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Abstract

Let G be a digraph where every node has preferences over its incoming edges. The preferences of a node extend naturally to preferences over *branchings*, i.e., directed forests; a branching B is *popular* if B does not lose a head-to-head election (where nodes cast votes) against any branching. Such popular branchings have a natural application in liquid democracy. The popular branching problem is to decide if G admits a popular branching or not. We give a characterization of popular branchings in terms of *dual certificates* and use this characterization to design an efficient combinatorial algorithm for the popular branching problem. When preferences are weak rankings, we use our characterization to formulate the *popular branching polytope* in the original space and also show that our algorithm can be modified to compute a branching with *least unpopularity margin*. When preferences are strict rankings, we show that "approximately popular" branchings always exist.

1 Introduction

Let G be a directed graph where every node has preferences (in partial order) over its incoming edges. When G is simple, the preferences can equivalently be defined on in-neighbors. We define a *branching* as a subgraph of G that is a directed forest where any node has in-degree at most 1; a node with in-degree 0 is a *root*. The problem we consider here is to find a branching that is *popular*.

Given any pair of branchings, we say a node u prefers the branching where it has a more preferred incoming edge (being a root is u's worst choice). If neither incoming edge is preferred to the other, then u is indifferent between the two branchings. So any pair of branchings, say B and B', can be compared by asking for the majority opinion, i.e., every node opts for the branching that it prefers, and it abstains if it is indifferent between them. Let $\phi(B, B')$ (resp., $\phi(B', B)$) be the number of nodes that

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prefer B (resp., B') in the B-vs-B' comparison. If $\phi(B', B) > \phi(B, B')$, then we say B' is *more popular* than B.

Definition 1. A branching B is popular in G if there is no branching that is more popular than B. That is, $\phi(B, B') \geq \phi(B', B)$ for all branchings B' in G.

An application in computational social choice. We see the main application of popular branchings within *liquid democracy*. Suppose there is an election where a specific issue should be decided upon, and there are several proposed alternatives. Every individual voter has an opinion on these alternatives, but might also consider certain other voters as being better informed than her. Liquid democracy is a novel voting scheme that provides a middle ground between the feasibility of representative democracy and the idealistic appeal of direct democracy [4]: Voters can choose whether they delegate their vote to another, well-informed voter or cast their vote themselves. As the name suggests, voting power flows through the underlying network, or in other words, delegations are transitive. During the last decade, this idea has been implemented within several online decision platforms such as *Sovereign* and *LiquidFeedback*[1] and was used for internal decision making at Google [22] and political parties, such as the German *Pirate Party* or the Swedish party *Demoex*.

In order to circumvent *delegation cycles*, e.g., a situation in which voter x delegates to voter y and vice versa, and to enhance the expressiveness of delegation preferences, several authors proposed to let voters declare a set of acceptable representatives [20] together with a preference relation among them [5, 22, 29]. Then, a mechanism selects one of the approved representatives for each voter, avoiding delegation cycles. Similarly as suggested in [6], we additionally assume that voters accept themselves as their least preferred approved representative.

This reveals the connection to branchings in simple graphs (with loops), where nodes correspond to voters and the edge (x, y) indicates that voter x is an approved representative of voter y.[2] Every root in the branching casts a weighted vote on behalf of all her descendants. What is a good mechanism to select representatives for voters? A crucial aspect in liquid democracy is the *stability* of the delegation process [3, 14]. For the model described above, we propose popular branchings as a new concept of stability, i.e., the majority of the electorate will always weakly prefer to delegate votes along the edges of a popular branching as opposed to delegating along the edges of any other branching.

Not every directed graph admits a popular branching. Consider the following simple graph on four nodes a, b, c, d where a, b (similarly, c, d) are each other's top choices, while a, c (similarly, b, d) are each other's second choices. There is no edge between a, d (similarly, b, c). Consider the branching $B = \{(a, b), (a, c), (c, d)\}$. A more popular branching is $B' = \{(d, c), (c, a), (a, b)\}$. Observe that a and c prefer B' to B, while d prefers B to B' and b is indifferent between B and B'. We can similarly obtain a branching $B'' = \{(b, a), (b, d), (d, c)\}$ that is more popular than B'. It is easy to check that this instance has no popular branching.

[1] See www.democracy.earth and www.interaktive-demokratie.org, respectively.

[2] Typically, such a delegation is represented by an edge (y, x); for the sake of consistency with downward edges in a branching, we use (x, y).

1.1 Our Problem and Results

The popular branching problem is to decide if a given digraph G admits a popular branching or not, and if so, to find one. We show that determining whether a given branching B is popular is equivalent to solving a min-cost arborescence problem in an extension of G with appropriately defined edge costs (these edge costs are a *function* of the arborescence). The dual LP to this arborescence problem gives rise to a laminar set system that serves as a certificate for the popularity of B if it is popular. This dual certificate proves crucial in devising an algorithm for efficiently solving the popular branching problem.

Theorem 2. *Given a directed graph G where every node has preferences in arbitrary partial order over its incoming edges, there is a polynomial-time algorithm to decide if G admits a popular branching or not, and if so, to find one.*

The proof of Theorem 2 is presented in Section 3; it is based on a characterization of popular branchings that we develop in Section 2. In applications like liquid democracy, it is natural to assume that the preference order of every node is a *weak ranking*, i.e., a ranking of its incoming edges with possible ties. In this case, the proof of correctness of our popular branching algorithm leads to a formulation of the *popular branching polytope* $\mathcal{B}_G$, i.e., the convex hull of incidence vectors of popular branchings in G.

Theorem 3. *Let G be a digraph on n nodes and m edges where every node has a weak ranking over its incoming edges. The popular branching polytope of G admits a formulation of size $O(2^n)$ in $\mathbb{R}^m$. Moreover, this polytope has $\Omega(2^n)$ facets.*

We also show an extended formulation of $\mathcal{B}_G$ in $\mathbb{R}^{m+mn}$ with $O(mn)$ constraints. When G has edge costs and node preferences are weak rankings, the min-cost popular branching problem can be efficiently solved. So we can efficiently solve extensions of the popular branching problem, such as finding one that minimizes the largest rank used or one with given forced/forbidden edges.

Relaxing popularity. Since popular branchings need not always exist in G, this motivates relaxing popularity to *approximate popularity*—do approximately popular branchings always exist in any instance G? An approximately popular branching B may lose an election against another branching, however the extent of this defeat will be bounded. There are two measures of unpopularity: *unpopularity factor* $u(\cdot)$ and *unpopularity margin* $\mu(\cdot)$. These are defined as follows:

$$u(B) = \max_{\phi(B',B)>0} \frac{\phi(B',B)}{\phi(B,B')} \quad \text{and} \quad \mu(B) = \max_{B'} \phi(B',B) - \phi(B,B').$$

A branching B is popular if and only if $u(B) \leq 1$ or $\mu(B) = 0$. We show the following results (Theorems marked by an asterisk ($\star$) are proved in the Appendix).

Theorem 4 ($\star$)**.** *A branching with minimum unpopularity margin in a digraph where every node has a weak ranking over its incoming edges can be efficiently computed. In contrast, when node preferences are in arbitrary partial order, the minimum unpopularity margin problem is* NP*-hard.*

Theorem 5 ($\star$). *Let G be a digraph where every node has a strict ranking over its incoming edges. Then there always exists a branching B in G with $u(B) \le \lfloor \log n \rfloor$. Moreover, for every n, we can show an instance G_n on n nodes with strict rankings such that $u(B) \ge \lfloor \log n \rfloor$ for every branching B in G_n.*

Hardness results for restricted popular branching problems. A natural optimization problem here is to compute a popular branching where no tree is large. In liquid democracy, a large-sized tree shows a high concentration of power in the hands of a single voter, and this is harmful for social welfare [20]. When there is a fixed subset of root nodes in a directed graph, it was shown in [20] that it is NP-hard to find a branching that minimizes the size of the largest tree. To translate this result to popular branchings, we need to allow ties, whereas Theorem 6 below holds even for strict rankings. Another natural restriction is to limit the out-degree of nodes; Theorem 6 also shows that this variant is computationally hard.

Theorem 6 ($\star$). *Given a digraph G where each node has a strict ranking over its incoming edges, it is NP-hard to decide if there exists*

(a) a popular branching in G where each node has at most 9 descendants;

(b) a popular branching in G with maximum out-degree at most 2.

1.2 Background and Related Work

The notion of popularity was introduced by Gärdenfors [19] in 1975 in the domain of bipartite matchings. Algorithmic questions in popular matchings have been well-studied for the last 10-15 years [1, 2, 8, 9, 15, 16, 21, 23, 24, 25, 26, 27, 30].

Algorithms for popular matchings were first studied in the *one-sided* preferences model where vertices on only one side of the bipartite graph have preferences over their neighbors. Popular matchings need not always exist here and there is an efficient algorithm to solve the popular matching problem [1]. The functions unpopularity factor/margin were introduced in [30] to measure the *unpopularity* of a matching; it was shown in [30] that it is NP-hard to compute a matching that minimizes either of these quantities. In the domain of bipartite matchings with *two-sided* strict preferences, popular matchings always exist since stable matchings always exist [18] and every stable matching is popular [19].

The concept of popularity has previously been applied to (undirected) spanning trees [10, 11, 12]. In contrast to our setting, voters have rankings over the entire edge set. This allows for a number of different ways to derive preferences over trees, most of which lead to hardness results.

Techniques. We characterize popular branchings in terms of *dual certificates*. This is analogous to characterizing popular matchings in terms of *witnesses* (see [15, 24, 26]). However, witnesses of popular matchings are in $\mathbb{R}^n$ and these are far simpler than dual certificates. A dual certificate is an appropriate family of subsets of the node set V. A certificate of size k implies that the unpopularity margin of the branching is at most $n - k$. Our algorithm constructs a partition $\mathcal{X}'$ of V such that if G admits popular branchings, then there has to be *some* popular branching in G with a dual certificate of size n supported by $\mathcal{X}'$. Moreover, when nodes have weak rankings, $\mathcal{X}'$ supports some dual certificate of size n to *every* popular branching in G: this leads to the formulation of $\mathcal{B}_G$ (see Section 4). Our positive results on low unpopularity branchings are extensions of our algorithm.

Notation. The preferences of node v on its incoming edges are given by a strict partial order $\prec_v$, so $e \prec_v f$ means that v prefers edge f to edge e. We use $e \sim_v f$ to denote that v is indifferent between e and f, that is, neither $e \prec_v f$ nor $e \succ_v f$ holds. The relation $\prec_v$ is a *weak ranking* if $\sim_v$ is transitive. In this case, $\sim_v$ is an equivalence relation and there is a strict order on the equivalence classes. When each equivalence class has size 1, we call it a *strict ranking*.

2 Dual Certificates

We add a dummy node r to $G = (V_G, E_G)$ as the *root* and make (r, v) the least preferred incoming edge of any node v in G. Let $D = (V \cup \{r\}, E)$ be the resulting graph where $V = V_G$ and $E = E_G \cup \{(r, u) : u \in V\}$. An *$r$-arborescence* in D is an out-tree with root r (throughout the paper, all arborescences are assumed to be rooted at r and to span V, unless otherwise stated).

Note that there is a one-to-one correspondence between branchings in G and arborescences in D (simply make r the parent of all roots of the branching). A branching is popular in G if and only if the corresponding arborescence is popular among all arborescences in D.[3] We will therefore prove our results for arborescences in D. The corresponding results for branchings in G follow immediately by projection, i.e., removing node r and its incident edges.

Let A be an arborescence in D. There is a simple way to check if A is popular in D. Let $A(v)$ be the incoming edge of v in A. For $e = (u, v)$ in D, define:

$$c_A(e) := \begin{cases} 0, & \text{if } e \succ_v A(v), \quad \text{i.e., } v \text{ prefers } e \text{ to } A(v); \\ 1, & \text{if } e \sim_v A(v), \quad \text{i.e., } v \text{ is indifferent between } e \text{ and } A(v); \\ 2, & \text{if } e \prec_v A(v), \quad \text{i.e., } v \text{ prefers } A(v) \text{ to } e. \end{cases}$$

Observe that $c_A(A) = |V| = n$ since $c_A(e) = 1$ for every $e \in A$. Let A' be any arborescence in D and let $\Delta(A, A') = \phi(A, A') - \phi(A', A)$ be the difference in the number of votes for A and the number of votes for A' in the A-vs-A' comparison. Observe that $c_A(A') = \Delta(A, A') + n$. Thus, $c_A(A') \geq n = c_A(A)$ if and only if $\Delta(A, A') \geq 0$. So we can conclude the following.

