Blackbox PIT for Bounded Fan-in Depth 3 Circuits: the field doesn't matter

N. Saxena and C. Seshadri

Presented by Ramprasad Saptharishi

Second Mysore Park Workshop in Theoretical Computer Science:
Algorithms and Complexity
6th May, 2011

Outline

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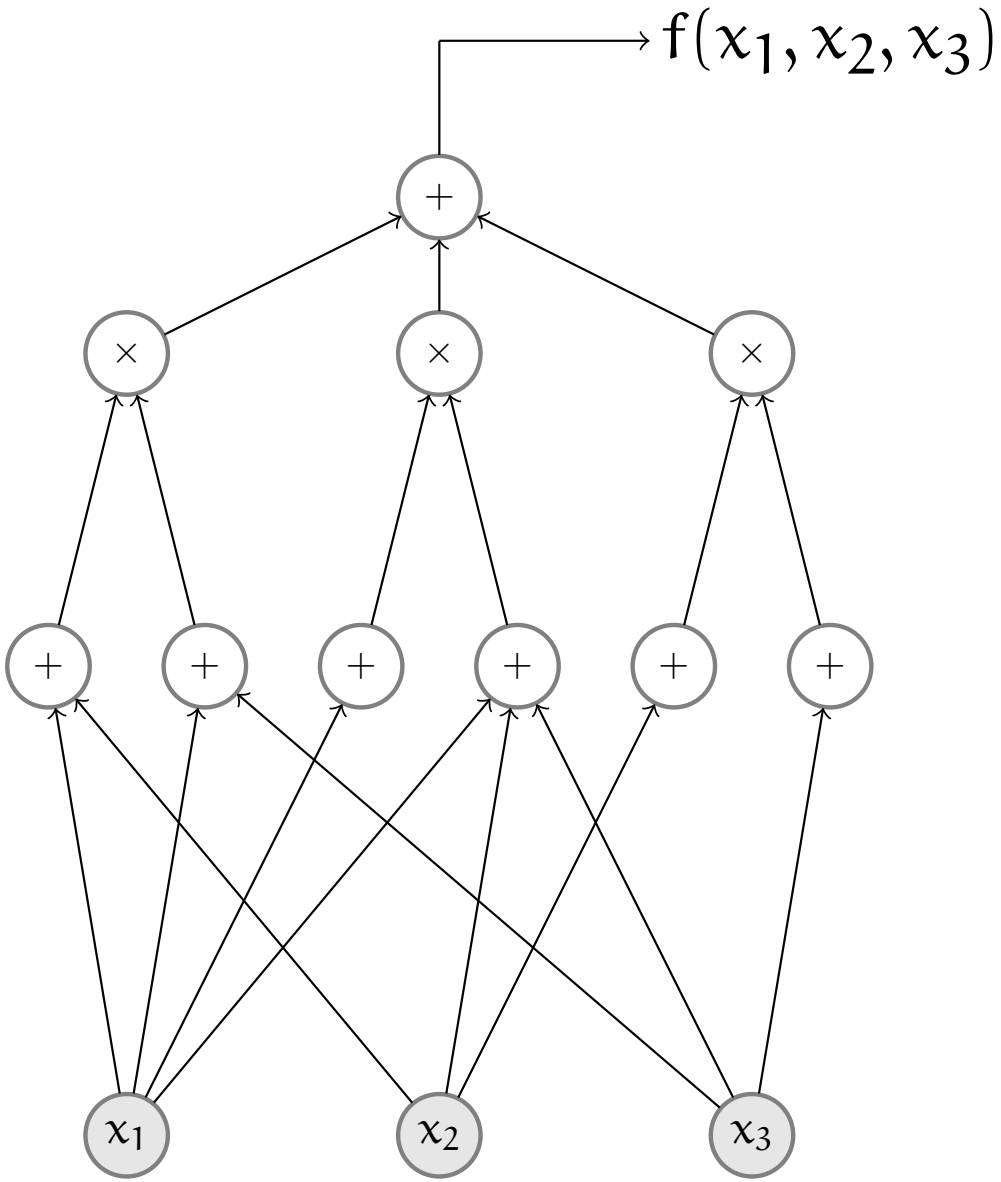
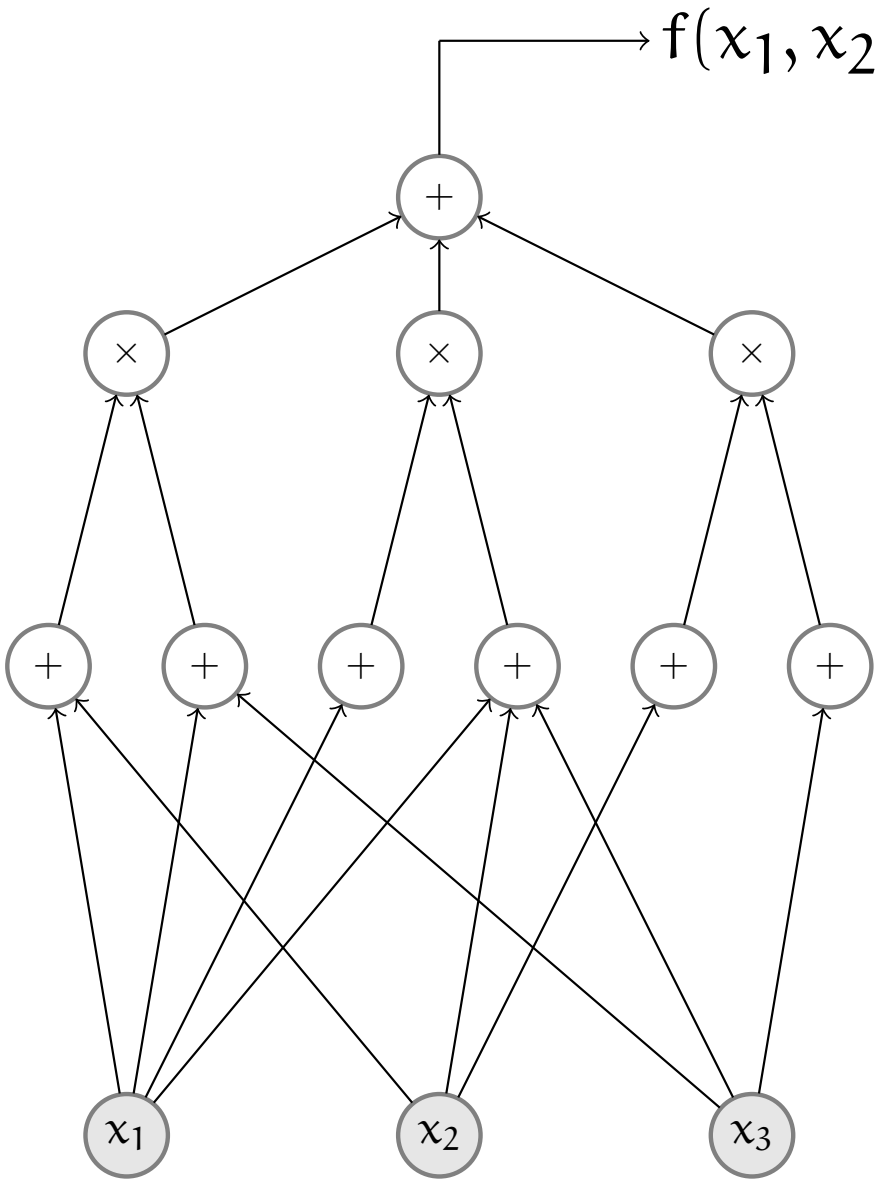
Arithmetic Circuits

