i) \in M'$ then $\mathsf{wt}_{M'}(e) = -2$ for every edge $e = (r', h_i)$ in $G'_{M'}$ where $r' \neq r$.*

Recall that $r_1 \succ_h \cdots \succ_h r_k$ are the partners of h in M. Let $r = r_\ell$, where r is the worst partner of h in M with a blocking edge incident to it in $G'_{M'}$.

Let $\widetilde{M} = \widetilde{M}_0 \cup \widetilde{M}_2$ where $\widetilde{M}_0$ is defined above in (2) and the matching $\widetilde{M}_2$ is defined as follows:

$$\widetilde{M}_2 = \{(r_i, h_{k-\ell+i})_- : 1 \leq i \leq \ell\} \cup \{(r_{\ell+i}, h_i)_+ : 1 \leq i \leq k - \ell\}.$$

So the *best* ℓ partners of h in M (these are $r_1, \ldots, r_\ell$) are matched in $\widetilde{M}$ along color $-$ edges to $h_{k-\ell+1}, \ldots, h_k$ respectively; note that these are the *worst* ℓ clones of h matched in any stable matching in $\widetilde{G}$. On the other hand, the *worst* $k - \ell$ partners of h in M (these are $r_{\ell+1}, \ldots, r_k$) are matched in $\widetilde{M}$ along color $+$ edges to the *best* $k - \ell$ clones of h: these are $h_1, \ldots, h_{k-\ell}$.

Lemma 14. *The matching $\widetilde{M}$ is stable in $\widetilde{G}$.*

The proof of Lemma 14 is given in the appendix. This finishes the proof of the theorem. $\square$

The following algorithm solves the min-cost popular matching problem in the hospitals/residents instance $G = (R \cup H, E)$ when $|R| < \sum_h \mathsf{cap}(h)$ and vertices have complete preferences.

1. Let S be a stable matching in G.
2. If there are two different hospitals not fully matched in S then
   - Return a min-cost stable matching in G.
3. Else compute the instance $\widetilde{G}$ and return $p(\widetilde{M})$ where $\widetilde{M}$ is a min-cost stable matching in $\widetilde{G}$.

The correctness of the above algorithm follows from Lemma 10 and Theorem 11. Thus we know how to solve the min-cost popular matching problem in $G = (R \cup H, E)$ when $|R| \neq \sum_h \mathsf{cap}(h)$.

The remaining case. What is left is the case $|R| = \sum_h \mathsf{cap}(h)$. Since preferences are complete, G admits a matching that fully matches all vertices in this case. Section 3 contains a polynomial-time algorithm for the min-cost popular *perfect* matching problem in a hospitals/residents instance $G = (R \cup H, E)$ that admits a perfect matching. We will reduce the min-cost popular perfect matching problem in G to the min-cost popular perfect matching problem in the corresponding marriage instance G'. There is a polynomial-time algorithm to solve the min-cost popular perfect matching problem in G' [26]. Thus this will yield a polynomial-time algorithm to solve the min-cost popular matching problem in $G = (R \cup H, E)$ when $|R| = \sum_h \mathsf{cap}(h)$ and preferences are complete.

Remark 15. Our reduction of the min-cost popular perfect matching problem from G to G' will not use complete preferences. For the min-cost popular matching problem in a hospitals/residents instance with incomplete preferences, note that a reduction to the corresponding marriage instance will not be interesting as it is NP-hard to solve the min-cost popular matching problem in a marriage instance with incomplete preferences. As mentioned earlier, this hardness holds even in marriage instances that admit perfect matchings [12].

3 Popular perfect matchings in a hospitals/residents instance

Let $G = (R \cup H, E)$ be a hospitals/residents instance where every vertex has a strict preference order over its neighbors. In this section note that preferences need not be complete, i.e., the underlying graph need not be the complete bipartite graph. Section 3.1 gives a characterization of popular perfect matchings in G. We will use this characterization in Section 3.2 to show helpful dual certificates for popular perfect matchings in G. These dual certificates will lead to our algorithm for the min-cost popular perfect matching problem in G.

3.1 A characterization of popular perfect matchings

We describe dual certificates for popular perfect matchings in a *marriage* instance $G_0 = (A \cup B, E_0)$ where vertices have strict (not necessarily complete) preferences. Let M' be a matching in G_0. Recall the edge weight function $\mathsf{wt}_{M'}$ from Section 2 where $\mathsf{wt}_{M'}(e) = 2$ if e blocks M', $\mathsf{wt}_{M'}(e) = -2$ if the endpoints of e prefer their partners in M' to each other, and $\mathsf{wt}_{M'}(e) = 0$ otherwise.

