Quantum algorithms

Pranab Sen

and

Quantum information and error correction

Naresh Sharma

Joint lecture 1

Information storage

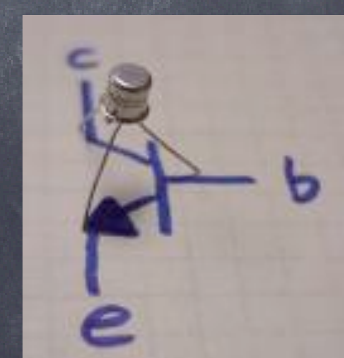
- **Classical:** Bit
- **Quantum:** Qubit

Bit

Physical device that can exist in exactly one of two distinguishable states at any point of time

Example: Transistor

- ON (saturation) state: bit "1"
- OFF (cut off) state: bit "0"



Bit, probabilistically

In a classical algorithm, at any point of time, the state of a bit is a probability distribution over "0" and "1".

Vector $\begin{pmatrix} 1-p \\ p \end{pmatrix}$, p: probability of state "1"

$$0 \leq p \leq 1$$

Quantum bit

Physical device that, if measured, gives exactly one of two distinguishable states at any point of time

Otherwise, it can be in an intermediate state, a superposition of "0" and "1".

Vector $\begin{pmatrix} e^{i\beta}\sqrt{1-p} \\ e^{i\alpha}\sqrt{p} \end{pmatrix}$

$e^{i\beta}\sqrt{1-p}$: amplitude of state "0"

$e^{i\alpha}\sqrt{p}$: amplitude of state "1"

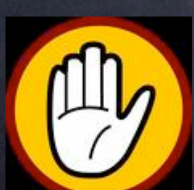
$0 \le p \le 1$

Amplitudes can be complex numbers!

Square of amplitude is probability of observing

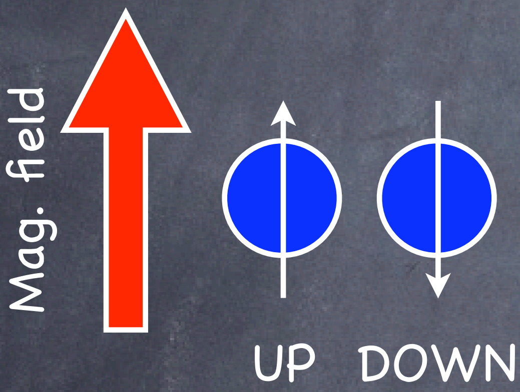
Bit versus Qubit

Bit	Qubit
$\begin{pmatrix} 1-p \\ p \end{pmatrix}$	$\begin{pmatrix} e^{i\beta}\sqrt{1-p} \\ e^{i\alpha}\sqrt{p} \end{pmatrix}$
p: prob. of "1"	p: prob. of "1"


$\begin{pmatrix} \sqrt{1-p} \\ \sqrt{p} \end{pmatrix}, \begin{pmatrix} \sqrt{1-p} \\ -\sqrt{p} \end{pmatrix}, \begin{pmatrix} \sqrt{1-p} \\ i\sqrt{p} \end{pmatrix}$ etc. are all different!

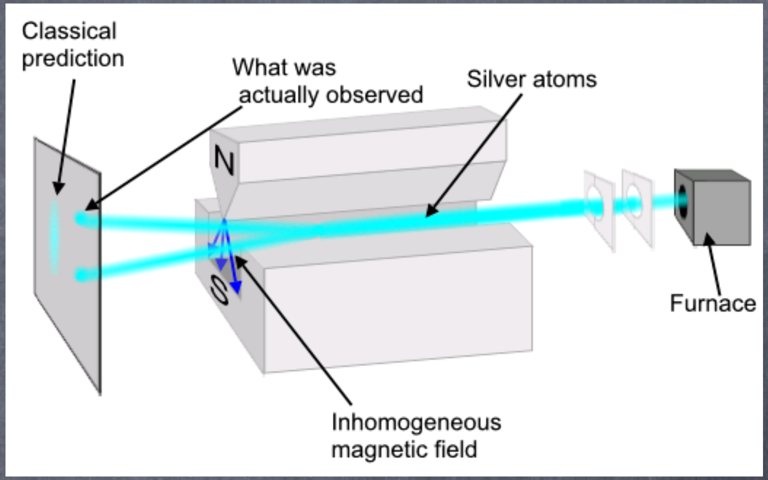
Electron spin: a qubit

- measured spin in a vertical magnetic field: UP (parallel) or DOWN (antiparallel), corr. to "0" and "1"