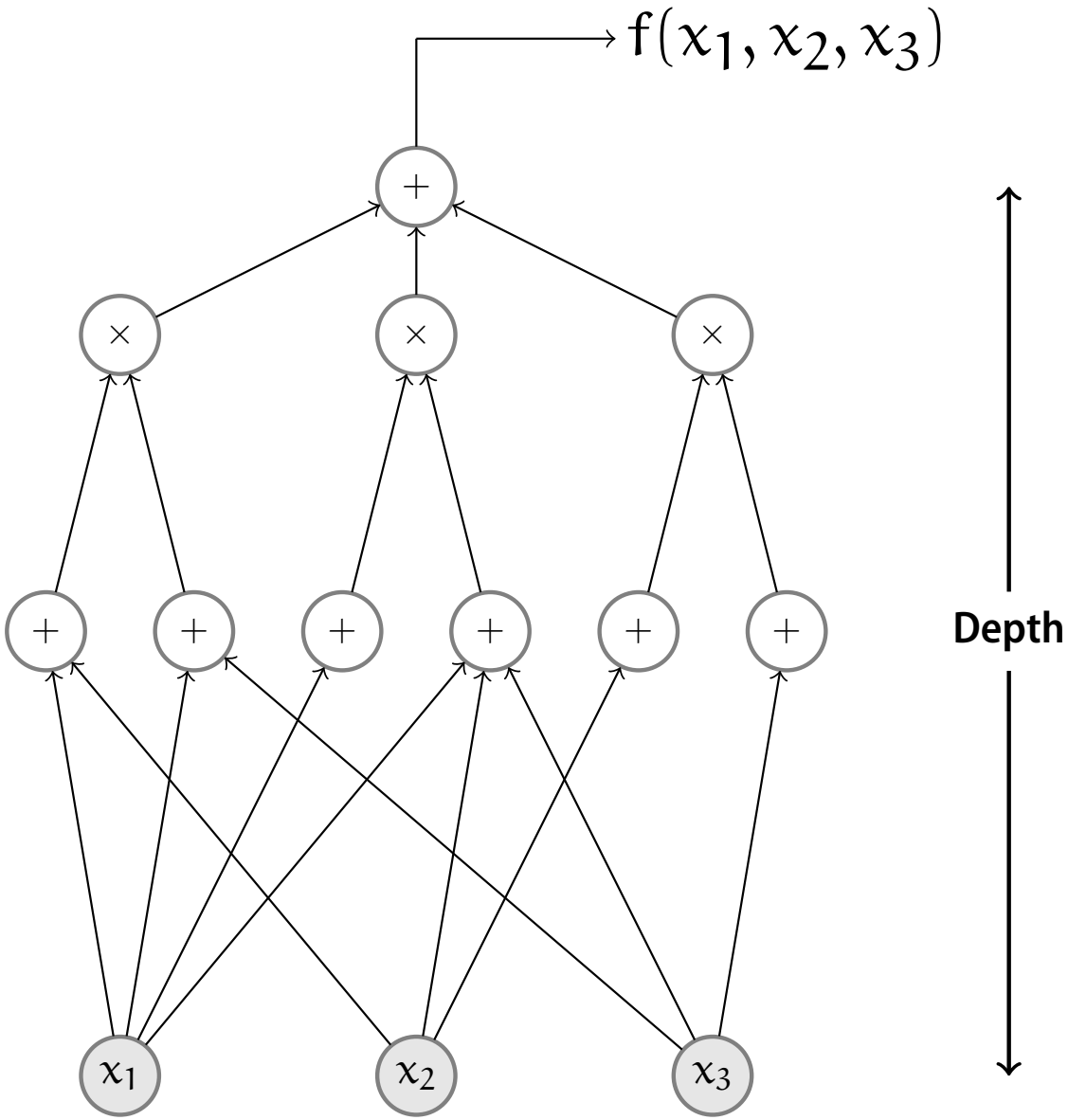


Arithmetic Circuits



Size = number of gates

Arithmetic Circuits



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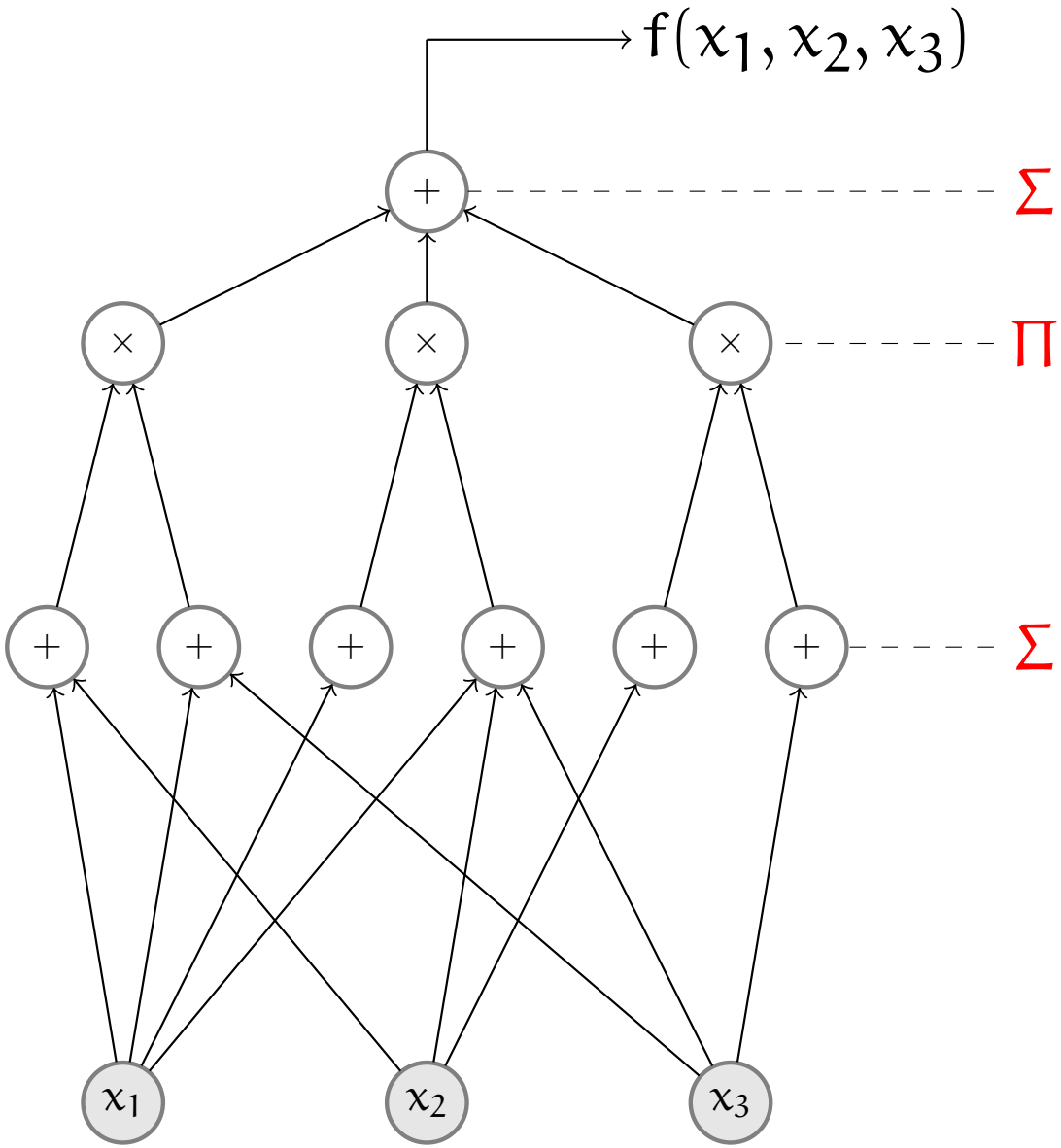
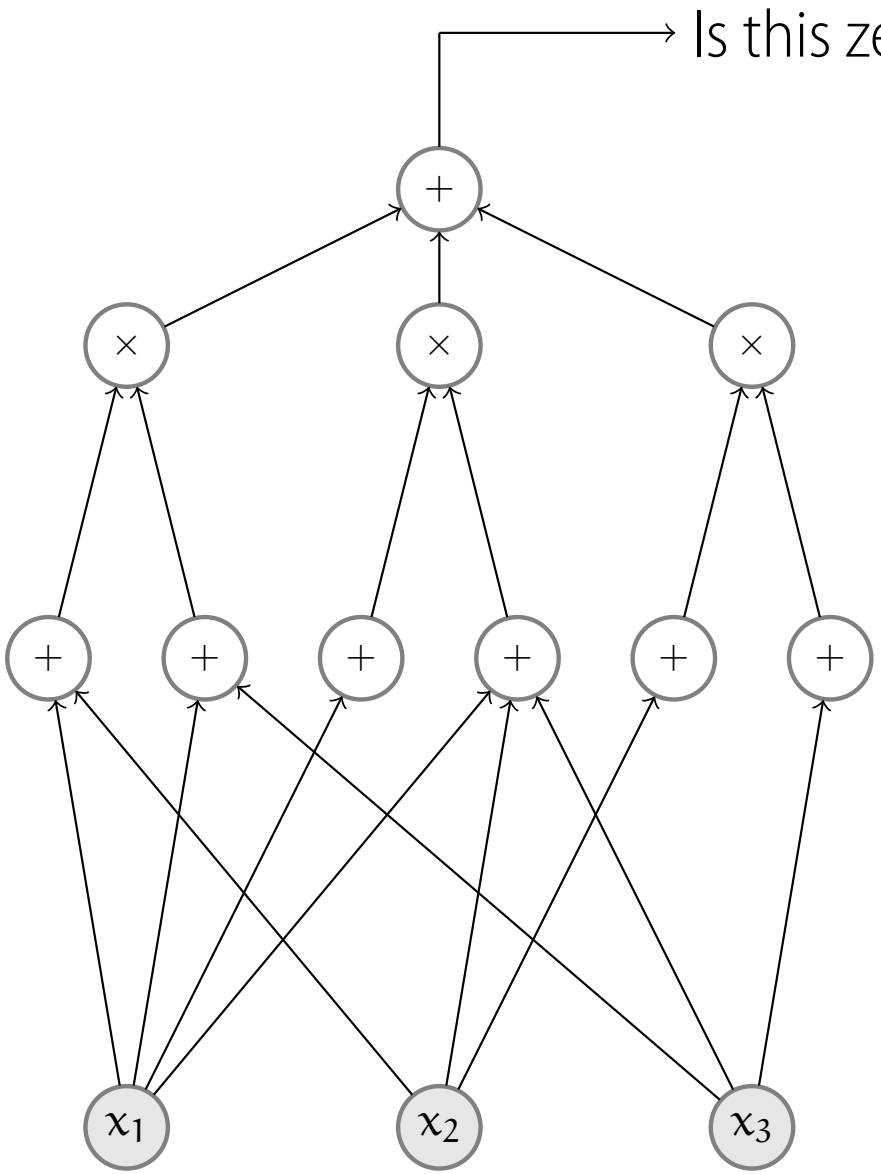
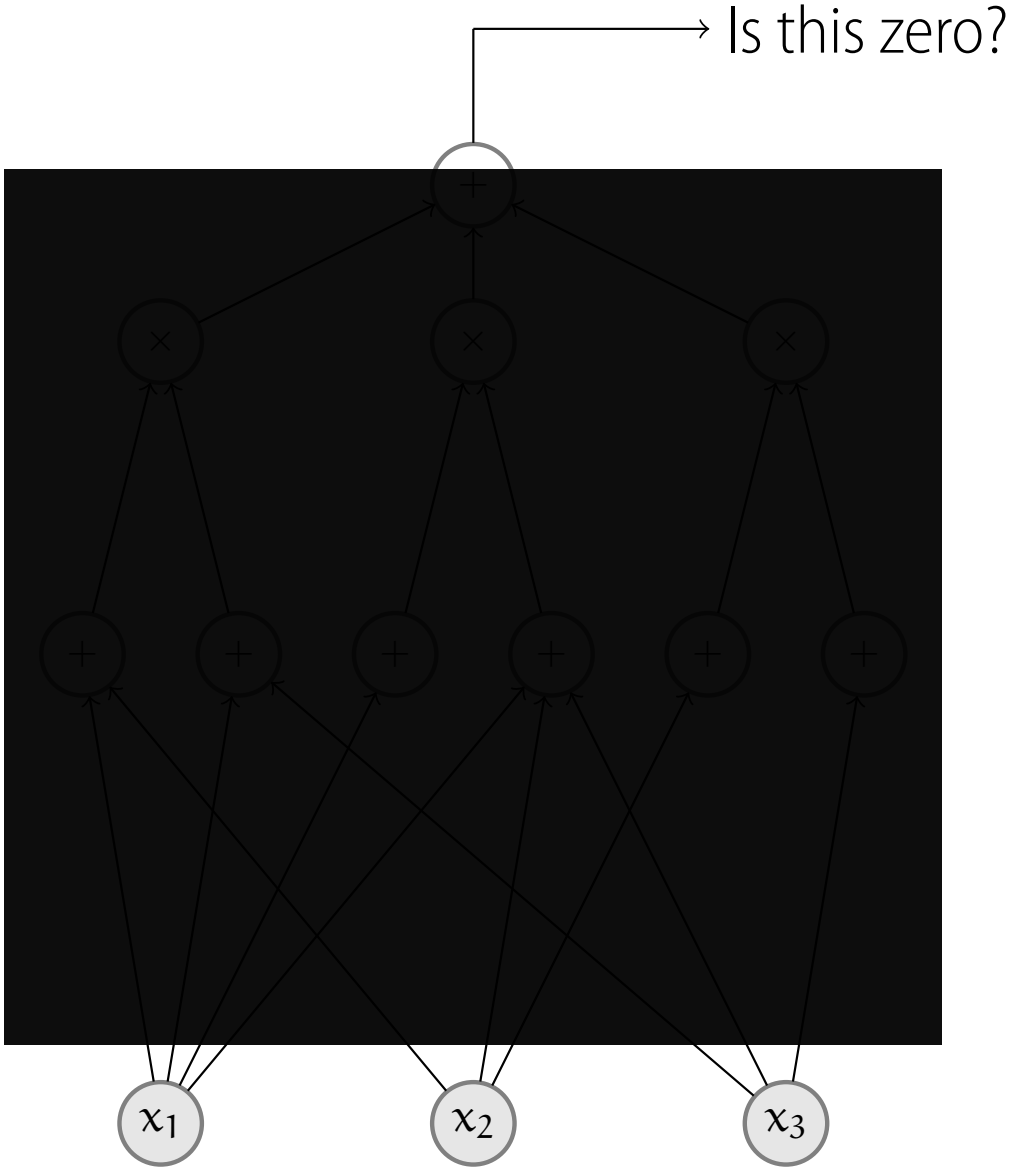


Identity Testing of Arithmetic Circuits



Blackbox Identity Testing of Arithmetic Circuits



Why do we care?

Part of many important results like IP $=$ PSPACE, the PCP theorem, AKS primality test etc.