It follows from the definition of $\mathsf{wt}_{M'}$ that for any perfect matching N' we have $\mathsf{wt}_{M'}(N') = \Delta(N', M')$. So for any popular perfect matching M', we have $\mathsf{wt}_{M'}(M') = 0 \geq \Delta(N', M') = \mathsf{wt}_{M'}(N')$ where N' is any perfect matching. Thus every popular perfect matching M' in a marriage instance $G_0 = (A \cup B, E_0)$ is a max-weight perfect matching in G_0 under the edge weight function $\mathsf{wt}_{M'}$. The linear program LP1 is the max-weight perfect matching LP in G_0 (under the edge weight function $\mathsf{wt}_{M'}$) and LP2 is the dual LP.

$$\begin{aligned} \max \sum_{e \in E_0} \mathsf{wt}_{M'}(e) \cdot x_e & \qquad \text{(LP1)} \\ \text{s.t.} \quad \sum_{e \in \delta(v)} x_e = 1 \ \ \forall v \in A \cup B & \\ x_e \geq 0 \ \ \forall e \in E_0. & \end{aligned}$$

$$\begin{aligned} \min \sum_{v \in A \cup B} y_v & \qquad \text{(LP2)} \\ \text{s.t.} \quad y_a + y_b \geq \mathsf{wt}_{M'}(a,b) \ \ \forall (a,b) \in E_0. & \end{aligned}$$

For any $v \in A \cup B$, note that $\delta(v)$ is the set of edges incident to v in the graph G_0. The following result (from [26]) shows that LP2 admits a special optimal solution. Here $|A \cup B| = n$ and $|A| = n_0$.

Lemma 16 ([26]). *Let M' be a perfect matching in $G_0 = (A \cup B, E_0)$. Then M' is a popular perfect matching in G_0 if and only if there exists $\vec{\alpha} \in \mathbb{R}^n$ that satisfies conditions (i)-(iii) given below:*

(i) $\alpha_a \in \{0, -2, -4, \ldots, -2(n_0 - 1)\}$ for all $a \in A$ and $\alpha_b \in \{0, 2, 4, \ldots, 2(n_0 - 1)\}$ for all $b \in B$.
(ii) $\alpha_a + \alpha_b \geq \mathsf{wt}_{M'}(a,b)$ for all $(a,b) \in E_0$.
(iii) $\sum_{v \in A \cup B} \alpha_v = 0$.

Definition 17. *For any popular perfect matching M' in a marriage instance G_0, a vector $\vec{\alpha} \in \mathbb{R}^n$ that satisfies properties (i)-(iii) in Lemma 16 will be called a dual certificate for M'.*

A hospitals/residents instance $G = (R \cup H, E)$. Let M be a perfect matching in G. The proof of Theorem 7 shows that M is a popular *perfect* matching in G if and only if there is a realization M' of M such that M' satisfies *property 1* (as given in Theorem 7) in the subgraph $G'_{M'}$. This is because for any perfect matching N' in $G'_{M'}$, the symmetric difference $M' \oplus N'$ consists only of alternating cycles as there are no unmatched vertices now. We state this characterization of popular perfect matchings as Proposition 18.

Proposition 18. *A perfect matching M is a popular perfect matching in the hospitals/residents instance $G = (R \cup H, E)$ if and only if there is a realization M' of M such that there is no alternating cycle C with respect to M' in $G'_{M'}$ such that $\mathsf{wt}_{M'}(C) > 0$.*

It follows from [26, Theorem 2] that a perfect matching M' in a marriage instance $G'_{M'}$ is a popular perfect matching if and only if there is no alternating cycle C with respect to M' such that $\mathsf{wt}_{M'}(C) > 0$. Thus we have the following corollary.

Corollary 19. *A perfect matching M in the hospitals/residents instance $G = (R \cup H, E)$ is a popular perfect matching if and only if there exists a realization M' of M such that M' is a popular perfect matching in the marriage instance $G'_{M'} = (R \cup H', E'_{M'})$.*

3.2 Constructing a helpful dual certificate

We know that for a perfect matching M to be a popular perfect matching in G, it is not necessary that it has a realization M' that is a popular perfect matching in the marriage instance G'. It suffices for the matching M' to be a popular perfect matching in the subgraph $G'_{M'}$ of G' (by Corollary 19).

Remark 20. Observe that being a popular perfect matching in the subgraph $G'_{M'}$ is a more relaxed condition than being a popular perfect matching in G' since the *edge covering constraints* in Lemma 16, i.e., the constraints in condition (ii), have to be satisfied only for the edges in $G'_{M'}$ rather than all the edges in G' (recall that $E' \supseteq E'_{M'}$).

Our goal is to find a *min-cost* popular perfect matching in G. Though we know how to find a min-cost popular perfect matching in a marriage instance, we do not know in which marriage instance we should run this algorithm (since $G'_{M'}$ depends on the matching M that we seek).

Let M be any popular perfect matching in G. Since there is a realization M' of M that is a popular perfect matching in $G'_{M'} = (R \cup H', E'_{M'})$, there is a vector $\vec{\gamma} \in \mathbb{R}^n$ (see Lemma 16) that satisfies the following conditions (i)-(iii) where $|R \cup H'| = n$ and $|R| = n_0$.

(i) $\gamma_r \in \{0, -2, -4, \ldots, -2(n_0 - 1)\}$ for all $r \in R$ and $\gamma_{h_i} \in \{0, 2, 4, \ldots, 2(n_0 - 1)\}$ for all $h_i \in H'$.
(ii) $\gamma_r + \gamma_{h_i} \geq \mathsf{wt}_{M'}(r, h_i)$ for all $(r, h_i) \in E'_{M'}$.
(iii) $\sum_{v \in R \cup H'} \gamma_v = 0$.

Let $\vec{\gamma}$ be a dual certificate for M' that minimizes the sum $\sum_{h_i \in H'} \gamma_{h_i}$. The following lemma will be very useful to us.