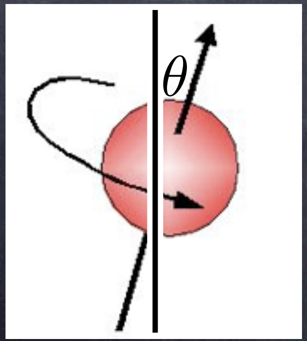


$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{Pr[UP] = 1}$$

$$|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{Pr[DOWN] = 1}$$



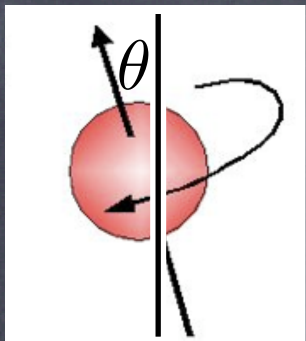
- unmeasured spin: can be aligned along any axis in 3D



$$\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} = \cos(\theta/2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\theta/2) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

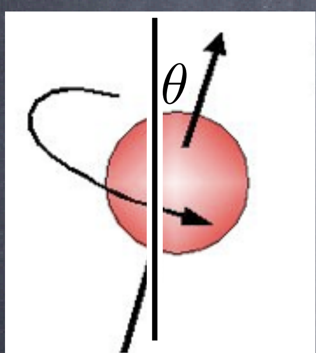
$$= \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$$

Non-positive amplitudes



$$\begin{pmatrix} \cos(\theta/2) \\ -\sin(\theta/2) \end{pmatrix} = \cos(\theta/2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin(\theta/2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos(\theta/2)|0\rangle - \sin(\theta/2)|1\rangle$$

$$\neq$$

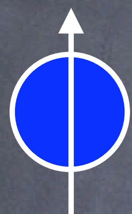
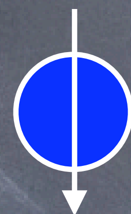
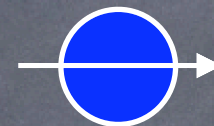
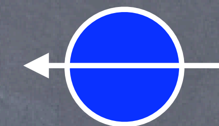


$$\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} = \cos(\theta/2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(\theta/2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$$