Connections with lower bounds. [Kabanets-Impagliazzo03], [Agrawal05]: "Efficient PIT algorithms imply lower bounds"

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Of course, it is a natural problem!

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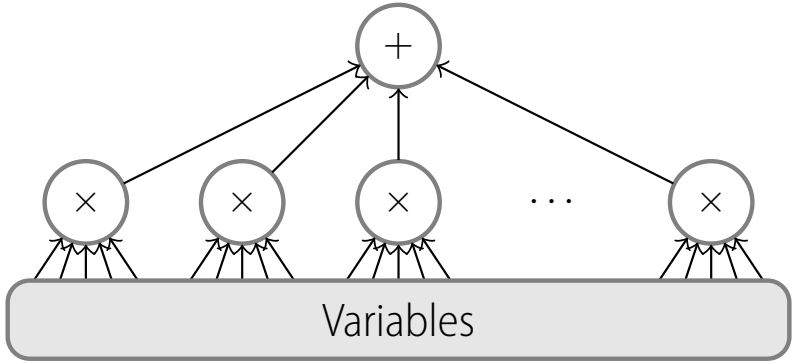
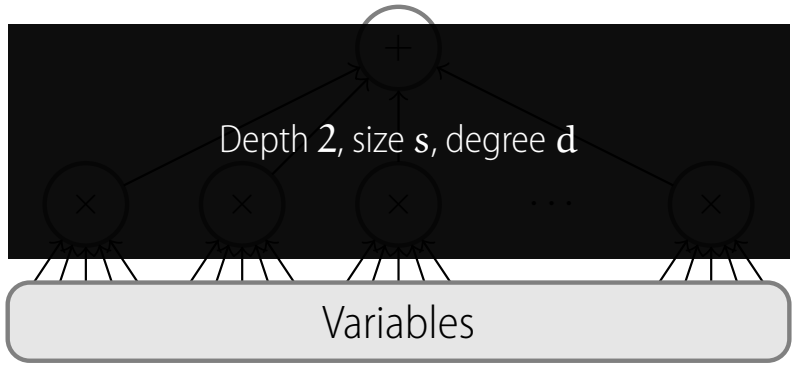
State of affairs: Depth 2



$$f = \sum_{i=1}^{\text{poly}} \text{monomial}_i$$

Depth 2 is easy (sparse polynomials)

State of affairs: Depth 2



Blackbox easy as well.