Lemma 21. *For any two clones h_i and h_j of the same hospital h, we have $\gamma_{h_i} \leq \gamma_{h_j} + 2$.*

Proof. Observe that except for their partners in M', the neighborhoods in $G'_{M'}$ of the two clones h_i and h_j are identical. Consider any edge $(r, h_i) \in E'_{M'}$ such that $(r, h_i) \notin M'$. So $(r, h'_t) \in M'$ for some $h' \neq h$. We have:

$$\begin{aligned}
\mathsf{wt}_{M'}(r, h_i) &\leq \mathsf{vote}_r(h_i, M'(r)) + \mathsf{vote}_h(r, M'(h_i)) && (4)\\
&= \mathsf{vote}_r(h_j, M'(r)) + \mathsf{vote}_h(r, M'(h_i)) && (5)\\
&\leq \mathsf{vote}_r(h_j, M'(r)) + \mathsf{vote}_h(r, M'(h_j)) + 2 && (6)\\
&= \mathsf{wt}_{M'}(r, h_j) + 2 && (7)\\
&\leq \gamma_r + \gamma_{h_j} + 2. && (8)
\end{aligned}$$

In the third constraint, we have $\mathsf{vote}_h(r, M'(h_i)) \leq \mathsf{vote}_h(r, M'(h_j)) + 2$ since $\mathsf{vote}_h(r, M'(h_i)) \leq 1$ and $\mathsf{vote}_h(r, M'(h_j)) \geq -1$.

Suppose $\gamma_{h_i} > \gamma_{h_j} + 2$. Let $(s, h_i) \in M'$. Since M' and $\vec{\gamma}$ are optimal solutions of LP1 and LP2 respectively, we have $\gamma_s + \gamma_{h_i} = \mathsf{wt}_{M'}(s, h_i) = 0$ by complementary slackness. Let us update $\vec{\gamma}$ to $\vec{\gamma}'$ as follows:

- $\gamma'_{h_i} = \gamma_{h_j} + 2$ and $\gamma'_s = -\gamma'_{h_i}$.
- $\gamma'_v = \gamma_v$ for all other vertices v in $R \cup H'$.

Observe that $\sum_{v \in R \cup H'} \gamma'_v = 0$. Thus $\vec{\gamma}'$ satisfies condition (iii) given in Lemma 16.

Constraints (4)-(8) tell us that $\gamma'_r + \gamma'_{h_i} = \gamma_r + \gamma_{h_j} + 2 \geq \mathsf{wt}_{M'}(r, h_i)$ for any $r \neq s$. Since $\gamma'_s + \gamma'_{h_i} = 0 = \mathsf{wt}_{M'}(s, h_i)$, we have $\gamma'_r + \gamma'_{h_i} \geq \mathsf{wt}_{M'}(r, h_i)$ for all $r \in R$. Thus all edges in $E'_{M'}$ that are incident to h_i are covered by $\vec{\gamma}'$.

Since $\gamma'_{h_i} = \gamma_{h_j} + 2 < \gamma_{h_i}$, our update ensures that $\gamma'_s > \gamma_s$. Because $\gamma'_v = \gamma_v$ for all vertices v other than s and h_i, we have $\gamma'_p + \gamma'_{q_j} \geq \mathsf{wt}_{M'}(p, q_j)$ for every (p, q_j) in $E'_{M'}$. Thus $\vec{\gamma}'$ satisfies condition (ii) given in Lemma 16.

So conditions (i)-(iii) given in Lemma 16 are satisfied by $\vec{\gamma}'$. Thus $\vec{\gamma}'$ is a dual certificate for M' in $G'_{M'}$. Moreover, $\sum_{h_i \in H'} \gamma'_{h_i} < \sum_{h_i \in H'} \gamma_{h_i}$, contradicting the choice of $\vec{\gamma}$ as a dual certificate for M' that minimizes this sum. Hence it has to be the case that $\gamma_{h_i} \leq \gamma_{h_j} + 2$. ◻

For any $v \in R \cup H'$, if $|\gamma_v| = 2\ell$ then we say v is in level ℓ in $\vec{\gamma}$. By Lemma 21, for any $h \in H$, all of $h_1, \ldots, h_{\mathsf{cap}(h)}$ are either in the same level (say, ℓ) or in two successive levels (say, ℓ and $\ell+1$) in $\vec{\gamma}$. For any $h \in H$, let $r^h_1, \ldots, r^h_{\mathsf{cap}(h)}$ be the $\mathsf{cap}(h)$ partners of h where $r^h_1 \succ_h \cdots \succ_h r^h_{\mathsf{cap}(h)}$. Let $M' = \cup_{h \in H}\{(r^h_i, h_{\sigma_h(i)}) : 1 \leq i \leq \mathsf{cap}(h)\}$ for some permutation σ_h on $\{1, \ldots, \mathsf{cap}(h)\}$.

Lemma 22. *For any $h \in H$, if k' clones of h are in level ℓ and $k - k'$ clones of h are in level $\ell + 1$ (where $k = \mathsf{cap}(h)$) then the k' clones of h in level ℓ have to be matched in M' to $r^h_1, \ldots, r^h_{k'}$. That is, $r^h_1, \ldots, r^h_{k'}$ are in level ℓ and $r^h_{k'+1}, \ldots, r^h_k$ are in level $\ell + 1$ in $\vec{\gamma}$.*

Proof. Suppose one or more of $r^h_1, \ldots, r^h_{k'}$ is not in level ℓ. Then r^h_i, for some $i \leq k'$, is in level $\ell+1$ and r^h_j, for some $j > k'$, is in level ℓ. So $h_{i'} = M'(r^h_i)$ is in level $\ell + 1$ while $h_{j'} = M'(r^h_j)$ is in level ℓ.