Proposition 7. Let A' be a min-cost arobrescence in D with respect to c_A. Then $\mu(A) = n - c_A(A')$. In particular, A is popular in D if and only if it is a min-cost arborescence in D with edge costs given by c_A.

Consider the following linear program LP1, which computes a min-cost arborescence in D, and its dual LP2. For any non-empty $X \subseteq V$, let $\delta^-(X)$ be the set of edges

[3] Note that, by the special structure of D, this is equivalent to A being a popular branching in D.

entering the set X in the graph D.

$$\begin{aligned} & \text{minimize} \sum_{e\in E} c_A(e)\cdot x_e & \text{(LP1)} \\ \text{subject to} \quad & \sum_{e\in\delta^-(X)} x_e \ \geq\ 1 \quad \forall\, X\subseteq V,\ X\neq\emptyset \\ & x_e \ \geq\ 0 \quad \forall\, e\in E. \end{aligned}$$

$$\begin{aligned} & \text{maximize} \sum_{X\subseteq V,\, X\neq\emptyset} y_X & \text{(LP2)} \\ \text{subject to} \quad & \sum_{X:\,\delta^-(X)\ni e} y_X \ \leq\ c_A(e) \quad \forall\, e\in E \\ & y_X \ \geq\ 0 \quad \forall\, X\subseteq V,\ X\neq\emptyset. \end{aligned}$$

For any feasible solution y to LP2, let $\mathcal{F}_y := \{X\subseteq V : y_X > 0\}$ be the support of y. Inspired by Edmonds' branching algorithm [13], Fulkerson [17] gave an algorithm to find an optimal solution y to LP2 such that y is integral. From an alternative proof in [28], we obtain the following useful lemma.

Lemma 8. *There exists an optimal, integral solution y^* to LP2 such that $\mathcal{F}_{y^*}$ is laminar.*

Let y be an optimal, integral solution to LP2 such that $\mathcal{F}_y$ is laminar. Note that for any nonempty $X\subseteq V$, there is an $e\in A\cap\delta^-(X)$ and thus $y_X\leq c_A(e)=1$. This implies that $y_X\in\{0,1\}$ for all X. We conclude that $\mathcal{F}_y$ is a dual certificate for A in the sense of the following definition.

Definition 9. A dual certificate for A is a laminar family $\mathcal{Y}\subseteq 2^V$ such that $|\{X\in\mathcal{Y} : e\in\delta^-(X)\}|\leq c_A(e)$ for all $e\in E$.

For the remainder of this section, let $\mathcal{Y}$ be a dual certificate maximizing $|\mathcal{Y}|$.

Lemma 10. *Arborescence A has unpopularity margin $\mu(A)=n-|\mathcal{Y}|$. Furthermore, the following three statements are equivalent:*

(1) A is popular.

(2) $|\mathcal{Y}|=n$.

(3) $|A\cap\delta^-(X)|=1$ for all $X\in\mathcal{Y}$ and $|\{X\in\mathcal{Y} : e\in\delta^-(X)\}|=1$ for all $e\in A$.

Proof. Let x and y be the characteristic vectors of A and $\mathcal{Y}$, respectively. By Proposition 7, A is popular if and only if x is an optimal solution to LP1. This is equivalent to (2) because $c_A(A)=n$. Note also that (3) is equivalent to x and y fulfilling complementary slackness, which is equivalent to x being optimal. □

Lemma 10 establishes the following one-to-one correspondence between the nodes in V and the sets of $\mathcal{Y}$: For every set $X\in\mathcal{Y}$, there is a unique edge $(u,v)\in A$ that enters X. We call v the *entry-point* for X. Conversely, we let Y_v be the unique set in $\mathcal{Y}$ for which v is the entry-point; thus $\mathcal{Y}=\{Y_v : v\in V\}$.

Observation 11. *For every $v \in V$ we have $|\{X \in \mathcal{Y} : v \in X\}| \leq 2$.*

Observation 11 is implied by the fact that $e = (r, v)$ is an edge in D for every $v \in V$ and $c_A(e) \leq 2$. Laminarity of $\mathcal{Y}$ yields the following corollary:

Corollary 12. *If $|\mathcal{Y}| = n$, then $w \in Y_v \setminus \{v\}$ for some $v \in V$ implies $Y_w = \{w\}$.*

The following definition of the set of *safe* edges $S(X)$ with respect to a subset $X \subseteq V$ will be useful. $S(X)$ is the set of edges (u, v) in $E[X] := E \cap (X \times X)$ such that properties 1 and 2 hold:

1. (u, v) is *undominated* in $E[X]$, i.e., $(u, v) \not\prec_v (u', v)\ \forall\, (u', v) \in E[X]$.
2. (u, v) *dominates* $(w, v)\ \forall\, w \notin X$, i.e., $(u, v) \succ_v (w, v)\ \forall\, (w, v) \in \delta^-(X)$.

The interpretation of $S(X)$ is the following. Suppose that the dual certificate $\mathcal{Y}$ proves the popularity of A. Let $X \in \mathcal{Y}$ with $|X| > 1$. By Corollary 12, for every node $v \in X$ other than the entry-point in X we have $\{v\} = Y_v \in \mathcal{Y}$. So edges in $\delta^-(v)$ within $E[X]$ enter exactly one dual set, i.e., $\{v\}$, while any edge (w, v) where $w \notin X$ enters two of the dual sets: X and $\{v\}$. This induces exactly the constraints (1) and (2) given above on $(u, v) \in A$ (see LP2), showing that the edge $A(v)$ must be safe, as stated in Observation 13.

Observation 13. *If A is popular, then $A \cap E[X] \subseteq S(X)$ for all $X \in \mathcal{Y}$.*

3 Popular Branching Algorithm

We are now ready to present our algorithm for deciding if D admits a popular arborescence or not. For each $v \in V$, step 1 builds the largest set X_v such that v can reach all nodes in X_v using edges in $S(X_v)$. The collection $\mathcal{X} = \{X_v : v \in V\}$ will be laminar (see Lemma 14). To construct the sets X_v we make use of the *monotonicity* of S: $X \subseteq X'$ implies $S(X) \subseteq S(X')$.

In steps 2-3, the algorithm contracts each maximal set in $\mathcal{X}$ into a single node and builds a graph D' on these nodes and r. For each set X here that has been contracted into a node, edges incident to X in D' are undominated edges from other nodes in D' to the *candidate entry-points* of X, which are nodes $v \in X$ such that $X = X_v$. Our proof of correctness (see Theorems 15-16) shows that D admits a popular arborescence if and only if D' admits an arborescence.

Our algorithm for computing a popular arborescence in D is given below.

1. For each $v \in V$ do:
   - let $X_v^0 = V$ and $i = 0$;
   - while v does not reach all nodes in the graph $D_v^i = (X_v^i, S(X_v^i))$ do:

     $X_v^{i+1} =$ the set of nodes reachable from v in D_v^i; let $i = i + 1$.
   - let $X_v = X_v^i$.

2. Let $\mathcal{X} = \{X_v : v \in V\}$, $\mathcal{X}' = \{X_v \in \mathcal{X} : X_v$ is $\subseteq$ -maximal in $\mathcal{X}\}$, $E' = \emptyset$.
3. For every edge $e = (u, v)$ in D such that $X_v \in \mathcal{X}'$ and $u \notin X_v$ do:
   - if e is undominated (i.e., $e \not\prec_v e'$) among all edges $e' \in \delta^-(X_v)$, then
   $$f(e) = \begin{cases} (U, X_v) & \text{where } u \in U \text{ and } U \in \mathcal{X}', \\ (r, X_v) & \text{if } u = r; \end{cases}$$
   - let $E' := E' \cup \{f(e)\}$.
4. If $D' = (\mathcal{X}' \cup \{r\}, E')$ contains an arborescence $\tilde{A}$, then
   - let $A' = \{e : f(e) \in \tilde{A}\}$;
   - let $R = \{v \in V : |X_v| \geq 2$ and v has an incoming edge in $A'\}$;
   - for each $v \in R$: let A_v be an arborescence in $(X_v, S(X_v))$;
   - return $A^* = A' \cup_{v \in R} A_v$.
5. Else return "*No popular arborescence in D*".

Correctness of the above algorithm. We will first show the easy direction, that is, if the algorithm returns an edge set A^*, then A^* is a popular arborescence in D. The following lemma will be key to this. Note that the set X_u, for each $u \in V$, is defined in step 1. Lemmas marked by ($\circ$) are proved in the Appendix.

Lemma 14 ($\circ$)**.** *$\mathcal{X} = \{X_v : v \in V\}$ is laminar. If $u \in X_v$, then $X_u \subseteq X_v$.*

Theorem 15 ($\star$)**.** *If the above algorithm returns an edge set A^*, then A^* is a popular arborescence in D.*

Sketch of proof. It is straightforward to verify that A^* is an arborescence in D. To prove the popularity of A^*, we construct a dual certificate $\mathcal{Y}$ of size n for A^*, by setting $\mathcal{Y} := \{X_v : v \in R\} \cup \{\{v\} : v \in V \setminus R\}$.

Note that $|\mathcal{Y}| = |R| + |V \setminus R| = n$. It remains to show that any edge $(w, v) \in E$ satisfies the constraints in LP2; let (u, v) be the incoming edge of v in A^*.

Suppose $v \in R$; then $(u, v) \in A'$ and $u \notin X_v$. Consider any edge (w, v): this enters one set of $\mathcal{Y}$ iff $w \notin X_v$ and no set iff $w \in X_v$. Hence, it suffices to show that $c_{A^*}((w, v)) \in \{1, 2\}$ for $w \notin X_v$. By construction of E', (w, v) does not dominate (u, v) and therefore $c_{A^*}((w, v)) \in \{1, 2\}$.

Suppose $v \in V \setminus R$. Let s be v's local root, i.e., the unique $s \in R$ with $v \in X_s$. Then $(u, v) \in A_s \subseteq S(X_s)$ by construction of A_s. Any edge $(w, v) \in \delta^-(v)$ enters at most two sets of $\mathcal{Y}$: $\{v\}$ and possibly X_s. If, on the one hand, $(w, v) \in \delta^-(X_s)$, then $(u, v) \in S(X_s)$ dominates (w, v) by property 2 of $S(X_s)$, and hence $c_{A^*}((w, v)) = 2$. If, on the other hand, $w \in X_s$, then $(u, v) \in S(X_s)$ is not dominated by (w, v) by property 1 of $S(X_s)$, and hence $c_{A^*}((w, v)) \geq 1$. Thus, any edge satisfies the constraints in LP2, proving the theorem. $\square$

Theorem 16. *If D admits a popular arborescence, then our algorithm finds one.*

Before we prove Theorem 16, we need Lemma 17 and Lemma 18.

Lemma 17 (∘)**.** *Let A be a popular arborescence and $\mathcal{Y}$ a dual certificate for A of size n. Then $Y_v \subseteq X_v$ for any $v \in V$.*

Lemma 18 (∘)**.** *Let A be a popular arborescence in D and let $X \in \mathcal{X}'$. Then A enters X exactly once, and it enters X at some node v such that $X = X_v$.*

Proof of Theorem 16. Assume there exists a popular arborescence A in D; then there exists a dual certificate $\mathcal{Y}$ of size n for A. We will show there exists an arborescence in D'. By Lemma 18, for each $X \in \mathcal{X}'$ there exists exactly one edge $e_X = (u, v)$ of A that enters X, and moreover, v is a candidate entry-point of X.

We claim that (u, v) is not dominated by any $(u', v) \in \delta^-(X)$. Recall that by Lemma 17, we know $Y_v \subseteq X_v = X$. If some $(u', v) \in \delta^-(X)$ dominates $(u, v) \in A$, its cost must be $c_A((u', v)) = 0$. However, (u', v) clearly enters $Y_v \subseteq X$, and thus violates LP2, contradicting our assumption that $\mathcal{Y}$ is a dual solution. Hence, e_X is undominated among the edges of $\delta^-(X) \cap \delta^-(v)$ and therefore our algorithm creates an edge $f(e_X)$ in E' pointing to X. Using the fact that A is an arborescence in D, it is straightforward to verify that the edges $\{f(e_X) : X \in \mathcal{X}'\}$ form an arborescence $\tilde{A}$ in D'. Thus our algorithm returns an edge set A^*, which by Theorem 15 must be a popular arborescence in D. □

It is easy to see that step 1 (the bottleneck step) takes $O(mn)$ time per node. Hence the running time of the algorithm is $O(mn^2)$; thus Theorem 2 follows.