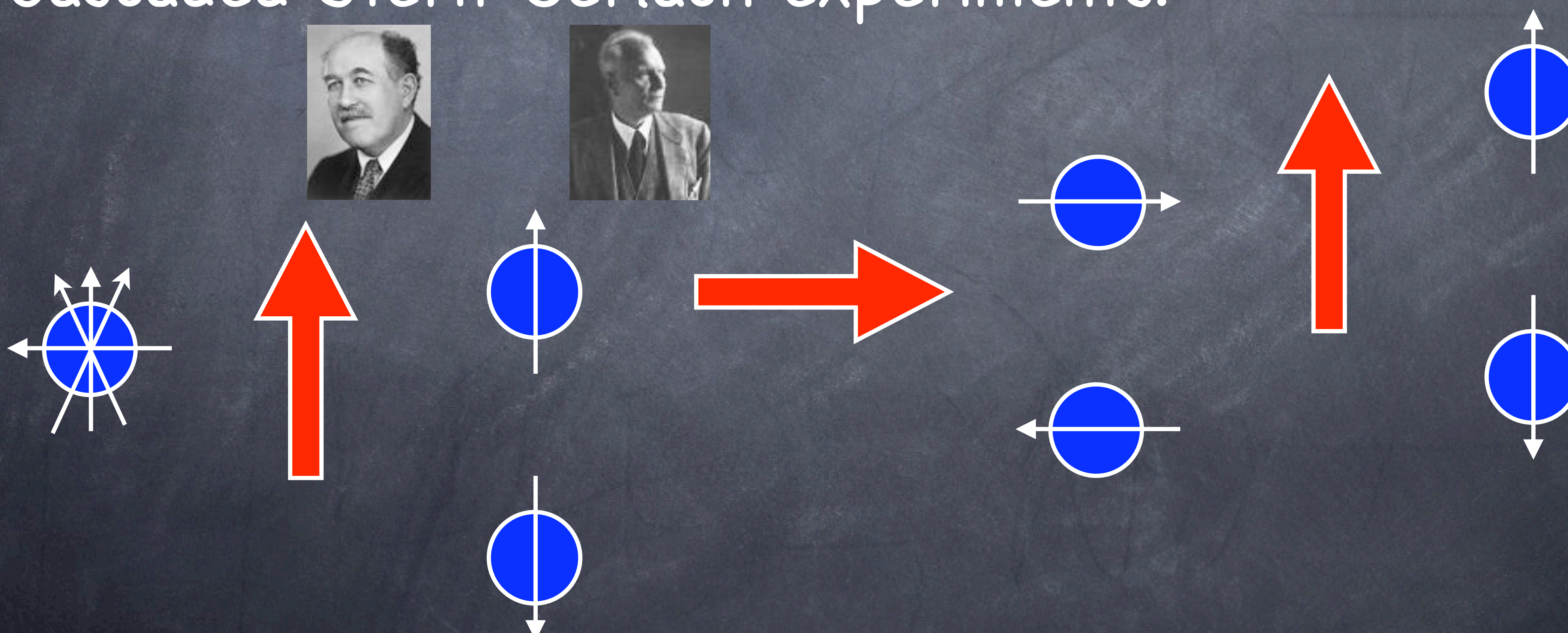
Note the factor of two in the angle above

Complex amplitudes denote spin axis "coming out" of the plane

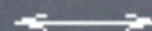
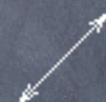
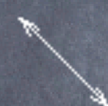
Qubit nature of spin

  $= |0\rangle$  $= |1\rangle$  $= \frac{|0\rangle + |1\rangle}{\sqrt{2}}$  $= \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

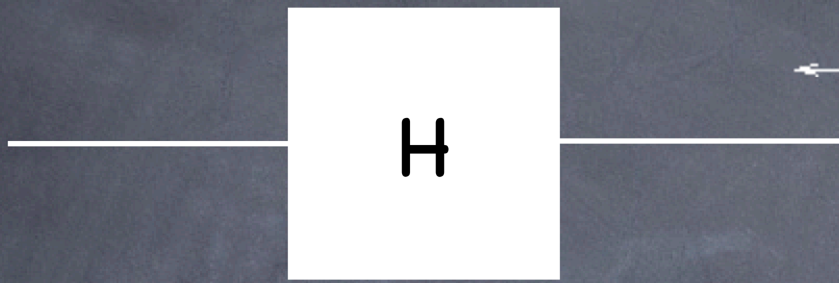
Cascaded Stern-Gerlach experiments:

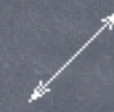


Photon polarisation

- Horizontal:  $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- Vertical:  $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Right diagonal:  $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$
- Left diagonal:  $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$
- Right circular:  $\frac{|0\rangle+i|1\rangle}{\sqrt{2}}$
- Left circular:  $\frac{|0\rangle-i|1\rangle}{\sqrt{2}}$