Let s be any neighbor of h in G such that $s \notin M(h)$. Then we have $\mathsf{wt}_{M'}(s, h_{i'}) \leq \mathsf{wt}_{M'}(s, h_{j'})$. This is because $\mathsf{vote}_s(h_{i'}, M'(s)) = \mathsf{vote}_s(h_{j'}, M'(s))$ while $\mathsf{vote}_h(s, r^h_i) \leq \mathsf{vote}_h(s, r^h_j)$ because h prefers r^h_i to r^h_j. Since $\gamma_s + \gamma_{h_{j'}} \geq \mathsf{wt}_{M'}(s, h_{j'}) \geq \mathsf{wt}_{M'}(s, h_{i'})$, let us update $\gamma'_{h_{i'}} = 2\ell$ and $\gamma'_{r^h_i} = -2\ell$ (so $\gamma'_{r^h_i} > \gamma_{r^h_i}$). For any vertex v other than r^h_i and h_i, let $\gamma'_v = \gamma_v$.

It is easy to see that $\vec{\gamma}'$ is a dual certificate for M' in $G'_{M'}$ and $\sum_{h_i \in H'} \gamma'_{h_i} < \sum_{h_i \in H'} \gamma_{h_i}$, contradicting the choice of $\vec{\gamma}$ as a dual certificate for M' that minimizes this sum. Hence it has to be the case that $r_1^h, \ldots, r_{k'}^h$ are in level ℓ and $r_{k'+1}^h, \ldots, r_k^h$ are in level $\ell + 1$. ☐

We are now ready to prove the main result in this section. For any $h \in H$, let $r_1^h, \ldots, r_k^h$ be all the partners of h in M (so $k = \mathsf{cap}(h)$) and $r_1^h \succ_h \cdots \succ_h r_k^h$.

Theorem 23. *Let M be any popular perfect matching in G. For each $h \in H$, there exists an appropriate permutation π_h on $\{1, \ldots, k\}$ (where $k = \mathsf{cap}(h)$) such that $M'' = \cup_{h \in H}\{(r_i^h, h_{\pi_h(i)}) : 1 \le i \le k\}$ is a popular perfect matching in G'.*

Proof. Let $h \in H$ and let $\mathsf{cap}(h) = k$. Consider the following two cases.

Case 1. All the clones of h are in the same level $\ell \in \{0, 1, \ldots, n_0 - 1\}$ in $\vec{\gamma}$.

This is the easy case—we will set $\pi_h : \{1, \ldots, k\} \to \{1, \ldots, k\}$ to be the identity function. So $(r_1^h, h_1), \ldots, (r_k^h, h_k)$ are in M''. For all $1 \le i \le k$: we have $\gamma_{r_i^h} = -2\ell$ and $\gamma_{h_i} = 2\ell$.

Case 2. All the clones of h are *not* in the same level.

So by Lemma 21, all the clones $h_1, \ldots, h_k$ are in two successive levels ℓ and $\ell + 1$ in $\vec{\gamma}$. Say, k' clones of h are in level ℓ and $k - k'$ clones of h are in level $\ell + 1$. The vertices $r_1^h, \ldots, r_k^h$ will be matched in M'' to $h_1, \ldots, h_k$ as follows:

- for $i \in \{1, \ldots, k'\}$, the vertex r_i^h is matched to $h_{k-k'+i}$;
- for $i \in \{k' + 1, \ldots, k\}$, the vertex r_i^h is matched to $h_{i-k'}$.

In more detail, we know from Lemma 22 that $r_1^h, \ldots, r_{k'}^h$ are in level ℓ and $r_{k'+1}^h, \ldots, r_k^h$ are in level $\ell + 1$. The residents $r_1^h, \ldots, r_{k'}^h$ are matched in M'' to $h_{k-k'+1}, \ldots, h_k$, respectively and the residents $r_{k'+1}^h, \ldots, r_k^h$ in level $\ell + 1$ are matched in M'' to $h_1, \ldots, h_{k-k'}$, respectively. Thus the matching M'' rearranges the clones of h so that the following holds:

- $h_1, \ldots, h_{k-k'}$ are placed in level $\ell + 1$.
- $h_{k-k'+1}, \ldots, h_k$ are placed in level ℓ.

Observe that $M'' = \cup_{h \in H}\{(r_i, h_{\pi_h(i)}) : 1 \le i \le k\}$ where in case 1 (i.e., all the clones of h are in the same level), $\pi_h(i) = i$ for all $i \in \{1, \ldots, k\}$ and for case 2, π_h is defined as follows:

$$\pi_h(i) = \begin{cases} k - k' + i & \text{if } 1 \le i \le k'; \\ i - k' & \text{if } k' + 1 \le i \le k. \end{cases}$$

The entire matching M'' is thus defined. We claim that M'' is the realization of M that we seek.

Lemma 24. *M'' is a popular perfect matching in G'.*

The proof of Lemma 24 is given below. This finishes the proof of Theorem 23. ☐

Proof of Lemma 24. We will prove M'' to be a popular perfect matching in G' by defining a dual certificate $\vec{\alpha}$ for M'' in G'. Recall that $\vec{\gamma}$ is a dual certificate for the realization M' to be a popular perfect matching in $G'_{M'}$.