3.1 A simple extension of our algorithm: Algorithm MinMargin

Our algorithm can be extended to compute an arborescence with minimum *unpopularity margin* when nodes have weak rankings. When D' does not admit an arborescence, algorithm MinMargin below computes a max-size branching $\tilde{B}$ in D' and adds edges from the root r to all root nodes in $\tilde{B}$ so as to make an arborescence of this branching in D'. This arborescence in D' is then transformed into an arborescence in D exactly as in our earlier algorithm.

1. Let D' be the graph constructed in our algorithm for Theorem 2, and let $\tilde{B}$ be a branching of maximum cardinality in D'.
2. Let $B' = \{e \mid f(e) \in \tilde{B}\}$, $R_1 = \{v \in V \mid \delta^-(v) \cap B' \neq \emptyset\}$, $R_2 = \emptyset$.
3. For each $X \in \mathcal{X}'$ which is a root in the branching $\tilde{B}$, select one arbitrary $v \in V$ such that $X_v = X$, add v to R_2 and (r, v) to B'.
4. For each $v \in R_1 \cup R_2$: let A_v be an arborescence in $(X_v, S(X_v))$.
5. Return $A^* := B' \bigcup_{v \in R_1 \cup R_2} A_v$.

Theorem 19 (⋆)**.** *When nodes have weak rankings, Algorithm* MinMargin *returns an arborescence with minimum unpopularity margin in $D = (V \cup \{r\}, E)$.*

4 The Popular Arborescence Polytope of D

We now describe the popular arborescence polytope of $D = (V \cup \{r\}, E)$ in $\mathbb{R}^m$. Throughout this section we assume that every node has a weak ranking over its incoming edges. The arborescence polytope $\mathcal{A}$ of D is described below [28].

$$\sum_{e \in E[X]} x_e \quad \le \quad |X| - 1 \qquad \forall\, X \subseteq V,\ |X| \ge 2. \tag{1}$$

$$\sum_{e \in \delta^-(v)} x_e \quad = \quad 1 \quad \forall\, v \in V \qquad \text{and} \qquad x_e \ \ge \ 0 \ \ \forall\, e \in E. \tag{2}$$

We will define a subgraph $D^* = (V \cup \{r\}, E_{D^*})$ of D: this is essentially the *expanded* version of the graph D' from our algorithm. The edge set of D^* is:

$$E_{D^*} = \bigcup_{X \in \mathcal{X}'} S(X) \cup \{(u,v) \in E : X_v \in \mathcal{X}',\ u \notin X_v, \text{ and } (u,v) \text{ is undominated in } \delta^-(X_v)\}.$$

Thus each set $X \in \mathcal{X}'$, which is a node in D', is replaced in D^* by the nodes in X and with edges in $S(X)$ between nodes in X. We also replace edges in D' between sets in $\mathcal{X}'$ by the original edges in E.

Lemma 20. *If every node has a weak ranking over its incoming edges, then every popular arborescence in D is an arborescence in D^* that includes exactly $|X| - 1$ edges from $S(X)$ for each $X \in \mathcal{X}'$.*

Proof. Let A be a popular arborescence in D and let $X \in \mathcal{X}'$. By Lemma 18 we know $|A \cap \delta^-(X)| = 1$; moreover, the proof of Theorem 16 tells us that the unique edge in $A \cap \delta^-(X)$ is contained in D^*. So A contains $|X| - 1$ edges from $E[X]$ for each $X \in \mathcal{X}'$. It remains to show that these $|X| - 1$ edges are in $S(X)$.

Let $u \in X$. Suppose $A(u) \in E[X] \setminus S(X)$. This means that either (i) $A(u)$ is dominated by some edge in $E[X] \cup \delta^-(X)$ or (ii) u is indifferent between $A(u)$ and some edge in $\delta^-(X)$. Let $\mathcal{Y}$ be a dual certificate of A. We know that $Y_u \subseteq X_u \subseteq X$ (by Lemma 17). Since the entry point of A into X is not in Y_u, there is an edge $e \in S(X) \cap \delta^-(Y_u)$.

Let e enter $w \in Y_u$. Since $e \in S(X)$, we have $e \succ_w A(w)$ or $e \sim_w A(w)$, hence $c_A(e) \in \{0, 1\}$. If $w \ne u$, then e enters two sets Y_u and $\{w\}$—thus the constraint in LP2 corresponding to edge e is violated. If $w = u$ then $e \succ_u A(u)$ (since $A(u) \in E[X] \setminus S(X)$, $e \in S(X)$, and u has a weak ranking over its incoming edges): so $c_A(e) = 0$. Since e enters one set Y_u, the constraint corresponding to e in LP2 is again violated. So $A(u) \in S(X)$, i.e., $A \cap E[X] \subseteq S(X)$. □

Hence, every popular arborescence in D satisfies constraints (1)-(2) along with constraints (3) given below, where E_{D^*} is the edge set of D^*.

$$\sum_{e \in E[X]} x_e \ = \ |X| - 1 \ \ \forall\, X \in \mathcal{X}',\ |X| \ge 2 \qquad \text{and} \qquad x_e \ = \ 0 \ \ \forall\, e \in E \setminus E_{D^*} \tag{3}$$

Note that constraints (3) define a face $\mathcal{F}$ of the arborescence polytope $\mathcal{A}$ of D. Thus every popular arborescence in D belongs to face $\mathcal{F}$.

Consider a vertex in face $\mathcal{F}$: this is an arborescence A in D of the form $A' \cup_{X \in \mathcal{X}'} A_X$ where (i) A_X is an arborescence in $(X, S(X))$ whose root is an entry-point of X and (ii) $A' = \{e_X : X \in \mathcal{X}'\}$ where e_X is an edge in D^* entering the root of A_X. Theorem 15 proved that such an arborescence A is popular in D. Thus we can conclude Theorem 21 which proves the upper bound in Theorem 3. The lower bound in Theorem 3 is given in the Appendix.

Theorem 21. *If every node has a weak ranking over its incoming edges, then face $\mathcal{F}$ (defined by constraints* (1)-(3)*) is the popular arborescence polytope of D.*

A compact extended formulation of this polytope and all missing proofs are in the Appendix. We also discuss popular *mixed branchings* (probability distributions over branchings) there.

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Appendix

A Missing proofs from Sections 3 and 4

Lemma 14 ($\circ$)**.** *$\mathcal{X} = \{X_v : v \in V\}$ is laminar. If $u \in X_v$, then $X_u \subseteq X_v$.*

Proof. We first show that $X_u^i \subseteq X_v^i$ for any i, where we set $X_v^i := X_v$ whenever X_v^i is not defined by the above algorithm. The claim clearly holds for $i = 0$. Let i be the smallest index such that $x \in X_u^i \setminus X_v^i$ for some node x; we must have $x \in X_u^{i-1} \cap X_v^{i-1}$. By the definition of X_u^i, x is reachable from u in $S(X_u^{i-1})$. Note that $X_u^{i-1} \subseteq X_v^{i-1}$ implies $S(X_u^{i-1}) \subseteq S(X_v^{i-1})$, which yields that x is reachable from u in $S(X_v^{i-1})$ as well. Moreover, u is reachable from v in $S(X_v^{i-1}) \supseteq S(X_v)$ because $u \in X_v$ and $S(\cdot)$ is monotone. Hence it follows that x is reachable from v in $S(X_v^{i-1})$ via u, contradicting the assumption that $x \notin X_v^i$. This proves the second statement of the lemma.

Now we will show the laminarity of $\mathcal{X}$. For contradiction, assume there exist $s, t \in V$ such that X_s and X_t *cross*, i.e., their intersection is non-empty, and neither contains the other. Then, by the second statement of the lemma, neither $s \in X_t$ nor $t \in X_s$ can hold. So we have that $s \notin X_t$ and $t \notin X_s$.

Let (x, y) be an edge in $S(X_t)$ such that $y \in X_s \cap X_t$ but $x \in X_t \setminus X_s$; since each node in X_t is reachable from t in $S(X_t)$, such an edge exists. Since $y \in X_s \setminus \{s\}$, there also exists an edge (u, y) in $S(X_s)$. As $x \notin X_s$ but $(u, y) \in S(X_s)$, we know that $(u, y) \succ_y (x, y)$ which contradicts $(x, y) \in S(X_t)$. $\square$

Theorem 15 ($\star$)**.** *If the above algorithm returns an edge set A^*, then A^* is a popular arborescence in D.*

Proof. We start by showing that A^* is an arborescence in D. Then, we construct a dual certificate of value n for A^*. This will prove the popularity of A^*.

The laminarity of $\mathcal{X}$ implies that sets in $\mathcal{X}'$ are pairwise disjoint. Moreover, by construction, each node in V is included in at least one set in $\mathcal{X}$, namely $v \in X_v$ for each $v \in V$. Hence, $\mathcal{X}'$ forms a partition of V. By construction of A^*, each node $v \in V$ can be reached in A^* from the *local root* $s \in R$ for which $v \in X_s$. It remains to show that the root r reaches all local roots $s \in R$ in A^*. This can be shown by induction over the distance of s from the root r within the arborescence A'. It remains to show that $|A^*| = n$. Let $k := |\mathcal{X}'|$. Since $\mathcal{X}'$ is a partition of V, we get that

$$|A^*| = |A'| + \textstyle\sum_{v \in R} |A_v| = k + \sum_{v \in R}(|X_v| - 1) = k + n - k = n.$$

We turn to the second part of the proof and show a dual certificate of size n for A^*. We claim that $\mathcal{Y} := \{X_v : v \in R\} \cup \{\{v\} : v \in V \setminus R\}$ is such a dual solution. Note that $|\mathcal{Y}| = |R| + |V \setminus R| = n$. We now show that for all $v \in V$, the incoming edges satisfy the constraints in LP2.

Suppose $v \in R$. An edge $(w, v) \in E$ enters one set of $\mathcal{Y}$ iff $w \notin X_v$ and no set iff $w \in X_v$. Hence, it suffices to show that $c_{A^*}((w, v)) \in \{1, 2\}$ for $w \notin X_v$. Let (u, v) be the incoming edge of v in arborescence A^*; note that $(u, v) \in A'$ and $u \notin X_v$. By construction of E', (w, v) does not dominate (u, v) and therefore $c_{A^*}((w, v)) \in \{1, 2\}$. The same argument works for $v \in V \setminus R$ and $\{v\} \in \mathcal{X}'$.

Suppose $v \in V \setminus R$. Let s be v's local root, i.e., the unique $s \in R$ with $v \in X_s$. Then $(u,v) \in A_s \subseteq S(X_s)$ by construction of A_s. Any edge $(w,v) \in \delta^-(v)$ enters at most two sets of $\mathcal{Y}$: $\{v\}$ and possibly X_s. If, on the one hand, $(w,v) \in \delta^-(X_s)$, then $(u,v) \in S(X_s)$ dominates (w,v) by property 2 of $S(X_s)$, and hence $c_{A^*}((w,v)) = 2$. If, on the other hand, $w \in X_s$, then $(u,v) \in S(X_s)$ is not dominated by (w,v) by property 1 of $S(X_s)$, and hence $c_{A^*}((w,v)) \geq 1$. This completes the proof that $\mathcal{Y}$ is a dual certificate of size n for A^*, thus A^* is popular. □

Lemma 17 (○)**.** *Let A be a popular arborescence and $\mathcal{Y}$ a dual certificate for A of size n. Then $Y_v \subseteq X_v$ for any $v \in V$.*

Proof. If $Y_v = \{v\}$, then $Y_v \subseteq X_v$ is trivial, so suppose that Y_v is not a singleton. We know from Corollary 12 that Y_w is a singleton set for each $w \in Y_v \setminus \{v\}$. Moreover, for every $(u,w) \in A$ with $w \in Y_v \setminus \{v\}$ it holds that $u \in Y_v$ since this edge would otherwise enter two sets; however, $c_A((u,w)) = 1$ as $(u,w) \in A$.

Assume for contradiction that $Y_v \setminus X_v \neq \emptyset$. Let i be the last iteration when $Y_v \subseteq X_v^i$. Then there exists a subset of Y_v which is not reachable by edges in $S(X_v^i)$, i.e., $\delta^-(Y_v \setminus X_v^{i+1}) \cap S(X_v^i) = \emptyset$. On the other hand, we know that the arborescence A can only enter nodes in $Y_v \setminus \{v\}$ by edges from $E[Y_v]$, and therefore, it needs to contain at least one edge from $\delta^-(Y_v \setminus X_v^{i+1}) \cap \delta^+(X_v^{(i+1)})$. Let (u,w) be this edge (see Fig. 1). By construction of X_v^i and X_v^{i+1}, we know that one of the following cases has to be true.