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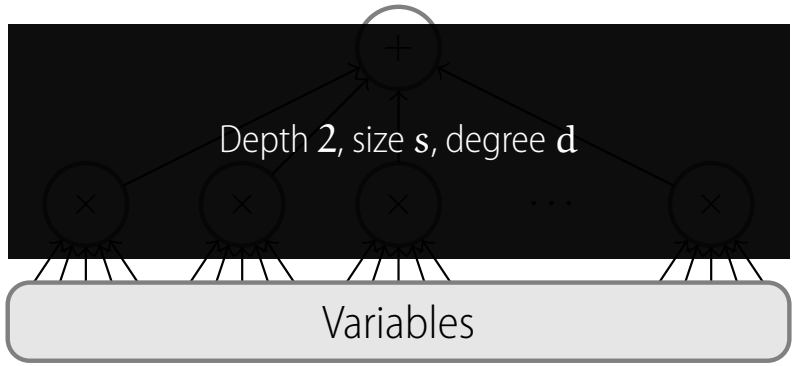

$$\Phi : \quad x_i \mapsto t^{(d+1)^i}$$

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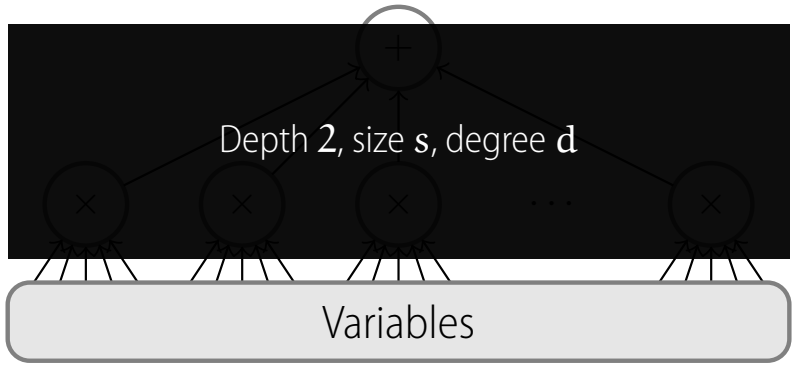

$$\Phi : \quad x_1^{a_1} \cdots x_n^{a_n} \quad \mapsto \quad t^A$$

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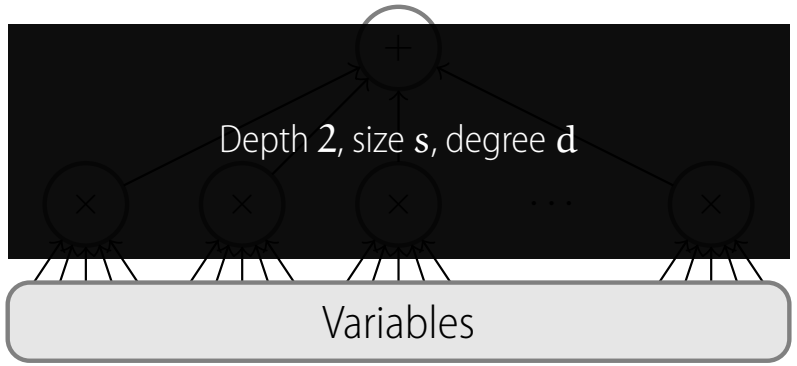

$$\Phi_r : \quad x_i \mapsto t^{(d+1)^i \bmod r}$$

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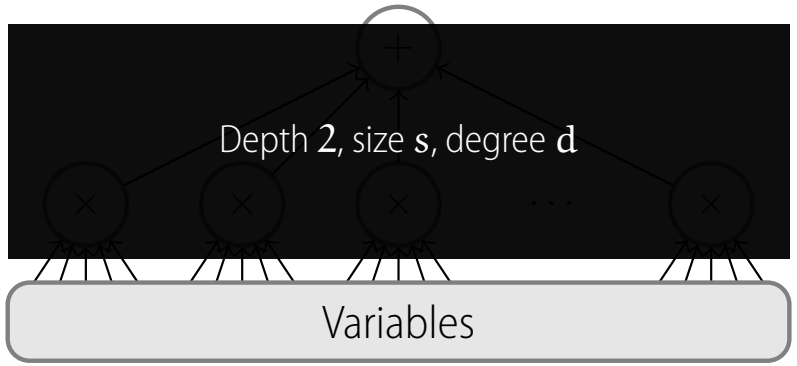

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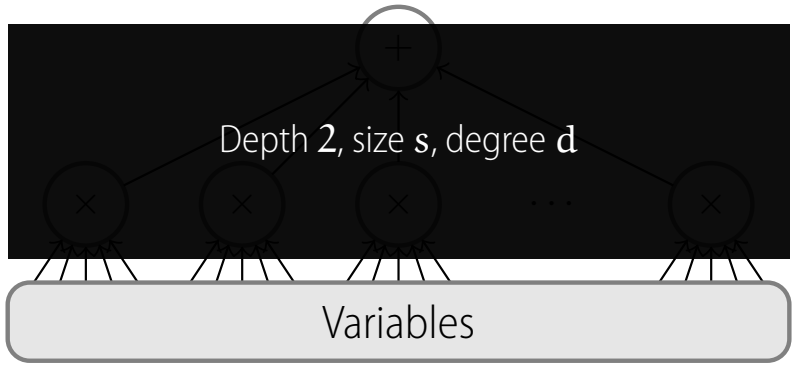

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$$\Phi_r(m_A) = \Phi_r(m_B) \quad \Longrightarrow \quad r \mid B - A$$

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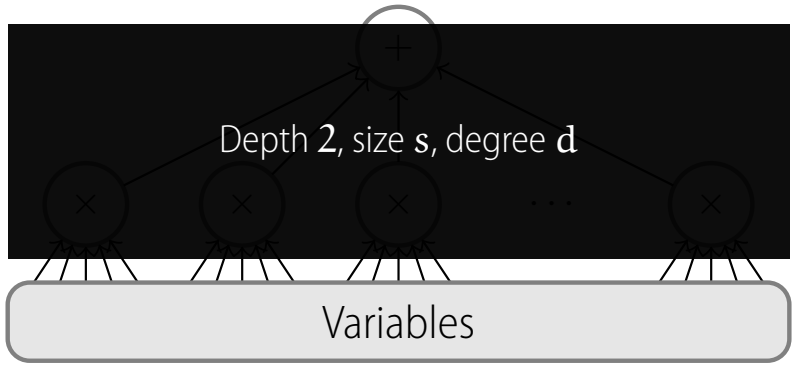

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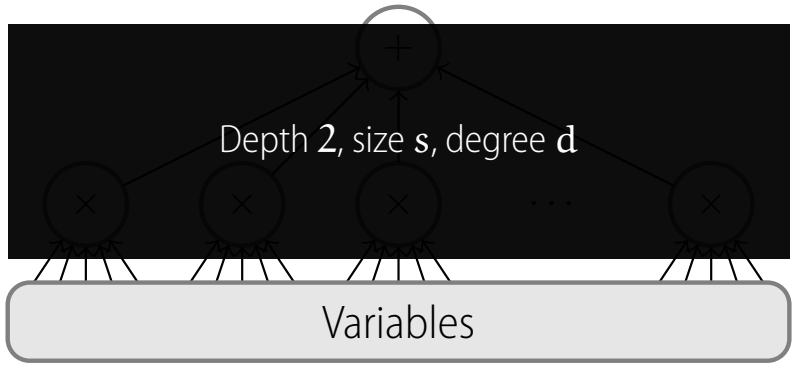
At most $(n \log d)$ bad r's for a pair A, B.

State of affairs: Depth 2



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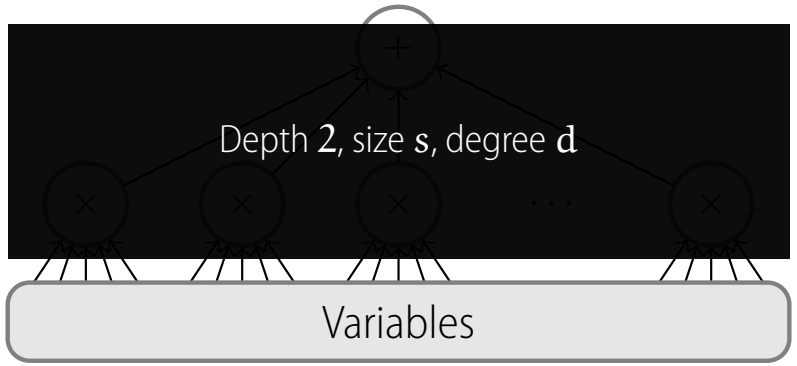
At most $s^2(n \log d)$ bad r's overall.