1. For any $r \in R$: set $\alpha_r = \gamma_r$.

2. For any $h \in H$ and $i \in \{1, \ldots, k\}$: set $\alpha_{h_i} = 2t$ where t is h_i's *level* as defined in the proof of Theorem 23.

So in case 1 of the proof of Theorem 23, when all the clones of h are in the same level ℓ, we have $\alpha_{h_i} = 2\ell$ for every $i \in \{1, \ldots, \mathsf{cap}(h)\}$. In case 2 of the proof of Theorem 23, the vertices $h_1, \ldots, h_{k-k'}$ are in level $\ell + 1$ and the vertices $h_{k-k'+1}, \ldots, h_k$ are in level ℓ where $k = \mathsf{cap}(h)$; so $\alpha_{h_i} = 2(\ell + 1)$ for $1 \le i \le k - k'$ and $\alpha_{h_i} = 2\ell$ for $k - k' + 1 \le i \le k$.

Let $h \in H$. We know the edges $(r_1^h, h_{\sigma_h(1)}), \ldots, (r_k^h, h_{\sigma_h(k)})$ are in M'. We will now show that $\alpha_{r_i^h} + \alpha_{h_j} = \mathsf{wt}_{M''}(r_i^h, h_j)$ for all $i, j \in \{1, \ldots, k\}$.

This is immediate to check for case 1 in the proof of Theorem 23 since $\alpha_{r_i^h} = -2\ell$ and $\alpha_{h_j} = 2\ell$ for all $i, j \in \{1, \ldots, k\}$. Recall that $r_1^h \succ_h \cdots \succ_h r_k^h$ and $h_1 \succ_r \cdots \succ_r h_k$ for all r adjacent to h in G. So we have $\alpha_{r_i^h} + \alpha_{h_j} = 0 = \mathsf{wt}_{M''}(r_i^h, h_j)$ for all $i, j \in \{1, \ldots, k\}$.

We will now check the equality $\alpha_{r_i^h} + \alpha_{h_j} = \mathsf{wt}_{M''}(r_i^h, h_j)$ for case 2 in the proof of Theorem 23, so $h_1, \ldots, h_{k-k'}$ are in level $\ell + 1$ and $h_{k-k'+1}, \ldots, h_k$ are in level ℓ.

(1) For $i \in \{1, \ldots, k'\}$ and $j \in \{k - k' + 1, \ldots, k\}$, observe that $\mathsf{wt}_{M''}(r_i^h, h_j) = 0$. Since $\alpha_{r_i^h} = -2\ell$ and $\alpha_{h_j} = 2\ell$, we have $\alpha_{r_i^h} + \alpha_{h_j} = 0 = \mathsf{wt}_{M''}(r_i^h, h_j)$ for all these pairs (i, j).

(2) Similarly, for $i \in \{k' + 1, \ldots, k\}$ and $j \in \{1, \ldots, k - k'\}$, we have $\mathsf{wt}_{M''}(r_i^h, h_j) = 0$. Since $\alpha_{r_i^h} = -2(\ell + 1)$ and $\alpha_{h_j} = 2(\ell + 1)$, we have $\alpha_{r_i^h} + \alpha_{h_j} = 0 = \mathsf{wt}_{M''}(r_i^h, h_j)$ for all these pairs (i, j).

(3) Let us now consider a pair (r_i^h, h_j) where $i \in \{1, \ldots, k'\}$ and $j \in \{1, \ldots, k - k'\}$. We have $\mathsf{wt}_{M''}(r_i^h, h_j) = 2$ for such an (i, j). Since $\alpha_{r_i^h} = -2\ell$ and $\alpha_{h_j} = 2(\ell + 1)$, we have $\alpha_{r_i^h} + \alpha_{h_j} = 2 = \mathsf{wt}_{M''}(r_i^h, h_j)$ for all these pairs (i, j).

(4) Finally, let us consider a pair (r_i, h_j) where $i \in \{k' + 1, \ldots, k\}$ and $j \in \{k - k' + 1, \ldots, k\}$. We have $\mathsf{wt}_{M''}(r_i^h, h_j) = -2$ for such an (i, j). Since $\alpha_{r_i^h} = -2(\ell + 1)$ and $\alpha_{h_j} = 2\ell$, we have $\alpha_{r_i^h} + \alpha_{h_j} = -2 = \mathsf{wt}_{M''}(r_i^h, h_j)$ for all these pairs (i, j).

Recall that the difference between the graphs G' and $G'_{M'}$ is that for every $h \in H$ where $(r_i^h, h) \in M$, the edges (r_i^h, h_j) for $h_j \neq M'(r_i^h)$ are missing in $G'_{M'}$. We just have checked all the edges (r_i^h, h_j) for all $i, j \in \{1, \ldots, \mathsf{cap}(h)\}$ are covered by $\vec{\alpha}$.

– We will now show that all the *non-matching* edges are covered by $\vec{\alpha}$ as well. That is, for any $(r', h) \in E \setminus M$, we need to show that $\alpha_{r'} + \alpha_{h_j} \ge \mathsf{wt}_{M''}(r', h_j)$ for all $i \in \{1, \ldots, k\}$.

For any $h \in H$ and $i \in \{1, \ldots, \mathsf{cap}(h)\}$, let $\sigma_h(i) = j$ and $\pi_h(i) = j'$. Observe that $\mathsf{wt}_{M'}(r', h_j) = \mathsf{wt}_{M''}(r', h_{j'})$. This is because $M'(h_j) = r_i^h = M''(h_{j'})$ and $\mathsf{vote}_{r'}(M'', h_j) = \mathsf{vote}_{r'}(M', h_{j'})$ since r' is matched in both M' and M'' to clones of the same hospital h'.