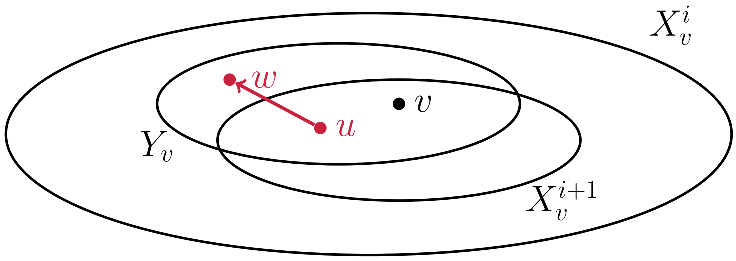

Figure 1: Illustration of the situation in the proof of Lemma 17.

Case 1. There exists an edge $(x,w) \in E[X_v^i]$ which dominates (u,w). Note that we do not know if $(x,w) \in E[Y_v]$ or not. However, $c_A((x,w)) = 0$ in either case, but by Corollary 12, (x,w) enters at least one set in $\mathcal{Y}$, namely $\{w\}$. This is a violation of LP2 and it contradicts $\mathcal{Y}$ being a dual certificate for A.

Case 2. There exists an edge $(x,w) \in \delta^-(X_v^i)$ which is not dominated by (u,w). Note that $c_A((x,w)) \in \{0,1\}$, but $(x,w) \in \delta^-(Y_v)$ and so the edge (x,w) enters two dual sets: Y_v and $\{w\}$. This contradicts $\mathcal{Y}$ being a dual solution. □

Lemma 18 (○)**.** *Let A be a popular arborescence in D and let $X \in \mathcal{X}'$. Then A enters X exactly once, and it enters X at some node v such that $X = X_v$.*

Proof. Let $X \in \mathcal{X}'$ and let A be a popular arborescence which enters X at some node $v \in V$ through an edge $(u,v) \in A \cap \delta^-(X)$. Moreover, let $\mathcal{Y}$ be a dual certificate for A, and let Y_v be the set whose entry-point is v.

Let $\mathsf{entry}(X) := \{w \in V : X_w = X\}$. We first show that $\mathsf{entry}(X) \subseteq Y_v$. Assume for contradiction that there exists $w \in \mathsf{entry}(X)$ such that $w \notin Y_v$. Since $X_w = X$ we know that there exists a w-v path P in $(X, S(X))$. Hence, there exists an edge $e \in P$ which enters Y_v. If the head of e is v, we know that e dominates $(u,v) \in \delta^-(X)$ and hence $c_A(e) = 0$, a contradiction to the feasibility of $\mathcal{Y}$. If v is not the head of e, then e not only enters Y_v, but also the singleton set corresponding to its head. However, $c_A(e) \leq 1$ since e is an undominated edge by $e \in S(X)$, a contradiction to the feasibility of $\mathcal{Y}$.

To prove that $v \in \mathsf{entry}(X)$, let us choose some $s \in \mathsf{entry}(X)$. By the previous paragraph and Lemma 17, we get $s \in Y_v \subseteq X_v$, from which Lemma 14 implies $X_s \subseteq X_v$. Because $s \in \mathsf{entry}(X)$, we have $X = X_s \subseteq X_v$. Because $X \in \mathcal{X}'$ is inclusionwise maximal in $\mathcal{X}$, we get $X = X_v$, proving $v \in \mathsf{entry}(X)$.

It remains to prove that A enters X only once. Suppose for contradiction that there exist two nodes $v, v' \in \mathsf{entry}(X)$ such that $(u,v),(u',v') \in A \cap \delta^-(X)$. By $\emptyset \neq \mathsf{entry}(X) \subseteq Y_v \cap Y_{v'}$ and the laminarity of $\mathcal{Y}$, we can assume w.l.o.g. that $Y_v \subseteq Y_{v'}$. Moreover, since $Y_{v'} \subseteq X$, the arborescence edge (u,v) enters both Y_v and $Y_{v'}$, a contradiction to the feasibility of the dual solution $\mathcal{Y}$. □

Lower bound for the popular arborescence polytope of D. Let $D = (V \cup \{r\}, E)$ be the complete graph where every node $v \in V$ regards all other nodes $u \in V$ as top-choice in-neighbors and r as its second-choice in-neighbor. Here $\mathcal{X}' = \{V\}$ and D^* is the complete bidirected graph on V along with edges (r,v) for all $v \in V$. We claim that in any minimal system contained in (1)-(3), the constraint $\sum_{e \in E[X]} x_e \leq |X| - 1$ for every $X \subset V$ with $|X| \geq 2$ has to be present. This is because a cycle on the nodes in X along with any rooted arborescence A on $V \setminus X$ plus (r,v), where v is the root of A, satisfies all the remaining constraints. Thus any minimal system of inequalities from (1)-(3) has to contain $2^n - n - 2$ inequalities from (1): one for every $X \subset V$ with $|X| \geq 2$. Since inequalities in a minimal system are in one-to-one correspondence with the facets of the polyhedron they describe [7, Theorem 3.30], the lower bound given in Theorem 3 follows.

A compact extended formulation. We now describe a compact extended formulation of the popular arborescence polytope of D when node preferences are weak rankings. We know from Lemma 20 that every popular arborescence in D is an arborescence in D^* that includes exactly $|X| - 1$ edges from $S(X)$ for each $X \in \mathcal{X}'$. Conversely, any such arborescence in D^* is a popular arborescence in D (by Theorem 15).

Thus the popular arborescence polytope of D is the face of the arborescence polytope of D^* that corresponds to the constraints $\sum_{e \in E_{D^*}[X]} x_e = |X| - 1$ for all $X \in \mathcal{X}'$. Let $\mathcal{A}_{D^*}$ be the arborescence polytope of $D^* = (V \cup \{r\}, E_{D^*})$. We will now use a compact extended formulation of $\mathcal{A}_{D^*}$.

Recall that $|V| = n$. Let $\mathcal{P}_{D^*}$ be the polytope defined by constraints (4)-(7) on variables x_e, f^v_e for $e \in E_{D^*}$ and $v \in V$. It is known [7] that $\mathcal{P}_{D^*}$ is a compact extended formulation of the arborescence polytope $\mathcal{A}_{D^*}$. Note that $\mathcal{A}_{D^*}$ is the projection of $\mathcal{P}_{D^*}$

on to x-space.

$$\begin{aligned}
x_e \;\geq\; f_e^v \;\geq\; 0 \quad & \forall v \in V \text{ and } e \in E_{D^*} && (4)\\
\sum_{e\in\delta^+(r)} f_e^v \;=\; 1 \quad & \forall v \in V && (5)\\
\sum_{e\in\delta^+(u)} f_e^v - \sum_{e\in\delta^-(u)} f_e^v \;=\; 0 \quad & \forall u, v \in V,\ u \neq v && (6)\\
\sum_{e\in E_{D^*}} x_e \;=\; n. & && (7)
\end{aligned}$$

For any $X \subseteq V$ with $|X| \geq 2$, the constraint $\sum_{e\in E_{D^*}[X]} x_e \leq |X| - 1$ is a valid inequality for $\mathcal{A}_{D^*}$ and also for $\mathcal{P}_{D^*}$. Thus the intersection of $\mathcal{A}_{D^*}$ along with the tight constraints $\sum_{e\in E_{D^*}[X]} x_e = |X| - 1$ for all $X \in \mathcal{X}'$ is a face of $\mathcal{A}_{D^*}$. Call this face $\mathcal{F}_{D^*}$—this is the popular arborescence polytope of D.

Consider the face of $\mathcal{P}_{D^*}$ that is its intersection with $\sum_{e\in E_{D^*}[X]} x_e = |X| - 1$ for all $X \in \mathcal{X}'$. This face of $\mathcal{P}_{D^*}$ is an extension $\mathcal{F}_{D^*}$. The total number of constraints used to describe this face of $\mathcal{P}_{D^*}$ is $O(mn)$.

B Branchings with minimum unpopularity margin

In this section we prove Theorem 4. Recall the definition of the *unpopularity margin* for branchings from Section 1. Again, instead of studying minimum unpopularity margin branchings within the digraph G, we look at r-arborescences of minimum unpopularity margin within the digraph D. It is easy to see that the unpopularity margin of a branching in G is the same as the unpopularity margin of the corresponding arborescence in D.[4] Thus we are looking for an arborescence of minimum unpopularity margin in D.

Furthermore, recall that by Proposition 7 and Lemma 10 in Section 2, the unpopularity margin $\mu(A)$ of an arborescence A fulfills

$$\mu(A) = n - c_A(A') = n - |\mathcal{Y}|,$$

where A' is a min-cost arborescence in D with respect to c_A and $\mathcal{Y}$ is a dual certificate of maximum cardinality for A.

Theorem 19 ($\star$)**.** *When nodes have weak rankings, Algorithm* MinMargin *returns an arborescence with minimum unpopularity margin in* $D = (V \cup \{r\}, E)$.

Algorithm MinMargin is described in Section 3.1. To prove Theorem 19, we first show in Lemma 22 that the size of the maximum cardinality branching $\tilde{B}$ bounds the unpopularity margin of the arborescence A^* returned by Algorithm MinMargin. Then we provide Lemma 23 and Observation 24 that will be helpful in showing the optimality of Algorithm MinMargin.

[4]Note that, due to the special structure of D, there always exists an arborescence A' such that $A' \in \arg\max_{B\in\mathcal{B}(\mathcal{D})} \phi(B, A) - \phi(A, B)$, where $\mathcal{B}(D)$ is the set of branchings in D.

Lemma 22. *If the number of roots in the branching $\tilde{B}$ is ℓ, then aborescence A^* returned by Algorithm* MinMargin *has unpopularity margin at most $\ell - 1$.*

Proof. We show that A^* has unpopularity margin at most $\ell - 1$ by constructing a dual certificate of size $n - \ell + 1$; by Lemma 10 this is sufficient. Define $\mathcal{Y} := \{X_v \mid v \in R_1\} \cup \{\{v\} \mid v \in V \setminus \{R_1 \cup R_2\}\}$. It is easy to see that $\mathcal{Y}$ contains $n - (|R_2| - 1)$ $= n - \ell + 1$ sets ($r \in R_2$ but $r \notin V$). It remains to show that any edge (w, v) satisfies the constraints in LP2; the argumentation is analogous to the one in the proof for Theorem 15.

First, if $v \in R_2$, then v is not contained in any set of $\mathcal{Y}$, so no constraints are violated by (w, v). Otherwise, let (u, v) be an incoming edge of v in A^*.

Suppose $v \in R_1$; then $(u, v) \in B'$ and $u \notin X_v$. Edge (w, v) enters one set of $\mathcal{Y}$ iff $w \notin X_v$ and no set iff $w \in X_v$. Hence, it suffices to show that $c_{A^*}((w, v)) \in \{1, 2\}$ for $w \notin X_v$. By construction of E' (recall that E' is the edge set of D'), (w, v) does not dominate (u, v) and therefore $c_{A^*}((w, v)) \in \{1, 2\}$.

Suppose now $v \in V \setminus (R_1 \cup R_2)$. Let s be v's local root, i.e., $s \in R_1 \cup R_2$, $v \in X_s$; then $(u, v) \in A_s$. Edge (w, v) enters two sets of $\mathcal{Y}$ iff $w \notin X_s$ and one set iff $w \in X_s$. If, on the one hand, $w \notin X_s$, then by construction of A_s and property 2 of $S(X_s)$, it holds that (w, v) is dominated by (u, v), and hence $c_{A^*}((w, v)) = 2$. If, on the other hand, $w \in X_s$, then by construction of A_s and property 1 of $S(X_s)$, (w, v) does not dominate (u, v), and hence $c_{A^*}((w, v)) \in \{1, 2\}$. Thus, any edge satisfies the constraints in LP2 and $\mathcal{Y}$ is a dual certificate for A^*. $\square$

Let $S \subseteq V$, $s \in S$ and $A \subseteq E$ be an arborescence rooted at s and spanning exactly the nodes in S. We say that A is *locally popular with respect to S*, if the set family $\mathcal{Y} := \{\{v\} \mid v \in S \setminus \{s\}\} \cup \{S\}$ fulfills the constraints of the dual LP induced by c_A, where we set $c_A(e) := 1$ for each edge $e \in \delta^-(s)$ and $c_A(e) := 0$ for every $e \in \delta^-(v)$ with $v \in V \setminus S$ (see LP2).