State of affairs: Depth 2



Hitting set:

$$\mathcal{H}_t = \left\{ (t^{(d+1) \bmod r}, \cdots, t^{(d+1)^n \bmod r}) \ : \ r \in [(s^2 n \log d)^2] \right\}$$

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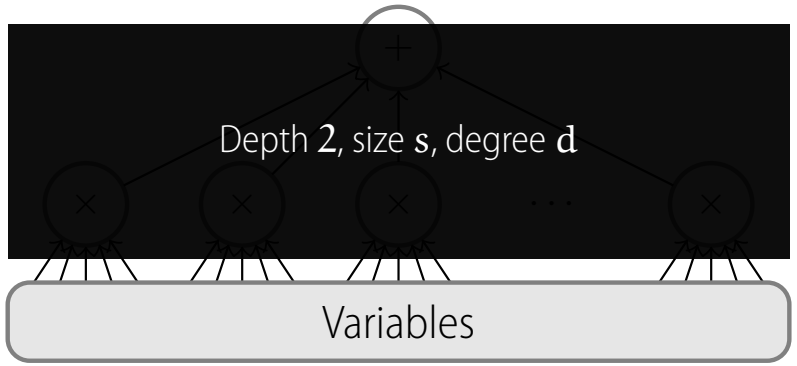

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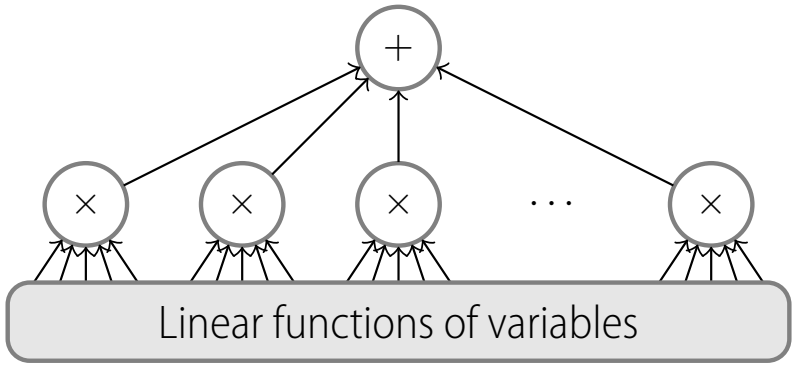
$$\mathcal{H} = \left\{ (t^{(d+1) \bmod r}, \cdots, t^{(d+1)^n \bmod r}) \; : \; \begin{array}{c} r \in [(s^2 n \log d)^2] \\ t \in [ndr + 1] \end{array} \right\}$$

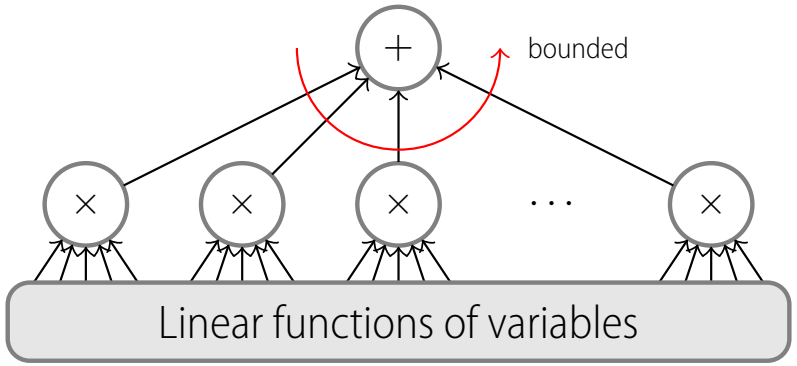
State of affairs: Depth 3



$$f = \sum_{i=1}^{k} \ell_{i1} \cdots \ell_{id}$$

PIT for even depth 3 circuits is open.

State of affairs: Depth 3



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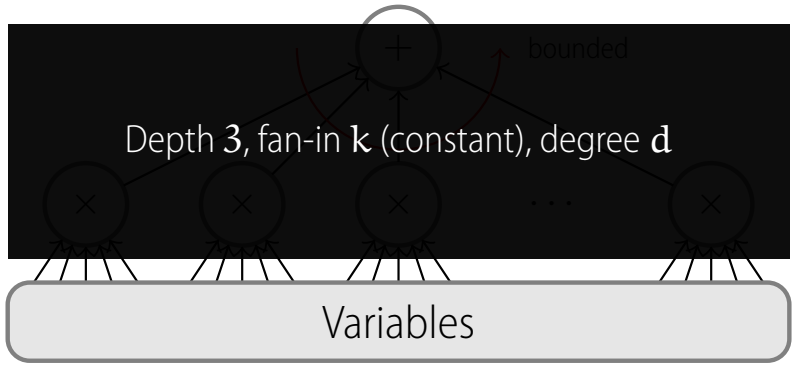
[KayalSaxena07] : polynomial time algorithm when k is a constant

State of affairs: Depth 3



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[KayalSaraf08] : Blackbox algorithm over the field $\mathbb{Q}$

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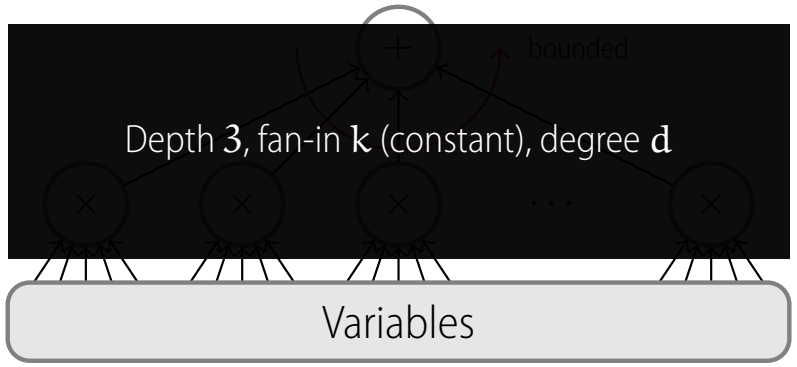

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This talk [SaxenaSeshadri11]: Blackbox algorithm over any field

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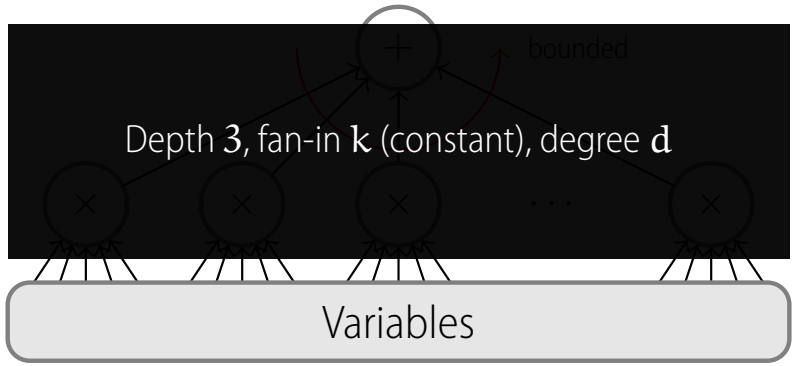

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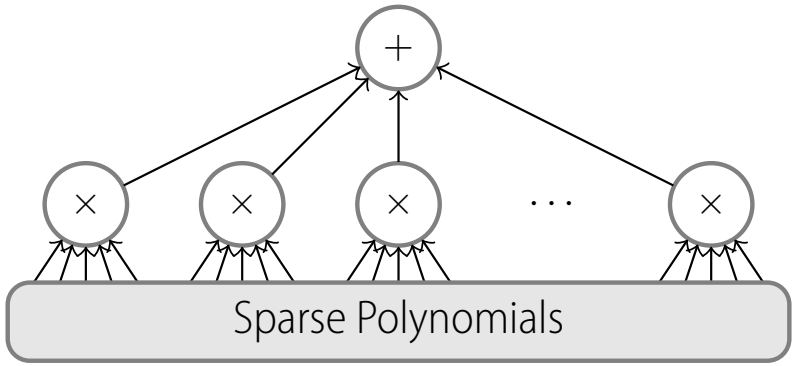
Main Ingredient: Rank bounds

State of affairs: Depth 4



$$f = \sum_{i=1}^{\text{poly}} g_{i1} \cdots g_{id}$$

[AgrawalVinay08] : Blackbox PIT for depth 4 implies $n^{O(\log n)}$ blackbox PIT for any depth!