* Since $(r_i^h, h_j) \in M'$ and $(r_i^h, h_{j'}) \in M''$, we have $\alpha_{h_{j'}} = -\alpha_{r_i^h} = -\gamma_{r_i^h} = \gamma_{h_j}$. Because $\gamma_{r'} + \gamma_{h_j} \ge \mathsf{wt}_{M'}(r', h_j)$, we get $\alpha_{r'} + \alpha_{h_{j'}} = \gamma_{r'} + \gamma_{h_j} \ge \mathsf{wt}_{M'}(r', h_j) = \mathsf{wt}_{M''}(r', h_{j'})$. Thus every non-matching edge $(r', h_{j'})$ is also covered by $\vec{\alpha}$.
* Since $\alpha_{r_i^h} + \alpha_{h_{j'}} = 0$ for any edge $(r_i^h, h_{j'}) \in M''$, we have $\sum_{v \in R \cup H'} \alpha_v = 0$. Hence conditions (i)-(iii) for dual certificates are satisfied by $\vec{\alpha}$; thus $\vec{\alpha}$ is a dual certificate for M'' in G'. In other words, M'' is a popular perfect matching in G' (by Lemma 16).

This finishes the proof of the lemma. ☐

This completes the proof of Theorem 5 stated in Section 1. It is now straightforward to show a polynomial-time algorithm to find a min-cost popular perfect matching in a hospitals/residents instance $G = (R \cup H, E)$.

1. Compute the corresponding marriage instance $G' = (R \cup H', E')$.
2. Find a min-cost popular perfect matching N' in G' using the algorithm in [26] for marriage instances.
3. Return the corresponding matching N in G by identifying all clones of the same hospital.

For any popular perfect matching M in G, we know there is some realization M'' such that M'' is a popular perfect matching in G' (by Theorem 5). Conversely, for any popular perfect matching M' in G', since M''s dual certificate obeys conditions (i)-(iii) in G', it also obeys these constraints in the subgraph $G'_{M'}$. Hence the corresponding matching $M = p(M')$ is a popular perfect matching in G (by Corollary 19). Thus solving the min-cost popular perfect matching problem in the marriage instance G' solves the min-cost popular perfect matching problem in the hospitals/residents instance G. So the matching M returned by the above algorithm is a min-cost popular perfect matching in G and we can conclude Theorem 6 stated in Section 1.

Completing the proof of Theorem 3. Recall that Section 2 has the algorithm for the min-cost popular matching problem in a hospitals/residents instance $G = (R \cup H, E)$ when $|R| \neq \sum_{h \in H} \mathsf{cap}(h)$. In a hospitals/residents instance $G = (R \cup H, E)$ with $|R| = \sum_{h \in H} \mathsf{cap}(h)$ and complete preferences, a matching M is popular if and only if M is a popular perfect matching. Thus our algorithm for min-cost popular perfect matching completes our polynomial-time algorithm for the min-cost popular matching problem in a hospitals/residents instance with complete preferences.

4 Conclusions and open problems

For any many-to-one instance $G = (R \cup H, E)$ where each vertex has a strict and complete preference order over vertices on the other side, we gave a polynomial-time algorithm to solve the min-cost popular matching problem in G when there is a function $\mathsf{cost} : E \to \mathbb{R}$. Our algorithm includes a subroutine for computing a min-cost popular perfect matching in a many-to-one instance with incomplete preferences. It is known that it is NP-hard to find a min-cost popular matching in a marriage (one-to-one) instance with incomplete preferences.

Our method of reducing the min-cost popular perfect matching problem in a many-to-one instance G to the min-cost popular perfect matching problem in the corresponding one-to-one instance G' does not work for the min-cost popular *maximum* matching problem. Section 1.1 describes a many-to-one instance G whose set of popular maximum matchings is a strict superset of the set of popular maximum matchings in the corresponding one-to-one instance G'. Settling the computational complexity of the min-cost popular maximum matching problem in a many-to-one instance with strict and incomplete preferences is an interesting open problem.

Another interesting open problem is to extend our results to the many-to-many setting. As remarked by a reviewer at MATCH-UP 2024, it is known that for stable matchings, the many-to-many setting cannot be reduced to the marriage case via cloning [18, Footnote 6]. Thus generalizing our results to the many-to-many setting is a challenging open problem.

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Missing proofs

Proof of Lemma 9. Let N be a maximum matching in G such that $|N(h)| < |M(h)|$ for some hospital h. We know from Theorem 7 that there is a realization M' of M such that properties 1 and 2 hold in $G'_{M'}$. Recall the realization N' of N in $G'_{M'}$ from the proof of Theorem 7. Since there is a hospital h such that $|N(h)| < |M(h)|$, the symmetric difference $M' \oplus N'$ cannot consist only of alternating cycles; thus there is an even length alternating path ρ_h in $M' \oplus N'$ with a clone of h as an endpoint. The following equality follows from the construction of N' in G'_M where the first summation is over alternating cycles in $M' \oplus N'$ and the second summation is over even length alternating paths in $M' \oplus N'$.

$$\Delta(M,N) = \sum_{v \in R \cup H} \mathsf{vote}_v(M,N) = -\sum_C \mathsf{wt}_{M'}(C) - \sum_\rho (\mathsf{wt}_{M'}(\rho) - 1).$$

Since M is a popular matching, Theorem 7 tells us that $\mathsf{wt}_{M'}(C) \le 0$ for all alternating cycles C in $M' \oplus N'$ and $\mathsf{wt}_{M'}(\rho) \le 0$ for all even length alternating paths ρ in $M' \oplus N'$ (recall that the endpoints of ρ cannot be clones of the same hospital—see Footnote 6). Furthermore, we know there is at least one alternating path ρ_h in $M' \oplus N'$. Thus $\Delta(M,N) \ge 1$.