Lemma 23. *Let $S \subseteq V$ such that there exists an arborescence $A \subseteq E$ which is rooted at $v \in S$ and locally popular with respect to S. Then, $S \subseteq X_v$.*

Proof. The proof of this lemma is a direct analog of the proof of Lemma 17: substituting S for Y_v and using the definition of local popularity instead of popularity, one can use the same arguments to obtain the statement of this lemma. $\square$

Observation 24. *Let $\tilde{B}$ be a branching of maximum cardinality in D' and $T \subseteq \tilde{B}$ be a maximal subarborescence of $\tilde{B}$ not containing r. Then, there exists $S \subseteq V(T)$ such that $\delta^-_{D'}(S) = \emptyset$.*

Proof. Assume for contradiction that $\delta^-_{D'}(S) \neq \emptyset$ for all $S \subseteq V(T)$. Hence, every $X \in V(T)$ is reachable from $\{r\} \cup \mathcal{X} \setminus V(T)$ in D'. Consequently, we can modify $\tilde{B}$ by attaching each $X \in V(T)$ to some node in $\{r\} \cup \mathcal{X}' \setminus V(T)$, one by one. This contradicts the maximality of $\tilde{B}$. $\square$

Let A be any arborescence, and $\mathcal{Y}$ a *dual certificate* for A. Since $c_A(e) = 1$ for every $e \in A$, we know that each edge in A enters at most one set in $\mathcal{Y}$. If an edge

$(u, v) \in A$ enters a set of $\mathcal{Y}$, we refer to this set as Y_v, and we say that Y_v *belongs* to v in $\mathcal{Y}$. In contrast to the case of popular arborescences, it can be the case that the same set belongs to two edges in $\mathcal{Y}$, i.e., $Y_v = Y_{v'}$ but $v \neq v'$. We say that $\mathcal{Y}$ is *complete* on $S \subseteq V$, if $|\{Y_v \mid v \in S\}| = |S|$; this concept will be crucial in the proof of Theorem 19. By a simple counting argument we obtain that if $\mathcal{Y}$ is complete on S, then $v, v' \in S, v \neq v'$ implies $Y_v \neq Y_{v'}$.

Proof of Theorem 19. By Lemma 22, the algorithm returns an arborescence with unpopularity margin at most $\ell - 1$, where ℓ is the number of maximal subtrees in $\tilde{B}$. Let A be an arborescence with minimum unpopularity margin and $\mathcal{Y}$ a corresponding dual certificate.

Take any maximal subtree T of $\tilde{B}$ not containing r. By Observation 24, there exists some $S \subseteq V(T)$ with $\delta^-_{D'}(S) = \emptyset$. Below we prove that $\mathcal{Y}$ is not complete on $S^* := \bigcup_{X \in S} X$. As there are $\ell - 1$ maximal subtrees of $\tilde{B}$ not containing r, and each contains a set of nodes on which $\mathcal{Y}$ is not complete, we get $|\mathcal{Y}| \leq n - (\ell - 1)$. This implies $\mu(A) \geq \ell - 1$ by Lemma 10, proving the theorem.

It remains to show that $\mathcal{Y}$ is not complete on S^*. Assume for contradiction that $\mathcal{Y}$ contains a set Y_x belonging to each $x \in S^*$. Recall that D^* is the expanded version of D'. Note that, by the construction of $\mathcal{X}$ in the algorithm, a most preferred edge (u, v) can enter a set $X \in \mathcal{X}$ only at a candidate entry node. Thus, $\delta^-_{D^*}(S^*) = \emptyset$ means that S^* is not entered by any most preferred edge.

Since A enters S^* but $\delta^-_{D^*}(S^*) = \emptyset$, there exists $(u, v) \in A \cap \delta^-_D(S^*)$ which is not included in D^*.

Claim. $Y_v \cap S^* \not\subseteq X_v$

Proof. Let $X \in S$ be the contracted node entered by (u, v).

Case 1: $v \notin \mathsf{entry}(X)$. Let $s \in \mathsf{entry}(X)$. Then there exists an s-v-path P in $(X, S(X))$; recall that every edge on P is most preferred and dominates all edges entering X. If $s \notin Y_v$, then there is an edge $(u', v') \in P$ entering Y_v. If $v' \neq v$, then the most-preferred edge (u', v') crosses two sets of $\mathcal{Y}$, a contradiction. If $v' = v$, then (u', v') dominates (u, v) but crosses Y_v, again a contradiction. We conclude that $s \in Y_v$. However, as v is not in $\mathsf{entry}(X)$, by Lemma 14 we know that $s \notin X_v$. We obtain $Y_v \cap S^* \not\subseteq X_v$.

Case 2: $v \in \mathsf{entry}(X)$, i.e., $X = X_v$. Since $(u, v) \notin D^*$, there exists an edge $(u', v) \in D^*$ which dominates (u, v) and $u' \in V \setminus X$. Hence, (u', v) must not enter any set in $\mathcal{Y}$ and we obtain $u' \in Y_v$. Clearly, we get that $Y_v \cap S^* \not\subseteq X_v$. □

Now, we are going to show that A induces a locally popular arborescence on $Y_v \cap S^*$, rooted at v. By the above claim, this contradicts Lemma 23.

Consider a node $x \in Y_v \cap S^* \setminus \{v\}$. If $f = (w, x)$ is a most-preferred edge in x, then $w \in Y_v \cap S^*$: indeed, f cannot enter S^* because $\delta^-_{D^*}(S^*) = \emptyset$, and moreover, f cannot enter Y_v because $c_A(f) \leq 1$ and thus cannot enter both Y_v and Y_x (recall that $\mathcal{Y}$ is complete on S^* which contains x, so $\mathcal{Y}$ contains a set Y_x corresponding to x). If x prefers f to $A(x)$, then $c_A(f) = 0$ and thus f cannot enter Y_x, implying $w \in Y_x$. However, this contradicts the fact that $\mathcal{Y}$ is two-layered: since $w \in S^*$, there exists a set $Y_w \in \mathcal{Y}$ corresponding to w, and so w is contained in three sets of $\mathcal{Y}$, Y_w, Y_x and Y_v. Hence, we obtain that $A(x)$ must be a most-preferred edge for x. Since this holds for each $x \in Y_v \cap S^* \setminus \{v\}$, $A' := A \cap E[Y_v \cap S^*]$ is an arborescence rooted at v, containing only most-preferred edges.

It remains to show that $\mathcal{Y}' := \{Y_v \cap S^*\} \cup \{\{u\} \mid u \in (Y_v \cap S^*) \setminus \{v\}\}$ fulfills all constraints in LP2 w.r.t. A' (with $c_{A'}(e) = 1$ for each $e \in \delta^-(v)$). Observe that we need to verify this only for edges that point from $Y_v \setminus S^*$ to $Y_v \cap S^*$, as all other edges enter the same number of sets in $\mathcal{Y}'$ as in $\mathcal{Y}$. So let $f = (w, x)$ be such an edge. If $x = v$, then f enters only $Y_v \cap S^*$ from $\mathcal{Y}'$; by $c_{A'}(f) = 1$ this satisfies LP2. If $x \neq v$, then f enters two sets $Y_v \cap S^*$ and $\{x\}$ from $\mathcal{Y}'$. Since $\delta^-_{D^*}(S^*) = \emptyset$, we know that f is not a most-preferred edge, so x prefers $A'(x)$ to f, yielding $c_{A'}(f) = 2$; note that here we need that $\succ_x$ is a weak ordering. This proves that all edges satisfy the constraints in LP2, so we can conclude that A' is indeed locally popular and spans $Y_v \cap S^*$. $\square$

The following theorem shows that Algorithm MinMargin cannot be extended for the case where each node v has a partial order over $\delta^-(v)$.

Theorem 25. *Given a directed graph where each node has a partial preference order over its incoming edges and an integer $k \leq n$, it is* NP*-hard to decide if there exists a branching with unpopularity margin at most k.*

Proof. We reduce from 3D-Matching where we are given disjoint sets X, Y, Z of equal cardinality and $T \subseteq X \times Y \times Z$, and we ask whether there exists $M \subseteq T$ with $|M| = |X|$ such that for distinct $(x, y, z), (x', y', z') \in M$ it holds that $x \neq x', y \neq y'$ and $z \neq z'$; such an M is called a 3D-*matching*. W.l.o.g. we assume that $|X| > 3$ and every $x \in X \cup Y \cup Z$ is in some $t \in T$.

We construct a digraph $D = (V \cup \{r\}, E)$ together with a partial order $\succ_v$ over the incoming edges of v for each $v \in V$ as follows. For every $x \in X \cup Y \cup Z$ we introduce a *node gadget* consisting of a lower node x_l and an upper node x_u. There exist two parallel edges, $d_x^{(1)}$ and $d_x^{(2)}$, from x_u to x_l, and there exist two parallel edges, $r_x^{(1)}$ and $r_x^{(2)}$, from r to x_l. Moreover, the upper node x_u has an incoming edge from the upper node of every other node gagdet, i.e., $(x'_u, x_u) \in E$ for all $x' \in X \cup Y \cup Z \setminus \{x\}$. Lastly, there exists an incoming edge from r to the upper node which we call $r_x^{(3)}$.

For each $t \in T$ we introduce a *hyperedge gadget* consisting of six edges in D. More precisely, for each $x \in t$ we introduce two parallel edges from x_l to x_u which we call $t_x^{(1)}$ and $t_x^{(2)}$. This finishes the definition of D.

Let us now define the preferences $\{\succ_v \mid v \in V\}$. A lower node x_l has the following preferences over its incoming edges:

$$d_x^{(1)} \succ r_x^{(1)}, \qquad d_x^{(2)} \succ r_x^{(2)},$$

and all other pairs are not comparable. Let $t = (x, y, z) \in T$ and $\bar{t} := \{x, y, z\}$. The preferences of an upper node x_u are as follows:

$$\begin{array}{ll} (x'_u, x_u) \succ r_x^{(3)} & \text{for each } x' \in X \cup Y \cup Z \setminus \{x\}, \\ t_x^{(1)} \succ (x'_u, x_u) & \text{for each } x' \in X \cup Y \cup Z \setminus \bar{t}, \\ t_x^{(2)} \succ (x'_u, x_u) & \text{for each } x' \in \bar{t} \setminus \{x\}, \\ t_x^{(1)} \succ r_x^{(3)}, & t_x^{(2)} \succ r_x^{(3)}, \end{array}$$

and all other pairs are not comparable. See Figure 2 for an illustration.

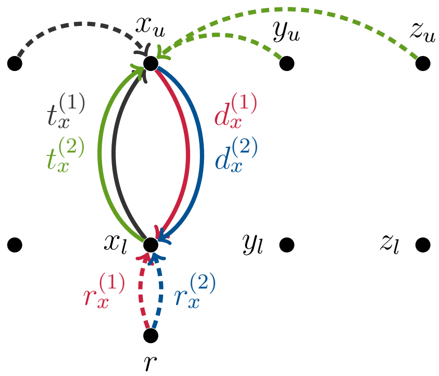

Figure 2: Construction within the reduction for Theorem 25. A solid edge of a certain color dominates the dashed edge(s) of the same color; the figure assumes $(x, y, z) \in T$.

Note that the digraph D has the special property that every node $v \in V$ has at least one incomming edge from r. As a consequence of this structure, any branching B in D minimizing $\mu(B)$ must in fact be an arborescence rooted at r. Moreover, we can apply Lemma 10 to any given arborescence A in D as usual. In the following we show that there exists a 3D-matching $M \subseteq T$ with $|M| = |X|$ iff there exists an r-arborescence in D with unpopularity margin at most $2|X|$.

First, let $M \subseteq T$ be a 3D-matching with $|M| = |X|$. We construct an arborescence A together with a feasible dual certificate $\mathcal{Y}$ with $|\mathcal{Y}| = 4|X|$. By Lemma 10, this suffices to show that A has unpopularity margin at most $6|X| - 4|X| = 2|X|$. We define

$$A := \{r_x^{(1)} \mid \text{for all } x \in X \cup Y \cup Z\} \cup \{t_w^{(1)} \mid \text{for all } t \in M \text{ and for all } w \in t\}$$

and

$$\mathcal{Y} := \{\{x_u\} \mid \text{for all } x \in X \cup Y \cup Z\} \cup \{\{x_u, y_u, z_u, x_l, y_l, z_l\} \mid \text{for all } (x, y, z) \in M\}.$$

Clearly, A is indeed a r-arborescence. It remains to show that $\mathcal{Y}$ is a feasible dual solution. First consider a node x_l for $x \in X \cup Y \cup Z$ which has four incoming edges. The edges $d_x^{(1)}$ and $d_x^{(2)}$ do not enter any set in $\mathcal{Y}$ and hence do not violate any constraint in the dual LP. Moreover, since node x_l is indifferent between $r_x^{(1)}$ and $r_x^{(2)}$, we obtain $c_A(r_x^{(1)}) = c_A(r_x^{(2)}) = 1$ and hence, none of the corresponding constraints is violated.