Depth 4 is (almost) as hard as the general case.

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The Schwartz-Zippel Lemma

Lemma

Let $f(x_1, \cdots, x_n)$ *be a non-zero polynomial with total degree bounded by* d. *Then,*

$$\Pr_{a_1,\cdots,a_n}[f(\bar{a}) = 0] \quad \leq \quad \frac{d}{|\mathbb{F}|}$$

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Proof.

$$f(x_1, \cdots, x_n) \quad = \quad \sum_{i=0}^{k} f_i(x_2, \cdots, x_n) \cdot x_1^i$$

□

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Question: Can we get reduce the number of variables?

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... essentially the number of variables the circuit truly depends on.

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- For $\Sigma\Pi\Sigma$ circuits:

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- For $\Sigma\Pi\Sigma\Pi$ circuits:

$$C = \sum_{i=1}^{k} f_{i1} \cdots f_{id}$$

[BeeckenMittmannSaxena11]: $\text{rank}(C) = \text{TrDeg}\{f_{ij}\}$

General Road map

Whitebox:

- Compute the rank r of the circuit C.
- (if the rank was small) Construct a map

$$\Phi : \mathbb{F}[x_1, \cdots, x_n] \longrightarrow \mathbb{F}[y_1, \cdots, y_r]$$

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  that *preserves the rank*. That is, $\text{rank}(C) = \text{rank}(\Phi(C))$. And use Schwartz-Zippel to get a $O(d^r)$-sized hitting set.
- For large rank, ... prove the following:

Meta-theorem for rank bounds

If the given circuit C has rank more than R, then C cannot be identically zero.*

* Conditions apply

Outline

Meta-theorem for $\Sigma\Pi\Sigma$ circuits

Meta theorem for rank bounds

Any $\Sigma\Pi\Sigma(n, k, d)$ circuit that has rank more than $R(n, k, d)$ cannot be identically zero.

Ways to cheat:

$$(x + y)^2 - x^2 - y^2 - 2xy$$

Meta-theorem for $\Sigma\Pi\Sigma$ circuits

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Meta-theorem for $\Sigma\Pi\Sigma$ circuits

Meta theorem for rank bounds

Any simple $\Sigma\Pi\Sigma(n, k, d)$ circuit that has rank more than $R(n, k, d)$ cannot be identically zero.

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- Circuit must be *simple*.

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Meta-theorem for $\Sigma\Pi\Sigma$ circuits

Meta theorem for rank bounds

Any simple, minimal $\Sigma\Pi\Sigma(n, k, d)$ circuit that has rank more than $R(n, k, d)$ cannot be identically zero.

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- Circuit must be *minimal*.

Rank bounds for $\Sigma\Pi\Sigma$ circuits

Theorem

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[KayalSaxena07]	$R(n, k, d) = \Omega(k \log d)$ over finite fields

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[SaxenaSeshadri09]	$R(n, k, d) = O(k^3 \log d)$

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[SaxenaSeshadri09]	$R(n, k, d) = O(k^3 \log d)$
[KayalSaraf09]	$R(n, k, d) = 2^{O(k \log k)}$ over the field $\mathbb{R}$

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Translates to a poly$(d^{R(n,k,d)}, n)$ whitebox PIT.

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What about blackbox?

Outline

Rank bounds to blackbox algorithms

Theorem (KarninShpilka08)

If R *is an upper bound on the rank of a simple minimal* $\Sigma\Pi\Sigma(n, k, d)$ *identity, then there is a blackbox polynomial identity test running in time* $\mathrm{poly}(d^R, n)$.

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General Idea:

- Find a linear map $\Phi : \mathbb{F}[x_1, \cdots, x_n] \to \mathbb{F}[y_1, \cdots, y_{R+1}]$ that preserves a *subspace* of dimension $R + 1$ (if it exists).
- Apply Schwartz-Zippel on this $(R + 1)$-variate circuit.

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- Large rank circuits stay non-identities
  Smaller rank circuits are transformed "isomorphically".

Rank preserving maps

Lemma (GabizonRaz05)

Given n, k*, there is a set of* $nk^2 + 1$ *of linear transformations* $\{\Phi_t\} : \mathbb{F}^n \to \mathbb{F}^k$ *such that for any subspace* $V \subset \mathbb{F}^n$ *of dimension* k*, there is at least one* Φ_t *that is an isomorphism between* V *and* $\mathbb{F}^k$*.*

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Rank preserving maps

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Main issue with rank bound approaches

- [KayalSaxena07]: $R(n, k, d) = \Omega(k \log d)$ over finite fields.
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*** SPOILER ***
Certificate is an ideal of small rank

Outline

$$\begin{aligned} C &= T_1 + \cdots + T_k \\ \text{where} \quad T_i &= \ell_{i1} \cdots \ell_{id} \end{aligned}$$

Reviewing the Kayal-Saxena test

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Outline

Some notation

$$
\begin{aligned}
T &= (x+y+z)(2y+3u+z)^2(3x+2y+2z)(3x+6u+2z) \\
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Cancellation Lemma

Lemma

Let I *be an ideal generated by multiplication terms, and let* $\ell \notin \mathit{radSpan}(\mathrm{I})$. *Then for any polynomial* $\mathfrak{g}$,

$$\ell\mathfrak{g} \in \mathrm{I} \quad \textit{if and only if} \quad \mathfrak{g} \in \mathrm{I}$$

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$$\sum g_i x_1^i \in I \quad \text{if and only if} \quad g_i \in I \text{ for each } i \qquad \square$$

Chinese Remainder Theorem

Theorem

Let I *be an ideal generated by multiplication terms. Let* f *and* g *be two multiplication terms such that*

$$\begin{aligned} \mathrm{L}(\mathrm{f}) \cap \mathit{radSpan}(\mathrm{I}) &= \emptyset \\ \mathrm{L}(\mathrm{g}) \cap \mathit{radSpan}(\mathrm{I}, \mathrm{f}) &= \emptyset \end{aligned}$$

Then, $\langle \mathrm{I}, \mathrm{f} \rangle \cap \langle \mathrm{I}, \mathrm{g} \rangle = \langle \mathrm{I}, \mathrm{fg} \rangle$.

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$$h = i_1 + af = i_2 + bg$$

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$$\therefore \quad b = i' + cf$$

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□

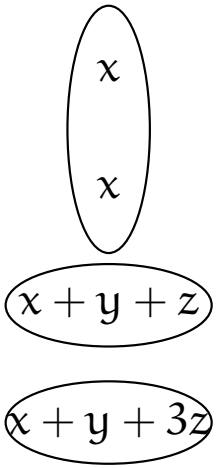
In search of a certificate

x	y	x	$3x+y$	x
x	$x+y$	y	$5x+y$	y
$x+y+z$	$x+z$	y	$x+y+z$	$x+y+z$
$x+y+3z$	$x+y+z$	$x+y+4z$	$x+y+z$	$x+y+z$
$\mathbf{T_1}$	$\mathbf{T_2}$	$\mathbf{T_3}$	$\mathbf{T_4}$	$\mathbf{T_5}$

$$I = \langle 0 \rangle$$

$$\mathsf{radSpan}(I) = \mathsf{span}(0)$$

In search of a certificate



T_1	T_2	T_3	T_4	T_5
x	y	x	$3x + y$	x
x	$x + y$	y	$5x + y$	y
$x + y + z$	$x + z$	y	$x + y + z$	$x + y + z$
$x + y + 3z$	$x + y + z$	$x + y + 4z$	$x + y + z$	$x + y + z$

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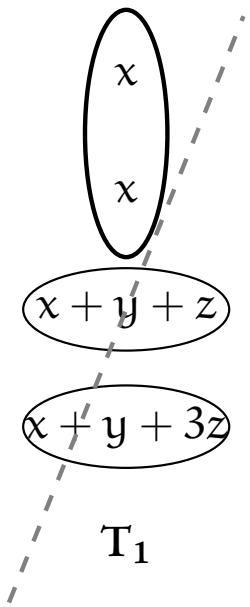