Observe that $\Delta(M,N) + \Delta(N,M) \le 0$. This is because $\Delta(M,N)$ compares M and N in the most adversarial way for M (equivalently, most favorably for N) while $\Delta(N,M)$ compares them in the most adversarial way for N. Thus $\Delta(M,N) \ge 1$ implies that $\Delta(N,M) \le -1$. In other words, N is *not* a popular matching in G. □

Proof of Lemma 10. Suppose M is a popular matching where neither h' nor h'' is fully matched for hospitals h', h'' where $h' \ne h''$. If M is not stable then there is a resident-hospital pair (r,h) that blocks M. So r prefers h to its partner $\hat{h}$ in M and either h is not fully matched or it prefers r to its least preferred partner r' in M. Let M' be a realization of M such that properties 1 and 2 hold in $G'_{M'}$ (see Theorem 7).

Let $(r, \hat{h}_i) \in M'$. If h is not fully matched in M, then some clone of h (say, h_t) is left unmatched in M'. As seen in the proof of Lemma 8, there is a length 2 alternating path $\rho = h_t - r \sim \hat{h}_i$ with respect to M' in $G'_{M'}$ such that $\mathsf{wt}_{M'}(\rho) = 2$. This contradicts the assumption that M' obeys property 2 in $G'_{M'}$. So assume h is fully matched in M and let $(r', h_t) \in M'$ where r' is the least preferred partner of h in M.

Recall that there are two different hospitals h' and h'' that are not fully matched in M. If $\hat{h} = h'$ then let $h^* = h''$; else let $h^* = h'$. Consider the length 4 alternating path $h^*_t - r' \sim h_t - r \sim \hat{h}_i$ with respect to M' in $G'_{M'}$ where h^*_t is left unmatched in M'. As seen in the proof of Theorem 7, $\mathsf{wt}_{M'}(\rho') \geq 2$. It follows from the definition of h^* that the endpoints of ρ' are not clones of each other. This again contradicts the fact that M' obeys property 2 in $G'_{M'}$. Hence M cannot admit a blocking edge, in other words, M is stable in G. ◻

Proof of Lemma 12. Consider any edge $e = (r, h_i) \in R_+ \times H'_-$. Let $e' = (r, h'_j)$ and $e'' = (s, h_i)$ be in M'. Since $r \in R_+$ and $h_i \in H'_-$, it follows that e'_+ and e''_- are in $\widetilde{M}$. Because $\widetilde{M}$ is a stable matching in $\widetilde{G}$, neither e_+ nor e_- blocks $\widetilde{M}$.

- Recall that h_i prefers any color $+$ edge to any color $-$ edge, so h_i prefers e_+ to e''_-. However e_+ does not block $\widetilde{M}$. Thus it has to be the case that r prefers e'_+ to e_+, in other words, r prefers its partner h'_j in M' to h_i.
- Recall that any resident prefers any color $-$ edge to any color $+$ edge, so r prefers e_- to e'_+. Since e_- does not block $\widetilde{M}$, it has to be the case that h_i prefers e''_- to e_-; in other words, h_i prefers its partner s in M' to r.

Thus both r and h_i prefer their respective partners in M' to each other. Hence $\mathsf{wt}_{M'}(e) = -2$.

We will now show that for any $e \in R_+ \times H'_+$, we have $\mathsf{wt}_{M'}(e) \leq 0$. If $e = (r, h'_i) \in M'$ then $\mathsf{wt}_{M'}(e) = 0$. Hence suppose $e \notin M'$. Then (i) $e'_+ \in \widetilde{M}$ where $e' = (r, h''_j)$ for some hospital $h''_j \in H'$ and (ii) either $e''_+ \in \widetilde{M}$ where $e'' = (s, h'_i)$ for some resident s or the vertex h'_i is unmatched in $\widetilde{M}$.[8] Since e_+ does not block $\widetilde{M}$ in $\widetilde{G}$, it has to be the case that (i) r prefers e'_+ to e_+ or (ii) h'_i is matched along e''_+ which it prefers to e_+. So it is the case that (i) r prefers its partner in M' to h'_i or (ii) h' prefers its partner in M' to r. Hence $\mathsf{wt}_{M'}(e) \leq 0$. An analogous argument shows that for any $e \in R_- \times H'_-$, we have $\mathsf{wt}_{M'}(e) \leq 0$. This finishes the proof of this lemma. ◻

Proof of Lemma 13. Suppose $\mathsf{wt}_{M'}(r', h_i) \geq 0$ for some edge (r', h_i) in $G'_{M'}$ where $r' \neq r$. Let (r, h'_j) be a blocking edge to M' in $G'_{M'}$ and let $M'(h'_j) = s$. If $s = r'$ then we have the length-4 alternating cycle $C = s \sim h'_j - r \sim h_i - s$ such that $\mathsf{wt}_{M'}(C) \geq 0 + 2 + 0 + 0 = 2$. Thus the cycle C violates property 1 given in Theorem 7, a contradiction. Hence $s \neq r'$.