Now, consider x_u for $x \in X \cup Y \cup Z$. Let (x, y, z) be the hyperedge in M containing x. We obtain that $c_A((y_u, x_u)) = c_A((z_u, x_u)) = c_A(t_x^{(2)}) = 1$ while for any other incoming edge e of x_u we get $c_A(e) = 2$. By construction of $\mathcal{Y}$, none of the constraints is violated. This suffices to show the first direction of the equivalence.

Now, let A be an r-arborescence of unpopularity margin at most $2|X|$. Let $\mathcal{Y}$ be a corresponding laminar certificate of size $|\mathcal{Y}| = 4|X|$.

Our first observation is that x_l for any $x \in X \cup Y \cup Z$ is included in at most one set in $\mathcal{Y}$. This can be seen by a simple case distinction over the incoming edge of x_l in A. No matter which of the four incoming edges to x_l is selected in A, it always holds that $c_A(r_x^{(1)}) = 1$ or $c_A(r_x^{(2)}) = 1$, while both of them enter two sets of $\mathcal{Y}$.

The first observation directly implies that a node gadget can intersect with at most two sets from $\mathcal{Y}$. Since the number of sets is greater than the number of node gadgets, there exist node gadgets which intersect with more than one set from $\mathcal{Y}$. Let $x \in X \cup Y \cup Z$ be a node such that the corresponding node gadget intersects with two sets from $\mathcal{Y}$. In the following we argue about the relation of these sets. Let Y_1 and Y_2 be the corresponding sets from $\mathcal{Y}$. We will show that w.l.o.g. $Y_2 \subseteq Y_1, \{x_u, x_l\} \subseteq Y_1, x_u \in Y_2, x_l \notin Y_2$.

First, assume for contradiction that $Y_1 \cap Y_2 = \emptyset$. Then, $\{r_x^{(1)}, r_x^{(2)}\} \cap A = \emptyset$ since otherwise $c_A(d_x^{(1)}) = 0$ or $c_A(d_x^{(2)}) = 0$, however, both of them enter a set from $\mathcal{Y}$. This implies that $\{t_x^{(1)}, t_x^{(2)}\} \cap A = \emptyset$ for all $t \in T$ such that $x \in t$ since otherwise A would contain a cycle. However, no matter which incoming edge of x_u is included, there exists one edge e pointing from x_l to x_u such that $c_A(e) = 0$ but e enters one set from $\mathcal{Y}$, a contradiction. We conclude that Y_1 and Y_2 need to intersect and since $\mathcal{Y}$ is laminar we can assume w.l.o.g. that $Y_2 \subseteq Y_1$. Second, assume for contradiction that $x_l \notin Y_1$. By the previous argumentation we know that x_u can only be entered by edges pointing from x_l to x_u, however, these enter two sets from $\mathcal{Y}$, a contradiction. We conclude that $Y_2 \subseteq Y_1, \{x_u, x_l\} \subseteq Y_1, x_u \in Y_2, x_l \notin Y_2$.

We define $\mathcal{S} := \{Y \in \mathcal{Y} \mid Y \text{ is } \subseteq\text{-maximal and there exists } Y' \in \mathcal{Y}, Y' \subseteq Y\}$. Elements in $\mathcal{S}$ are non-overlapping and by the above observations we know that $|\mathcal{S}| \geq |X|$. For every $Y_1 \in \mathcal{S}$ we select one representative $x(Y_1) \in X \cup Y \cup Z$ such that the node gadget of x intersects with Y_1 and one other set from $\mathcal{Y}$. Considering the node gadget of $x := x(Y_1)$, we observe that x_u can only be entered by edges pointing from x_l to x_u. We argue that in particular, x_u needs to be entered by $t_x^{(1)}$ for some $t \in T$. Assume for contradiction that x_u is entered by $t_x^{(2)}$ for some $t \in T$. Then, there exist $3|X|-3$ edges which are uncomparable with $t_x^{(2)}$ and hence their tails need to be included in Y_1. Hence, $\mathcal{S}$ contains at most 3 sets, a contradiction to the assumption that $|X| > 3$. Therefore, x_u is entered by $t_x^{(1)}$ for some $t = (x, y, z) \in T$. Then, we know that $\{x_u, y_u, z_u\} \subseteq Y_1$ since $c_A((y_u, x_u)) = c_A((z_u, x_u)) = 1$. We conclude that neither y nor z are included in any other set of $\mathcal{S}$. Hence, $M := \{t \in T \mid t_{x(Y_1)}^{(1)} \in A, Y_1 \in \mathcal{S}\}$ is a 3D-matching of size $|X|$. □

C Branchings with low unpopularity factor

Recall the definition of *unpopularity factor* from Section 1. As done in the previous section, instead of studying branchings within the digraph G, we look at r-arborescences within the digraph D. The unpopularity factor of any branching in G is the same as the unpopularity factor of the corresponding arborescence in D. Given any arborescence A and value t, there is a simple method to verify if $u(A) \leq t$ or not. This is totally analogous to our method from Section 2 to verify popularity and it involves computing a min-cost arborescence in D with the following edge costs. For $e = (u, v)$

in D, define:

$$c_A(e) := \begin{cases} 0 & \text{if } e \succ_v A(v) \\ 1 & \text{if } e \sim_v A(v) \\ t+1 & \text{if } e \prec_v A(v) \end{cases}$$

Lemma 26. *Arborescence A satisfies $u(A) \le t$ if and only if A is a min-cost arborescence in D with edge costs given by c_A defined above.*

Proof. For any arborescence A', we now have $c_A(A') = t \cdot \phi(A, A') - \phi(A', A) + n$. We also have $c_A(A) = n$. Suppose A is a min-cost arborescence in D. Then $c_A(A') \ge n$; so for any arborescence A' such that $\phi(A', A) > 0$, we have:

$$t \cdot \phi(A, A') - \phi(A', A) \ge 0, \quad \text{so} \quad \frac{\phi(A', A)}{\phi(A, A')} \le t, \quad \text{i.e.,} \quad u(A) \le t.$$

Conversely, if $u(A) \le t$, then $t \cdot \phi(A, A') \ge \phi(A', A)$ for all arborescences A'. Thus $c_A(A') = t \cdot \phi(A, A') - \phi(A', A) + n \ge n$. Since $c_A(A) = n$, A is a min-cost arborescence in D. □

Lemma 27 follows from Lemma 26 and LP-duality.

Lemma 27. *Arborescence A satisfies $u(A) \le t$ if and only if there exists a dual feasible solution y (see LP2 where $c_A(e)$ is as defined above) with $\sum_X y_X = n$.*

As before, $y_X \in \{0, 1\}$ for every non-empty $X \subseteq V$—thus we can identify y with the corresponding set family $\mathcal{F}_y = \{X \subseteq V : y_X > 0\}$. Moreover, the family $\mathcal{F}_y$ has at most $t+1$ levels now, i.e., if $X_1 \subset \cdots \subset X_k$ is a chain of sets in $\mathcal{F}_y$, then $k \le t+1$.

Proof of Theorem 5. We now assume node preferences are *strict* (thus we may assume the graph to be simple, and nodes have preferences over their in-neighbors) and modify our algorithm from Section 3 so that the new algorithm always computes an arborescence A in $D = (V \cup \{r\}, E)$ such that $u(A) \le \lfloor \log n \rfloor$.

1. Initially all nodes in V are active. Set $X_v^0 = \{v\}$ for all $v \in V$.
2. Initialize the current edge set $E' = \emptyset$; let $i = 1$.
3. Let $E' = E' \cup \{(u, v) : v \in V$ is active and u is v's most preferred in-neighbor such that $u \notin X_v^{i-1}\}$.
4. For every active node v do:
   - let X_v^i = set of nodes reachable from v using edges in E'.
5. Let $\mathcal{X} = \{X_v^i$ is $\subseteq$-maximal in $\mathcal{X}^i\}$ where $\mathcal{X}^i = \{X_v^i : v$ is active$\}$.

   $\{$*note that $\mathcal{X}^i$ is a laminar family*$\}$
6. For each $X \in \mathcal{X}$ do:
   - select any active node v such that $X_v^i = X$;

- *deactivate* all $u \in X \setminus \{v\}$. $\{$*now v is the only active node in X*$\}$
- if v is reachable from r using edges in E', then deactivate v.
  $\{$*this means all nodes in X are reachable from r*$\}$

7. If there exists any active node, then set $i = i + 1$ and go to step 3 above.
8. Compute an arborescence A in $(V \cup \{r\}, E')$ and return A.

We reach step 8 only when there is no active node. This means when we reach step 8, every node is reachable from r using the edges in E'. Thus there exists an arborescence A in $(V \cup \{r\}, E')$. Our task now is to bound its unpopularity factor.

Lemma 28. *The while-loop runs for at most $\lfloor \log n \rfloor + 1$ iterations.*

Proof. Every node v that is active at the start of some iteration either becomes reachable from r in this iteration or it forms a weakly connected component that contains two or more active nodes. At the end of each iteration, there is at most one active node in each weakly connected component.

So if k is the number of active nodes at the start of some iteration then the number of active nodes at the end of that iteration is at most $k/2$. Thus the number of active nodes at the end of the i-th iteration of the while-loop is at most $n/2^i$. Hence the while-loop can run for at most $\lfloor \log n \rfloor + 1$ iterations. □

Lemma 29. *If the while-loop runs for $t + 1$ iterations then $u(A) \leq t$.*

Proof. Let $\mathcal{Y} = \{X_v^{i-1} : v \in V \text{ and } v \text{ got deactivated in the } i\text{-th iteration}\}$. That is, $y_X = 1$ if $X \in \mathcal{Y}$ and $y_X = 0$ otherwise. For each node v, there is a corresponding set X_v^{i-1} in $\mathcal{Y}$—note that v is the *entry-point* for this set. We have $\sum_{X \subseteq V} y_X = n$.

Since our algorithm runs for $t+1$ iterations, $\mathcal{Y}$ has at most $t+1$ levels. For any node v, our algorithm ensures that the edge $(u^*, v) \in A$ is the most preferred edge entering v with its tail outside X_v^{i-1}. So every other edge $e = (u, v)$ with $u \notin X_v^{i-1}$ is ranked worse than $(u^*, v) \in A$, thus $c_A(e) = t + 1$. Hence we have $\sum_{X : \delta^-(X) \ni e} y_X \leq c_A(e)$ for every edge e. This proves that y is a feasible dual solution for A, so $u(A) \leq t$. □

Combining Lemmas 28 and 29, the first part of Theorem 5 follows.

A tight example. We now describe an instance $G = (V, E)$ on n nodes with strict preferences where every branching has unpopularity factor at least $\lfloor \log n \rfloor$. For convenience, let $n = 2^k$ for some integer k. Let $V = \{v_0, \ldots, v_{n-1}\}$. Every node will have in-degree $k = \log n$. This instance is a generalization of the instance on 4 vertices a, b, c, d given in Section 1.

- For $0 \leq i \leq n/2 - 1$, the nodes v_{2i} and v_{2i+1} are each other's top in-neighbors. Thus v_0, v_1 are each other's top choice in-neighbors, v_2, v_3 are each other's top choice in-neighbors, and so on.
- The nodes v_0, v_2 are each other's second choice in-neighbors, similarly, v_1, v_3 are each other's second choice in-neighbors, and so on. More generally, for any i, if $i \in \{4j, \ldots, 4j + 3\}$, then the node v_ℓ, where $\ell = 4j + (i + 2 \bmod 4)$, is v_i's second choice in-neighbor.

- For any i and any $t \in \{1, \ldots, k\}$, if $i \in \{j2^t, \ldots, (j+1)2^t - 1\}$ then the node v_ℓ, where $\ell = j2^t + (i + 2^{t-1} \bmod 2^t)$, is v_i's t-th choice in-neighbor.