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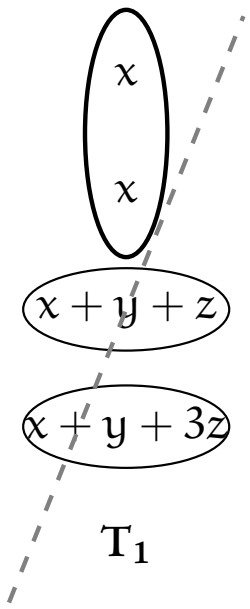
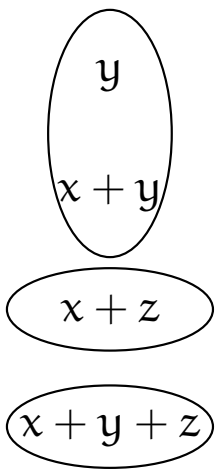
$$I = \langle x^2 \rangle$$

$$\mathsf{radSpan}(I) = \mathsf{span}(x)$$

In search of a certificate



T_1



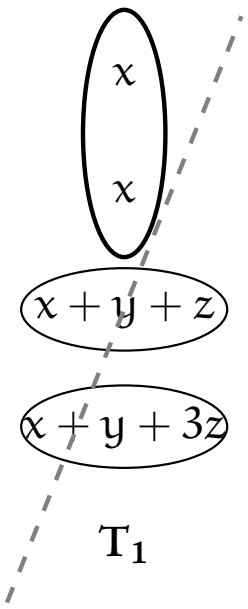
T_2

T_3	T_4	T_5
x	$3x + y$	x
y	$5x + y$	y
y	$x + y + z$	$x + y + z$
$x + y + 4z$	$x + y + z$	$x + y + z$

$$I = \langle x^2 \rangle$$

$$\mathsf{radSpan}(I) = \mathsf{span}(x)$$

In search of a certificate



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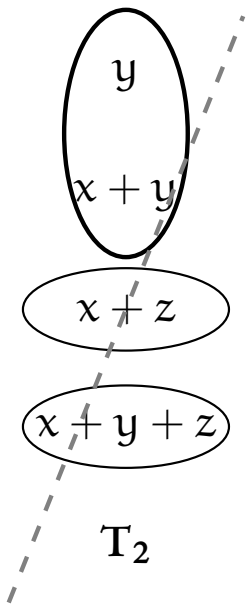


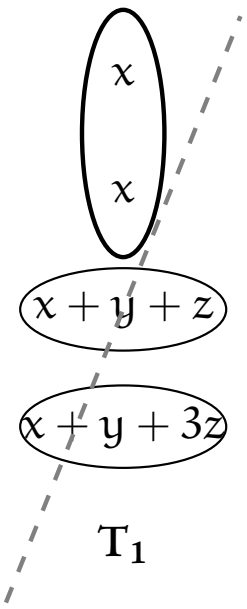
T_2

T_3	T_4	T_5
x	$3x + y$	x
y	$5x + y$	y
y	$x + y + z$	$x + y + z$
$x + y + 4z$	$x + y + z$	$x + y + z$

$$I = \langle x^2, y(x + y) \rangle$$

$$\mathsf{radSpan}(I) = \mathsf{span}(y, x + y)$$

In search of a certificate



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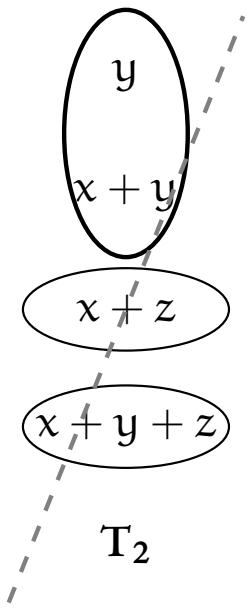
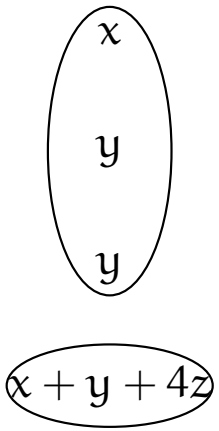


T_2



T_3

T_4	T_5
$3x + y$	x
$5x + y$	y
$x + y + z$	$x + y + z$
$x + y + z$	$x + y + z$

$$I = \langle x^2, y(x + y) \rangle$$

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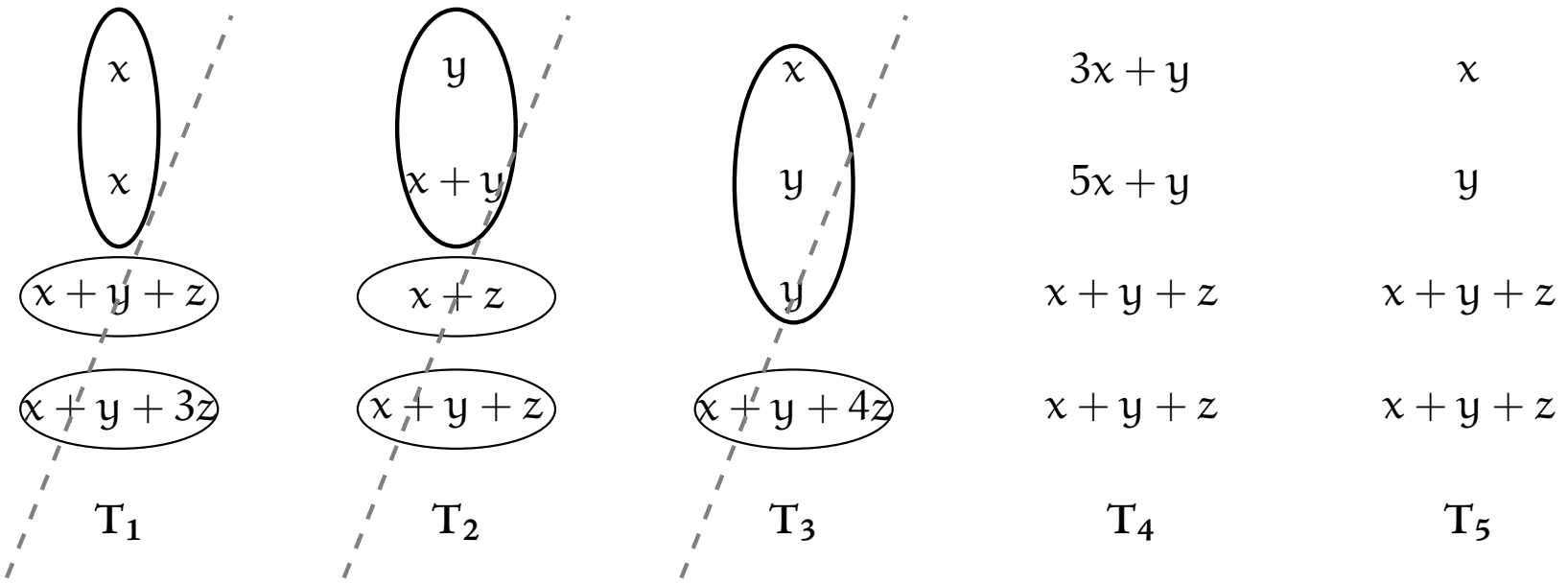

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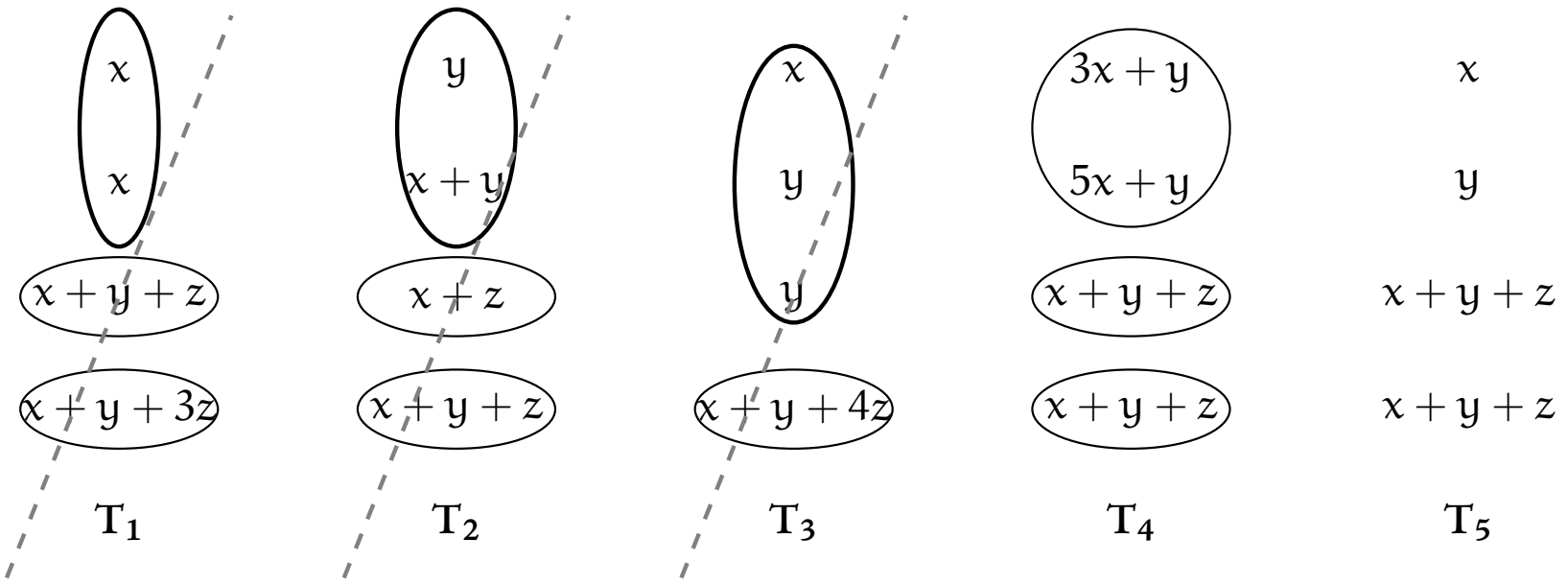