Consider the length 6 alternating path $\rho = h_t - s \sim h'_j - r \sim h_i - r' \sim M'(r')$ where h_t is a clone of h left unmatched in M'. Observe that $\mathsf{wt}_{M'}(\rho') \geq 0 + 0 + 2 + 0 + 0 + 0 = 2$. If $M'(r')$ is a clone of h then the edge (r', h_i) would not be present in $G'_{M'}$, hence $M'(r')$ cannot be a clone of h. Thus the path ρ violates property 2 given in Theorem 7, a contradiction. ◻

Proof of Lemma 14. It is residents $r_1, \ldots, r_\ell$ and hospitals $h_{k-\ell+1}, \ldots, h_k$ that are matched along color $-$ edges in $\widetilde{M}$. Define $R_- = \{r_1, \ldots, r_\ell\}$ and $H'_- = \{h_{k-\ell+1}, \ldots, h_k\}$; let $R_+ = R \setminus R_-$ and $H'_+ = H \setminus H'_-$ (see Fig. 1). The stability of $\widetilde{M}$ in $\widetilde{G}$ will be proved via the following three claims.

[8] For h'_i to be unmatched in $\widetilde{M}$, it has to be the case that $h' = h$ and $i \in \{k+1, \ldots, \mathsf{cap}(h)\}$.

Claim 1. There is no blocking edge incident to any vertex in R_-. It is easy to show there is no blocking edge incident to any of $r_1, \ldots, r_\ell$. Recall that each resident prefers any color $-$ edge to any color $+$ edge. Thus there is no edge $(r_i, h'_j)_+$ that can block $\widetilde{M}$ for any $h' \in H'$ and $i \in \{1, \ldots, \ell\}$. The only color $-$ edges in $\widetilde{G}$ are incident to $h_1, \ldots, h_k$. Since hospitals $h_1, \ldots, h_{k-\ell}$ are matched along color $+$ edges which they prefer to any color $-$ edge, there is no edge $(r_i, h_j)_-$ that blocks $\widetilde{M}$, where $j \in \{1, \ldots, k-\ell\}$. Furthermore, for $k - \ell + 1 \leq j \leq k$, the j-th best clone of h gets the j-th best partner in $\{r_1, \ldots, r_\ell\}$ in $\widetilde{M}$. Thus there is no edge $(r_i, h_j)_-$ that blocks $\widetilde{M}$, where $j \in \{k - \ell + 1, \ldots, k\}$.

Claim 2. There is no blocking edge incident to any vertex in H'_-. For any $r' \in R_+$, Lemma 13 tells us that $\mathsf{wt}_{M'}(r', M'(r_\ell)) = -2$. This implies that $M'(r_\ell)$ (and thus any clone of h) prefers r_ℓ to any resident that is outside $\{r_1, \ldots, r_\ell\}$. Thus each $h_j \in H'_-$ prefers its partner in $\widetilde{M}$ to any resident in R_+ (note that R_+ includes $r_{\ell+1}, \ldots, r_k$ which are h's worst $k - \ell$ partners in M). Similarly, we claim that each resident $r' \in R_+$ prefers its partner in $\widetilde{M}$ to any hospital in H_-. If $r' \notin M(h)$ then this claim follows from Lemma 13. Else $r' \in M(h) \cap R_+$, i.e., $r' \in \{r_{\ell+1}, \ldots, r_k\}$, and each of these residents prefers its partner in $\widetilde{M}$ (this is one of $h_1, \ldots, h_{k-\ell}$) to any of $h_{k-\ell+1}, \cdots, h_k$. Hence neither e_+ nor e_- blocks $\widetilde{M}$ where $e \in R_+ \times H'_-$. Recall that we already showed there is no blocking edge to $\widetilde{M}$ in $R_- \times H'_-$.

Claim 3. There is no blocking edge incident to any vertex in $R_+ \cup H'_+$. It follows from the definition of the resident r_ℓ that all blocking edges to M' in $G'_{M'}$ are incident to residents in $\{r_1, \ldots, r_\ell\}$, i.e., to residents in R_-. So no edge in $R_+ \times H'_+$ that is present in $G'_{M'}$ blocks M'. Thus e_+ (where $e \in R_+ \times H'_+$) does not block the matching $\widetilde{M}$ restricted to color $+$ edges. The edges present in $\widetilde{G}$ and missing in $G'_{M'}$ are between a resident r and clones of its partner in M'. In the matching $\widetilde{M}$, the hospital h'_i where $h' \neq h$ and $1 \leq i \leq \mathsf{cap}(h')$ gets the i-th best partner of h' in M. Thus there is no blocking edge in $\widetilde{G}$ between h'_i and any resident in R_+. Similarly, $h_i \in \{h_1, \ldots, h_{k-\ell}\}$ is matched in $\widetilde{M}$ to the i-th best partner of h in $\{r_{\ell+1}, \ldots, r_k\}$. Thus there is no blocking edge $(r_i, h_j)_+$ to $\widetilde{M}$ where $\ell + 1 \leq i \leq k$ and $1 \leq j \leq k - \ell$.

Hence there is no edge in $\widetilde{G}$ that blocks $\widetilde{M}$. So $\widetilde{M}$ is a stable matching in $\widetilde{G}$. $\square$