For example, v_0's preference order is: $v_1 \succ v_2 \succ v_4 \succ v_8 \succ \cdots \succ v_{n/2}$. The other preference orders are analogous. As a concrete example, let $n = 8$. So $V = \{v_0, v_1, \ldots, v_7\}$. The preference orders of all the nodes over their in-neighbors are given below.

$$\begin{array}{ll} v_0 :\ v_1 \succ v_2 \succ v_4 & \quad v_1 :\ v_0 \succ v_3 \succ v_5 \\ v_2 :\ v_3 \succ v_0 \succ v_6 & \quad v_3 :\ v_2 \succ v_1 \succ v_7 \\ v_4 :\ v_5 \succ v_6 \succ v_0 & \quad v_5 :\ v_4 \succ v_7 \succ v_1 \\ v_6 :\ v_7 \succ v_4 \succ v_2 & \quad v_7 :\ v_6 \succ v_5 \succ v_3. \end{array}$$

For any branching in the above instance (let us call it G_k) on 2^k nodes, we claim its unpopularity factor is at least k. We will prove this claim by induction on k. The base case, i.e., $k = 1$, is trivial. So let us assume that we have $u(\tilde{B}) \geq i$ for any branching $\tilde{B}$ in G_i.

Consider G_{i+1}. Note that v_{2j} and v_{2j+1} are each other's top choice in-neighbors for $0 \leq j \leq 2^i - 1$. Let B be any branching in G_{i+1}. Suppose it is the case that in B, for some j: neither v_{2j} is v_{2j+1}'s in-neighbor nor v_{2j+1} is v_{2j}'s in-neighbor. Then $u(B) = \infty$, because by making v_{2j} the in-neighbor of v_{2j+1}, no node is worse-off and v_{2j} is better-off. We assume $u(B) < \infty$. So it is enough to restrict our attention to the case where for each j we have in B:

($*$) either v_{2j} is v_{2j+1}'s in-neighbor or v_{2j+1} is v_{2j}'s in-neighbor.

For each $j \in \{0, \ldots, 2^i - 1\}$, contract the set $\{v_{2j}, v_{2j+1}\}$ into a single node in the graph G_{i+1}. The new graph (call it G'_i) is on 2^i nodes and it is exactly the same as G_i except that there are 2 parallel edges between every adjacent pair of nodes now – both these edges have the same rank.

Perform the same contraction step on the branching B as well. By ($*$), it follows that the contracted B (call it B') is a branching such that B' uses at most 1 edge in any pair of parallel edges in G'_i. Thus B' is a branching in G_i and we can use induction hypothesis to conclude that $u(B') \geq i$.

Claim. *There is a branching A' in G'_i such that $\phi(A', B') \geq i$ and $\phi(B', A') = 1$. Moreover, the lone vertex that prefers B' to A' is a root in A'.*

We will first assume the above claim and finish our proof on $u(B)$. Then we will prove this claim. Opening up the size-2 supernodes in B' will create B: let us run the same "opening up" step on A' to create a branching A in G_{i+1}. So $\phi(A, B) \geq i$ and $\phi(B, A) = 1$. We will now modify A to A^* so that $\phi(A^*, B) \geq i + 1$ and $\phi(B, A^*) = 1$.

Let v_{2j} be the lone vertex that prefers B to A. By the "opening up" step in B, v_{2j+1} has v_{2j} as its in-neighbor. The branching A^* will affect only the 2 nodes v_{2j} and v_{2j+1} in A. Every other node will have the same in-neighbor in A^* as in A. The above claim tells us that v_{2j} is a root in A. Make v_{2j+1} a root in A^* and v_{2j}'s in-neighbor will be v_{2j+1}. The node v_{2j} was the only node that preferred B to A and now v_{2j} prefers A^* to B. However there is one node that prefers B to A^*: this is v_{2j+1}. Recall that v_{2j+1}'s in-neighbor in B, just as in A, is its top-choice neighbor v_{2j} while v_{2j+1} is a root in A^*. Thus $\phi(A^*, B) \geq i+1$ and $\phi(B, A^*) = 1$.

Proof of Claim. Let $\tilde{A}$ be a branching that maximizes $\phi(\tilde{A}, B')/\phi(B', \tilde{A})$. Let $\{u_1, \ldots, u_j\}$ be the nodes that prefer B' to $\tilde{A}$. There is no loss in assuming that $u_1, \ldots, u_j$ are root nodes in $\tilde{A}$. For each i, let n_i be the number of nodes in the arborescence rooted at u_i in $\tilde{A}$ that have different in-neighbors in $\tilde{A}$ and B' – note that each of these nodes prefers $\tilde{A}$ to B' (since the ones who prefer B' to $\tilde{A}$ are root nodes in $\tilde{A}$).

Let $n_t = \max\{n_i : 1 \leq i \leq j\}$. Let $\tilde{A}_t$ be the maximal sub-arborescence of $\tilde{A}$ rooted at u_t, and let X be those n_t nodes in $\tilde{A}_t$ that prefer $\tilde{A}$ to B'. We construct a branching A'. Let us define an arborescence A'_t rooted at u_t by modifying $\tilde{A}_t$ as follows: for each $w \notin \tilde{A}_t$ that is the descendant of some $v \in \tilde{A}_t$ in B', we add $B'(w)$. We define A' as the branching that contains A'_t and for which $A'(v) = B'(v)$ for each $v \notin A'_t$. So each node in $\tilde{A}_t$ has the same in-neighbor in A' as in B', except for the nodes in $X \cup \{u_t\}$.

The n_t nodes in X prefer A' to B', and u_t prefers B' to A', so we have $\frac{\phi(A',B')}{\phi(B',A')} = n_t$. Moreover, by $u(B) < \infty$ we also have $u(B') < \infty$, which implies that every node that prefers $\tilde{A}$ to B' is contained in a sub-arborescence of $\tilde{A}$ rooted at one of the nodes $u_1, \ldots, u_j$. Therefore we have $\phi(\tilde{A}, B') = \sum_{i=1}^{j} n_i$, which yields

$$\frac{\phi(A', B')}{\phi(B', A')} = n_t \geq \frac{1}{j}\sum_{i=1}^{j} n_i = \frac{\phi(\tilde{A}, B')}{\phi(B', \tilde{A})}.$$

Thus the claim follows. □

D Hardness Results: Proof of Theorem 6

Theorem 6 ($\star$)**.** *Given a digraph G where each node has a strict ranking over its incoming edges, it is* NP*-hard to decide if there exists*

(a) a popular branching in G where each node has at most 9 descendants;

(b) a popular branching in G with maximum out-degree at most 2.

In fact, we will show that Theorem 6 holds for simple graphs, therefore in this section we will assume that nodes have preferences over their in-neighbors. Nevertheless, we will say that an edge (u, v) is a *top-choice edge*, if u is the best choice for v. The following lemma will be useful to prove Theorem 6.

Lemma 30. *Let D be a digraph where each node has a strict ranking over its in-neighbors, and let A be a popular arborescence in D with dual certificate $\mathcal{Y}$. If C is a directed cycle consisting of only top-choice edges, then A enters C exactly once. Let a be the unique edge in $A \cap \delta^-(C)$ (guaranteed by Lemma 10), and let Y_a be the unique set in $\mathcal{Y}$ entered by a. Then $C \subseteq Y_a$.*

Proof. Observe that $c_A(e) \leq 1$ for any edge e in C, as e is a top-choice edge. Let $c_1, \ldots, c_k$ be the nodes of C in this order, with a pointing to c_k. Since c_k prefers c_{k-1} in C to the tail of a, we get $c_A((c_{k-1}, c_k)) = 0$ and thus $c_{k-1} \in Y_a$ by the constraints of LP2. Supposing $c_{k-2} \notin Y_a$ we get that $(c_{k-2}, c_{k-1}) \notin A$ because exactly one edge of A enters Y_a, by Lemma 10. Using that c_{k-1} has strict linear preferences and prefers c_{k-2} most, we obtain $c_A((c_{k-2}, c_{k-1})) = 0$, but this contradicts the constraints of LP2. Hence we get $c_{k-2} \in Y_a$ as well. Repeatedly applying this argument, we get that $C \subseteq Y_a$. □

Proof of Theorem 6, part (a). The reduction is from the NP-hard problem 3-SAT where we are given a 3-CNF formula $\varphi = \bigwedge_{j=1}^{m} c_j$ over variables $x_1, \ldots, x_n$ with each clause c_j containing at most 3 literals; the task is to decide whether φ can be satisfied. It is well known that the special case where each variable occurs at most 3 times is NP-hard as well, so we assume this holds for φ.

We define a digraph D_φ as follows. For each variable x_i we define a variable-gadget consisting of a directed 9-cycle A_i on nodes $a_i^1, \ldots, a_i^9$, together with nodes t_i and f_i, both having in-degree 0 in D_φ. The top choice for any node a_i^k on A_i is its in-neighbor a_i^{k-1} on A_i, its second choice is t_i if $k = 1$ and f_i otherwise.[5] Next, for each clause c_j we define a clause-gadget as a directed cycle C_j on nodes $c_j^1, \ldots, c_j^h$ where h is the number of literals in c_j; we may assume $h \in \{2, 3\}$. The top choice for any node c_j^k on C_j is its in-neighbor on C_j. The second choice of c_j^k depends on the k-th literal ℓ_j^k in c_j: it is t_i if $\ell_j^k = x_i$, and it is f_i if $\ell_j^k = \overline{x}_i$. We claim that the digraph D_φ defined this way admits a popular branching where every node has at most 9 descendants if and only if φ is satisfiable.

First let us suppose that we have a satisfying truth assignment for φ; we create a branching B. If variable x_i is true, then we add to B the edge (f_i, a_i^2) and all edges of A_i except for (a_i^1, a_i^2); if x_i is false we add to B the edge (t_i, a_i^1) and all edges of A_i except for (a_i^9, a_i^1). For each $j \in [m]$ let us choose a literal ℓ_j^k in clause c_j that is true according to our truth assignment. If $\ell_j^k = x_i$, then we let B contain the edge (t_i, c_j^k); if $\ell_j^k = \overline{x}_i$, then we let B contain the edge (f_i, c_j^k). In either case, we also add to B all edges of C_j but the one going into c_j^k; this finishes the definition of B. Observe that if x_i is true, then the descendants of f_i in B are the nodes of A_i, and the descendants of t_i are among the nodes of those cycles C_j where x_i is a literal of C_j; the case when x_i is false is analogous. Hence, each node in B has at most 9 descendants as promised.

Let us prove that B is popular. To this end, we define the graph D'_φ by adding a new dummy root r_0 to D_φ and making it the worst choice for every node in D_φ; moreover, we define an arborescence A in D'_φ by adding an edge from r_0 to each root of B. Then B is a popular branching in D_φ if and only if A is a popular arborescence in D'_φ. To show the latter, we define a dual certificate $\mathcal{Y}$ that contains the set $V(A_i)$ for each $i \in [n]$, the set $V(C_j)$ for each $j \in [m]$, and a singleton for each node except for those at which an edge of B enters some cycle A_i or C_j. It is straightforward to check that $\mathcal{Y}$ is indeed a dual solution proving the popularity of A in D'_φ, and therefore of B in D_φ.

[5] Throughout the rest of the proof, we treat superscripts in a circular way, that is, modulo length of the cycle in question.

Let us now suppose that we have a popular branching B with each node having at most 9 descendants; we are going to define a satisfying truth assignment for φ. Note that the only possible roots in B are the nodes in $R = \bigcup_{i \in [n]} \{t_i, f_i\}$, since any other node v has an in-neighbor in R (assuming v to be a root in B, adding an edge from R to v results in a branching more popular than B). Let A be the popular arborescence corresponding to B, and let $\mathcal{Y}$ be a dual certificate proving the popularity of A.

Let e_i be the edge entering A_i in B, and let Y_{e_i} be the unique set in $\mathcal{Y}$ entered by e_i. Similarly, let e'_j be the edge entering C_j in B, and let $Y_{e'_j}$ be the unique set in $\mathcal{Y}$ entered by e'_j. By Lemma 30, we know that $A_i \subseteq Y_{e_i}$ and $C_j \subseteq Y_{e'_j}$.

Let us define a truth assignment by setting x_i true if and only if the head of e_i is f_i. Note that all 9 nodes of A_i are descendants of the head of e_i. Hence, the head of an edge e'_j can only be f_i if x_i is false, and similarly, it can only be t_i if x_i is true. Thus, any cycle C_j must be the descendant of a node representing a true literal (where t_i and f_i represent x_i and $\overline{x}_i$, respectively). By the construction of D_φ, we have that any clause contains a literal set to true by the truth assignment, so φ is satisfiable, proving the theorem. $\square$

Proof of Theorem 6, part (b). We give a reduction from the variant of the Directed Hamiltonian Path problem where the input digraph has a root r with in-degree 0 that is the parent of all other nodes; it is easy to see that this version is also NP-hard. Let $G = (V \cup \{r\}, E)$ be our given input. For each node we fix an arbitrary ordering on its in-neighbors, and we denote by $n(v,i)$ the i-th in-neighbor of a node $v \in V$.