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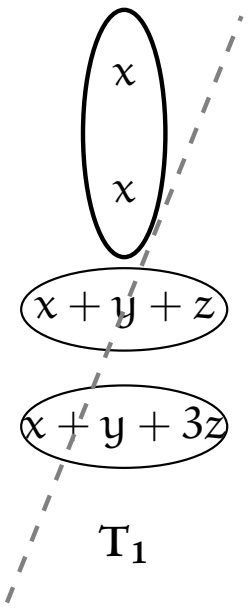

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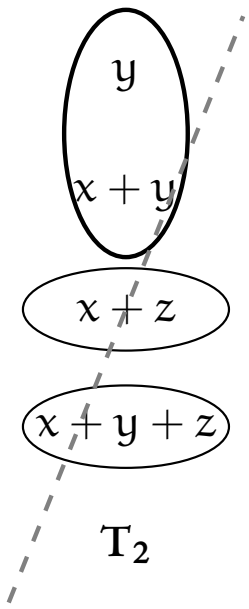
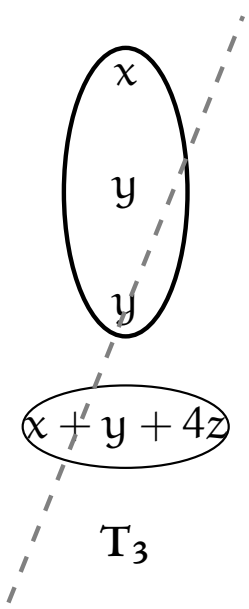


T_2



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$x + y + z$	$x + y + z$

T_4 T_5

$$I = \langle x^2, y(x+y) \rangle$$

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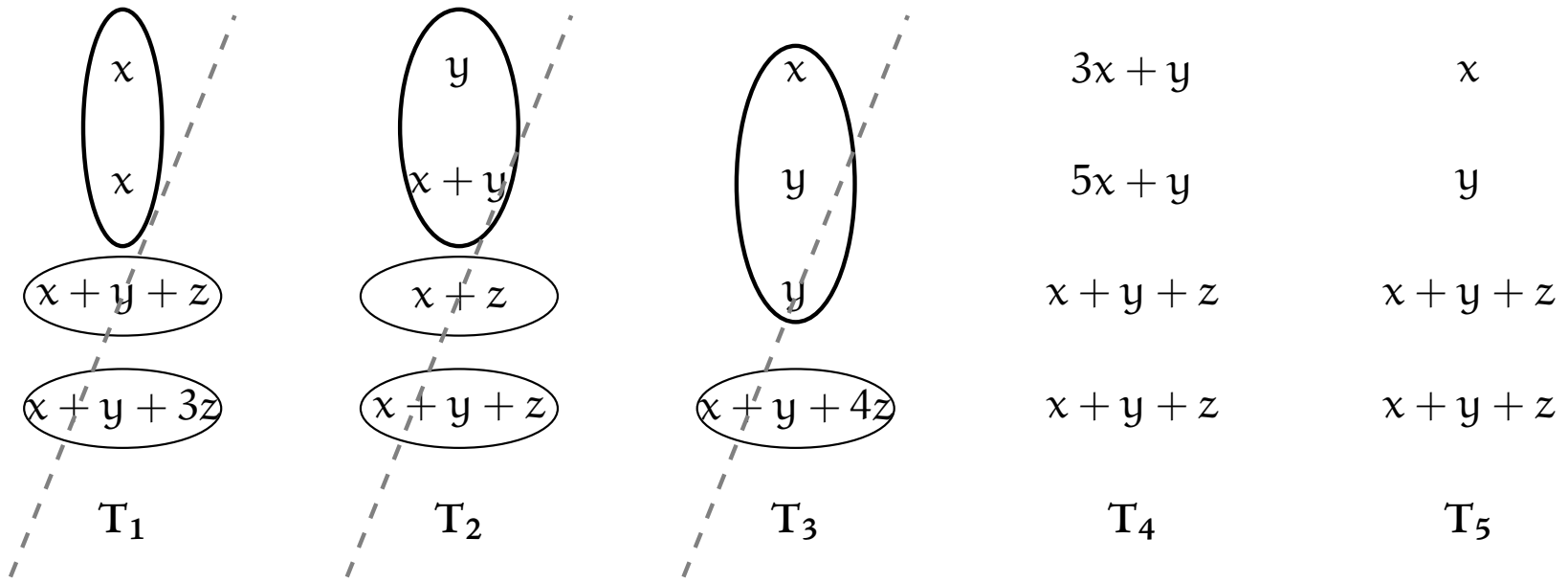
$$C = \alpha T_4 \mod I$$

In search of a certificate



Theorem ([KayalSaxena07] rephrased)

For any non-zero $\Sigma\Pi\Sigma(n, k, d)$ *circuit* C*, then there is a path certificate* $p = \langle v_1, \cdots, v_{k'} \rangle$ *and a* T_i *such that*

$$C = \alpha T_i \bmod p \quad \textit{(for } \alpha \in \mathbb{F}^*\textit{)}$$

Outline

$$0 \neq C \quad = \quad T \bmod \langle v_1, \cdots, v_{k'} \rangle$$

$$0 \neq C \quad = \quad T \bmod \langle v_1, \cdots, v_{k'} \rangle$$

We know $T \notin I$

Preserving the certificate

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Properties that Φ must satisfy:

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We know $T \notin I$ $\Leftrightarrow$ $v_T \notin I$

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Question: Do we know of such a Φ?

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[SaxenaSeshadri11]: The same [GabizonRaz05] map preserves low rank ideals!

Vandermonde works

Lemma

Let $f_0, f_1, \cdots, f_m$ *be multiplication terms with span* $\{\bigcup L(f_i)\}$ *has rank at most* k. *Let* Φ *be a linear map the preserves the space* V *generated by* $\bigcup L(f_i)$. *Then,*

$$f_0 \in \langle f_1, \cdots, f_m \rangle \quad \Longleftrightarrow \quad \Phi(f_0) \in \langle \Phi(f_1), \cdots, \Phi(f_m) \rangle$$

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Proof overview.

Since Φ preserves V, it can be shown that Φ induces an isomorphism between the two algebras $\mathbb{F}[\ell_1, \cdots, \ell_k]$ and $\mathbb{F}[y_1, \cdots, y_k]$... □

... and we are nearly done

$$0 \neq C \quad = \quad T \bmod \langle v_1, \cdots, v_{k'} \rangle$$

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Besides possibly nk^2d bad α's, the map Φ_α ensures that

$$T \in \langle v_1, \cdots, v_{k'} \rangle \quad \text{if and only if} \quad \Phi_\alpha(T) \in \Phi_\alpha(\langle v_1, \cdots, v_{k'} \rangle)$$

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- The Vandermonde preserves ideals with small radical span.
- Certificate can be preserved by mapping (via the Vandermonde) to just a $\mathbf{k}$-variate polynomial ring.
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Whitebox PITs are known. [Saxena08], [Kayal10]

Thank you!

Questions?