A quantum gate

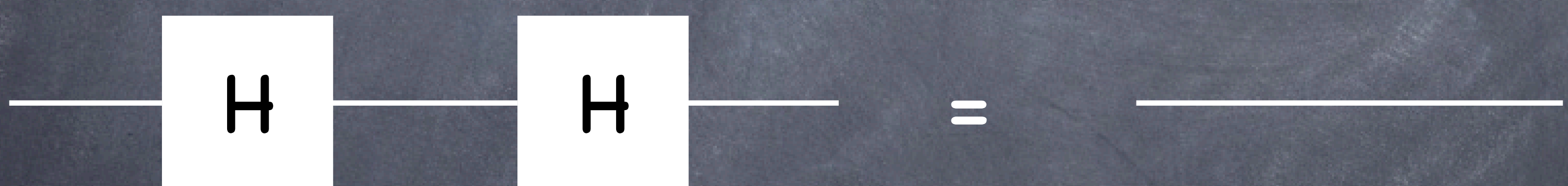


 $= |0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} =$ 

 $= |1\rangle \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}} =$ 

Extend H by linearity to all superpositions

$$\alpha_0|0\rangle + \alpha_1|1\rangle \mapsto \alpha_0\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \alpha_1\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

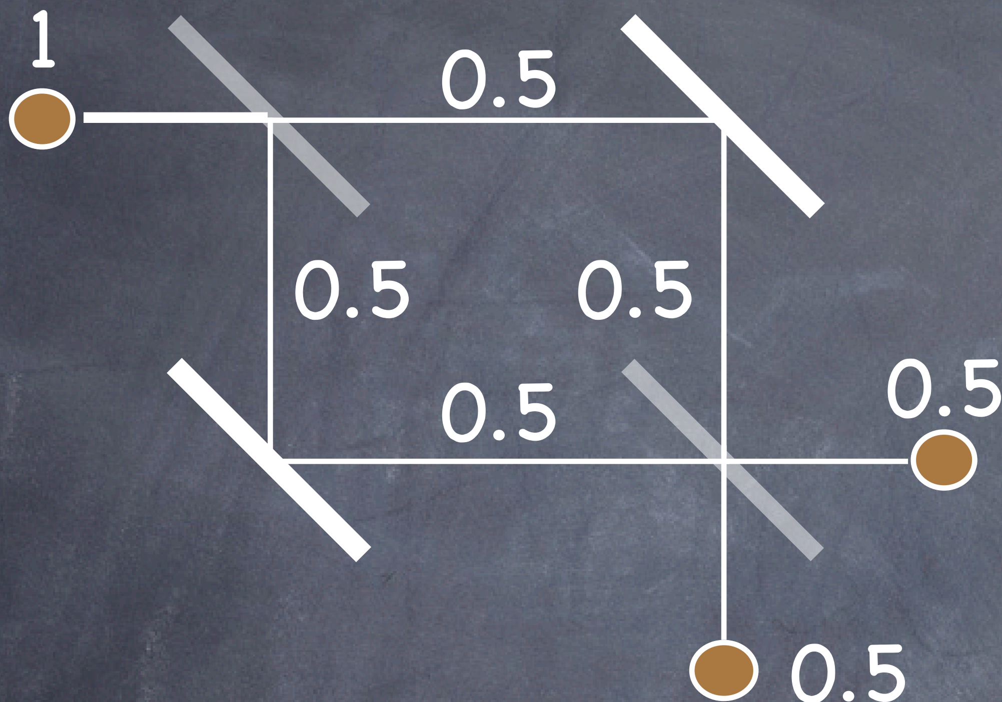


Applying Hadamard twice does nothing!