We are going to construct a digraph D that consists of a node gadget $\mathcal{G}_v$ for each $v \in V$, together with extra nodes r (having in-degree 0) and r'. The gadget $\mathcal{G}_v$ consists of a *core cycle* C_v together with *pendant cycles* $P_{v,1}, \dots, P_{v,d_v}$, each of length d_v, where d_v denotes the in-degree of v in G. The nodes in the core cycle are $c_v^1, \dots, c_v^{d_v}$, those in the i-th pendant cycle $P_{v,i}$ are $p_{v,i}^1, \dots, p_{v,i}^{d_v}$; we treat superscripts modulo d_v. The top choice for any node on these cycles is its in-neighbor within the cycle. The preferences are as follows, where for simplicity we define $c_r^1 := r$.

$$\begin{aligned} c_v^j &: \quad c_v^{j-1} \succ c^1_{n(v,j)} \succ c^1_{n(v,j+1)} \succ \cdots \succ c^1_{n(v,d_v)} \succ c^1_{n(v,1)} \succ \cdots \succ c^1_{n(v,j-1)}; \\ p_{v,i}^j &: \quad p_{v,i-1} \succ c_v^j; \\ r' &: \quad r. \end{aligned}$$

This finishes the definition of D.

We claim that G has a Hamiltonian path if and only if D has a popular branching with out-degree at most 2.

For the first direction, suppose that D has such a branching B. Clearly, r is a root of B, since it has in-degree 0. We claim that B is an arborescence with root r. First observe that any pendant cycle must be entered by A once, as otherwise there exists a root of B in the cycle, and adding the second-choice edge of this root node to B (coming form a core cycle unreachable from the pendant cycle) we obtain a branching B' that is more popular than B. Since $(r,v) \in E$ for each $v \in V$, each node c_v^j in a core cycle has an incoming edge from r, so such a node cannot be a root in B either, proving that B is indeed an arborescence. In particular, $\delta^-(C_v) \cap B \neq \emptyset$ for each $v \in V$.

By Lemma 30 we know that B enters any core cycle C_v exactly once, and therefore $|B \cap C_v| = d_v - 1$. In addition, there are exactly d_v edges of B pointing from C_v to the pendant cycles $P_{v,j}$, $j \in [d_v]$, because B is an arborescence. This implies that there can be at most 1 edge of B leaving C_v and pointing to another core cycle C_u, as otherwise the d_v nodes in C_v would together have more than $2d_v$ outgoing edges in B, yielding that at least one of them would have out-degree 3 in B, a contradiction.

Let us now define a set H of edges in G as follows: for each $u, v \in V$, we add (u, v) to H if and only if there is an edge from C_u to C_v in B. Furthermore, we add the edge (r, v) to H if and only if there is an edge from r to C_v in B; note that there can be at most one such edge, because $(r, r') \in B$ and B has out-degree at most 2. Observe that by the construction of D, we have $H \subseteq E$. Recall that by the previous paragraph, $|\delta^+(v) \cap H| \leq 1$ for each $v \in V \cup \{r\}$, and that $|\delta^-(v) \cap H| \geq 1$ for each $v \in V$. Moreover, H is acyclic, since any cycle in H would imply the existence of a cycle in B as well. Therefore, H must be a Hamiltonian path.

For the other direction, let H be a Hamiltonian path in G, starting from r. We define a popular branching B that happens to be an arborescence. First, for each $(u, v) \in H$ we add (c_u^1, c_v^j) to B where u is the j-th in-neighbor of v, and we also add all edges of C_v to B except for the one pointing to c_v^j; note that here we cover the case where $u = r$ as well. Notice that for each $v \in V$ there are at most d_v edges of B whose tail is in C_v. Hence, there exist d_v edges in $\delta^+(C_v)$ whose addition to B does not violate our bound on the out-degree and such that each of these edges points to a distinct pendant cycle $P_{v,j}$ (note that any pendant cycle can be connected to any node on C_v). Let us add these edges to B as well, together with all edges in $P_{v,j}$ except for the one whose head already has an incoming edge in B, for each $j \in [d_v]$. Finally, we add the edge (r, r') to B. It is easy to verify that the edge set B obtained this way is indeed an arborescence with root r, and has out-degree at most 2.

It remains to show that B is popular. To this end, we define the graph D' by adding a new dummy root r_0 to D and making it the worst choice for every node in D; moreover, we define an arborescence A in D' by adding the edge (r_0, r) to B. Then B is a popular branching in D if and only if A is a popular arborescence in D'. To prove the latter, we provide a dual certificate $\mathcal{Y}$ as follows. For each core or pendant cycle C, we put the set $V(C)$ into $\mathcal{Y}$, together with a singleton $\{v\}$ for each $v \in V(C)$ except for the node at which B enters C. We also add singletons $\{r'\}$ and $\{r\}$. The set system $\mathcal{Y}$ so obtained contains exactly $|V(D)|$ sets, so it remains to show that it fulfills the conditions of LP2. First note that any edge may enter at most two sets from $\mathcal{Y}$. Note also that if v is a node such that B enters a core or pendant cycle C at v, then $\delta^-(v) \cap B$ is the second choice for v (and its best choice is within C, the set of $\mathcal{Y}$ corresponding to v); otherwise $\delta^-(v) \cap B$ is the best choice for v. From these facts it is straightforward to verify the constraints of LP2, so the popularity of B and hence the theorem follows. $\square$

E Popular mixed branchings

A *mixed branching* P is a probability distribution (or lottery) over branchings in G, i.e., $P = \{(B_1, p_1) \dots, (B_k, p_k)\}$, where B_i is a branching in G for each i and $\sum_{i=1}^{k} p_i = 1$, $p_i \geq 0$ for all i. Popular mixed *matchings* were studied in [27] where it was shown that popular mixed matchings always exist and can be efficiently computed. Using the proof and method in [27], we now show that popular mixed branchings also always exist and such a mixed branching can be computed in polynomial time.

The function $\phi(B, B')$ that allowed us to compare two branchings B, B' generalizes to mixed branchings in a natural way. For mixed branchings $P = \{(B_1, p_1) \dots, (B_k, p_k)\}$ and $Q = \{(B'_1, q_1) \dots, (B'_\ell, q_\ell)\}$, the function $\phi(P, Q)$ is the expected number of nodes that prefer B to B' where B and B' are drawn from the probability distributions P and Q respectively; in other words,

$$\phi(P, Q) = \sum_{i=1}^{k} \sum_{j=1}^{l} p_i \, q_j \, \phi(B_i, B'_j).$$

Definition 31. A mixed branching P is popular if $\phi(P, Q) \geq \phi(Q, P)$ for all mixed branchings Q.

Consider the instance on 4 nodes a, b, c, d described in Section 1 that did not admit any popular branching. Let $B_1 = \{(a, b), (b, d), (d, c)\}$, $B_2 = \{(b, a), (a, c), (c, d)\}$, $B_3 = \{(c, d), (d, b), (b, a)\}$, and $B_4 = \{(d, c), (c, a), (a, b)\}$. It can be verified that the mixed matching $P = \{(B_1, 1/4), (B_2, 1/4), (B_3, 1/4), (B_4, 1/4)\}$ is popular.

Proposition 32. Every instance G admits a popular mixed branching.

The proof of the above proposition is the same as the one given in [27] for popular mixed matchings. Consider a two-player zero-sum game where the rows and columns of the payoff matrix M are indexed by all branchings $B_1, \dots, B_N$ in G. The (i, j)-th entry of the matrix M is $\Delta(B_i, B_j) = \phi(B_i, B_j) - \phi(B_j, B_i)$. A mixed strategy of the row player is a probability distribution $\langle p_1, \dots, p_N \rangle$ over the rows of M; similarly, a mixed strategy of the column player is a probability distribution $\langle q_1, \dots, q_N \rangle$ over the columns of M.

The row player seeks to find a mixed branching P that maximizes $\min_Q \Delta(P, Q)$. The column player seeks to find a mixed branching Q that minimizes $\max_P \Delta(P, Q)$. We have:

$$0 \;\leq\; \min_Q \max_P \Delta(P, Q) \;=\; \max_P \min_Q \Delta(P, Q) \;\leq\; 0,$$

where the first inequality follows by taking $P = Q$, the last inequality follows by taking $Q = P$, and the (middle) equality follows from Von Neumann's minimax theorem. Thus $\max_P \min_Q \Delta(P, Q) = 0$, i.e., there exists a probability distribution P over branchings such that $\Delta(P, Q) \geq 0$ for all mixed branchings Q. In other words, P is a popular mixed branching.

Computing a popular mixed branching. Since branchings in G and r-arborescences in $D = (V \cup \{r\}, E)$ are equivalent with respect to popularity, we will work in the graph D now. Analogous to [27], instead of *mixed* arborescences, it will be more convenient to deal with *fractional* arborescences.

A fractional arborescence x is a point in the arborescence polytope $\mathcal{A}$ of D, i.e., x is a point that satisfies constraints (1)-(2). So x is a convex combination of arborescences in D, i.e., it is a mixed arborescence $\{(A_1, \alpha_1) \ldots, (A_k, \alpha_k)\}$ where $x = \sum_j \alpha_j I_{A_j}$ (note that there may be multiple ways of expressing x as a mixed arborescence).

Conversely, every mixed arborescence $P = \{(A'_1, p_1) \ldots, (A'_t, p_t)\}$ maps to a fractional arborescence $\sum_k p_k I_{A'_k}$, where $I_{A'_k}$ is the incidence vector of arborescence A'_k. Thus there is a many-to-one mapping between mixed arborescences and fractional arborescences. Given a fractional arborescence x, we can efficiently find an equivalent mixed arborescence whose support is at most m using Carathéodory's theorem.

For any two fractional arborescences x, y, define $\Delta(x, y)$ as follows:

$$\Delta(x,y) = \sum_{u \in V} \sum_{\substack{e \in \delta^-(u) \\ e' \in \delta^-(u)}} x_e \, y_{e'} \, \mathsf{vote}_u(e, e'),$$

where $\mathsf{vote}_u(e, e')$ is 1, 0, -1 corresponding to $e \succ_u e'$, $e \sim_u e'$, and $e \prec_u e'$, respectively. Let P, Q be two mixed arborescences and let x, y be their corresponding fractional arborescences. It is easy to show that $\Delta(P, Q) = \Delta(x, y)$.

A popular fractional arborescence x is popular if $\Delta(x, y) \geq 0$ for all fractional arborescences y. It follows from Proposition 32 that popular fractional arborescences always exist in D. The following linear program finds a popular fractional arborescence x.

$$\begin{aligned} & \text{minimize } 0 && \text{(LP3)} \\ \text{subject to} \quad & \Delta(x, A) \;\geq\; 0 \quad \forall \text{ arborescences } A \text{ in } D \\ & x \;\in\; \mathcal{A} \end{aligned}$$

The feasible region of LP3 is the set of fractional arborescences that do not lose to any *integral* arborescence. This immediately implies that such a fractional arborescence is a popular fractional arborescence.

There are 2 sets of exponentially many constraints in LP3. Both sets of constraints admit efficient separation oracles: to decide if $x \in \mathcal{A}$ or not, a min r-cut needs to be computed in D with edge capacities given by x. If this cut $(S \cup \{r\}, V \setminus S)$ has value less than 1, then the set $V \setminus S$ forms a violating constraint w.r.t. (1); else $x \in \mathcal{A}$.

To decide if $\Delta(x, A) \geq 0$ for all arborescences A, we compute a min-cost arborescence in D with the following edge costs:

$$c_x(e) \;=\; \sum_{e' \succ_u e} x_{e'} - \sum_{e' \prec_u e} x_{e'} \quad \forall e \in E.$$

It is simple to check that for any arborescence A, we have $c_x(A) = \Delta(x, A)$. Thus x is *unpopular* if and only if there is an arborescence A with $c_x(A) < 0$.

Since a min-cost arborescence can be computed in polynomial time [13, 28], we can efficiently find a violating constraint $\Delta(x, A) < 0$ if x is unpopular. Thus we can compute a popular mixed arborescence in polynomial time using the ellipsoid method. Hence we have shown the following theorem.

Theorem 33. *A popular mixed branching in a digraph G where every node has preferences in arbitrary partial order over its incoming edges can be computed in polynomial time.*