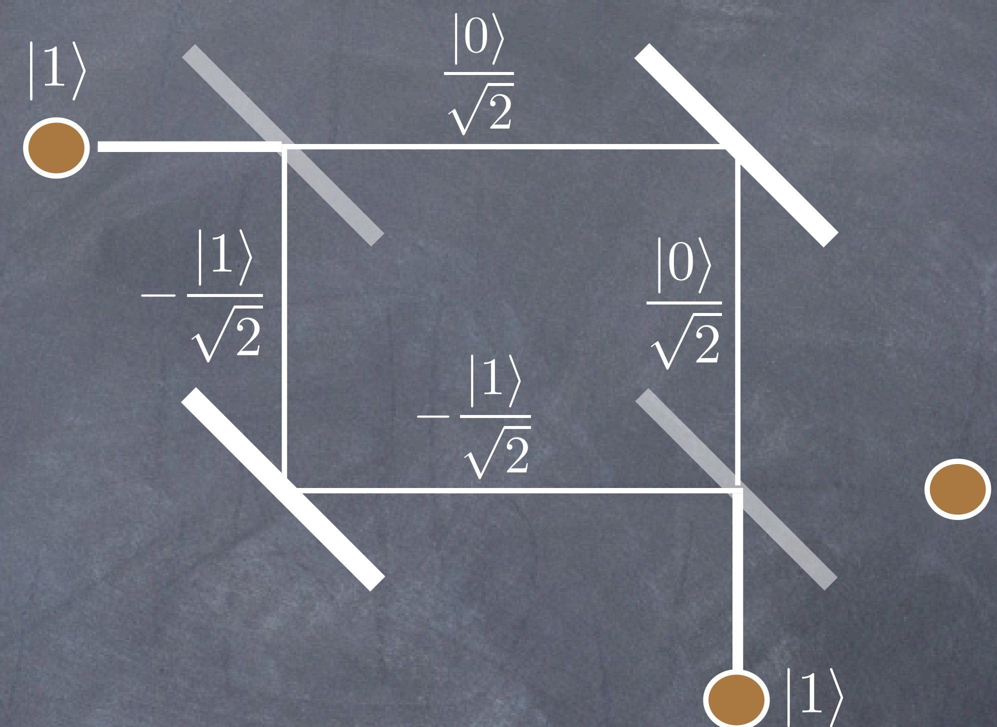
Constructive and destructive interference!

Quantum effect, no classical analogue

Quantum interference

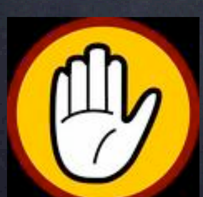


If reality were classical



Reality is in fact quantum

behaves as the Hadamard gate H
on polarisation qubit



From one to two

One qubit

Measured states: $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Two qubits

Measured states:

$$|00\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |01\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$|10\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |11\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

From one to two (contd)

One qubit:

Unmeasured states:

$$\alpha_0|0\rangle + \alpha_1|1\rangle$$
$$\alpha_0, \alpha_1 \in \mathbb{C}, |\alpha_0|^2 + |\alpha_1|^2 = 1$$
$$\Pr[\text{“0”}] = |\alpha_0|^2, \Pr[\text{“1”}] = |\alpha_1|^2$$

Two qubits:

Unmeasured states:

$$\alpha_{00}|00\rangle + \alpha_{10}|10\rangle + \alpha_{01}|01\rangle + \alpha_{11}|11\rangle$$
$$\alpha_{00}, \alpha_{10}, \alpha_{01}, \alpha_{11} \in \mathbb{C}$$
$$|\alpha_{00}|^2 + |\alpha_{10}|^2 + |\alpha_{01}|^2 + |\alpha_{11}|^2 = 1$$
$$\Pr[\text{“00”}] = |\alpha_{00}|^2, \Pr[\text{“10”}] = |\alpha_{10}|^2,$$
$$\Pr[\text{“01”}] = |\alpha_{01}|^2, \Pr[\text{“11”}] = |\alpha_{11}|^2$$

n qubits

Measured states: $|x\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$

$$x \in \{0,1\}^n$$

Unmeasured states: $\sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$

$$\alpha_x \in \mathbb{C}, \ \sum_{x \in \{0,1\}^n} |\alpha_x|^2 = 1, \ \Pr[x] = |\alpha_x|^2$$

State vector of n qubits is a unit length vector in

$$\mathbb{C}^{2^n} \cong \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n \text{ times}}$$

Computers, physically

- Physics experiment turned on its head
- Memory: physical system
- Input: initial condition
- Algorithm: dynamics
- Output: final state
- Classical computer: classical physics experiment
- Quantum computer: quantum physics experiment

Quantum algorithm

- Input: Intialised to a bit string $|x\rangle \otimes |\bar{0}\rangle, x \in \{0,1\}^n$
- Algorithm: Unitary transformation on input state
- Output: Bit string obtained by measuring state of computer at the end
- Require: With high probability, output is the correct answer