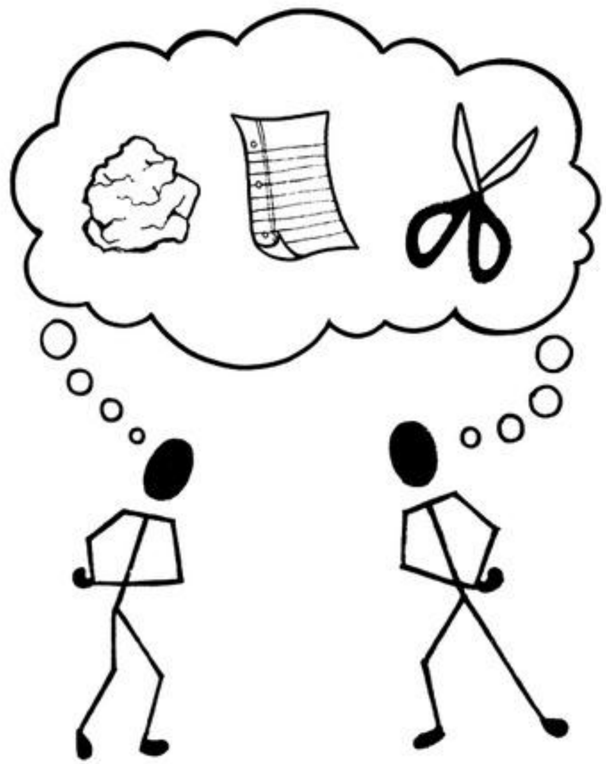
Algorithmic Game Theory and Economics: A very short introduction

Mysore Park Workshop

August, 2012

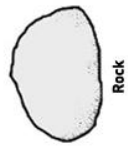
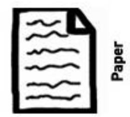
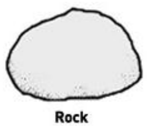
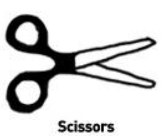
PRELIMINARIES

Game

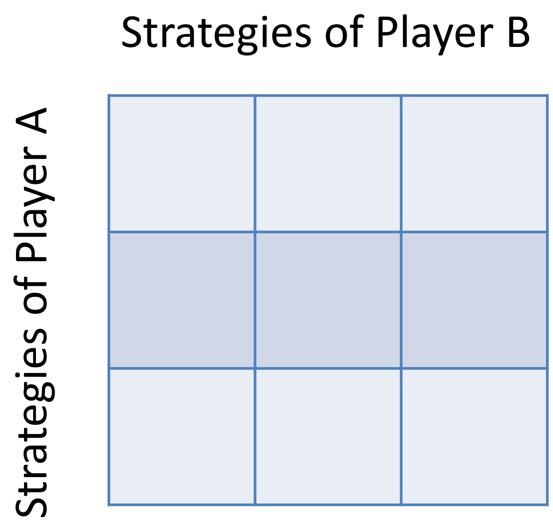


Choose Your Weapon

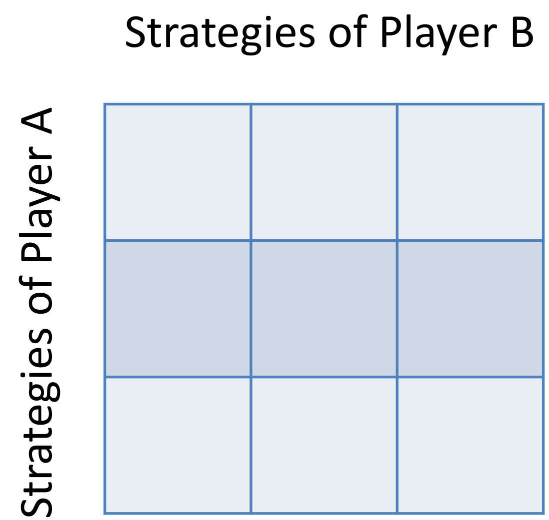
Rock Paper Scissors

			
	0,0	-1,1	1,-1
	1,-1	0,0	-1,1
	-1,1	1,-1	0,0

Bimatrix Notation



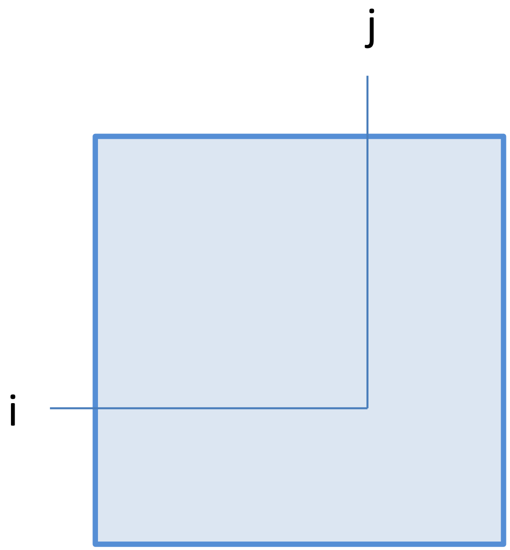
A



B

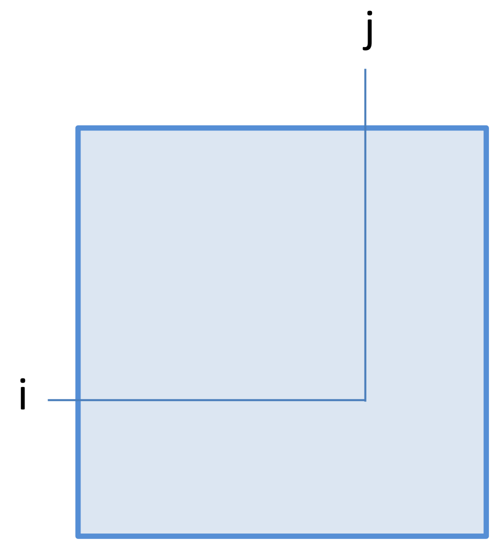
Zero Sum Game: A + B = all zero matrix

The Notion of Equilibrium



A

$$A_{ij} \geq A_{i'j} \quad \forall i'$$



B

$$B_{ij} \geq B_{ij'} \quad \forall j'$$

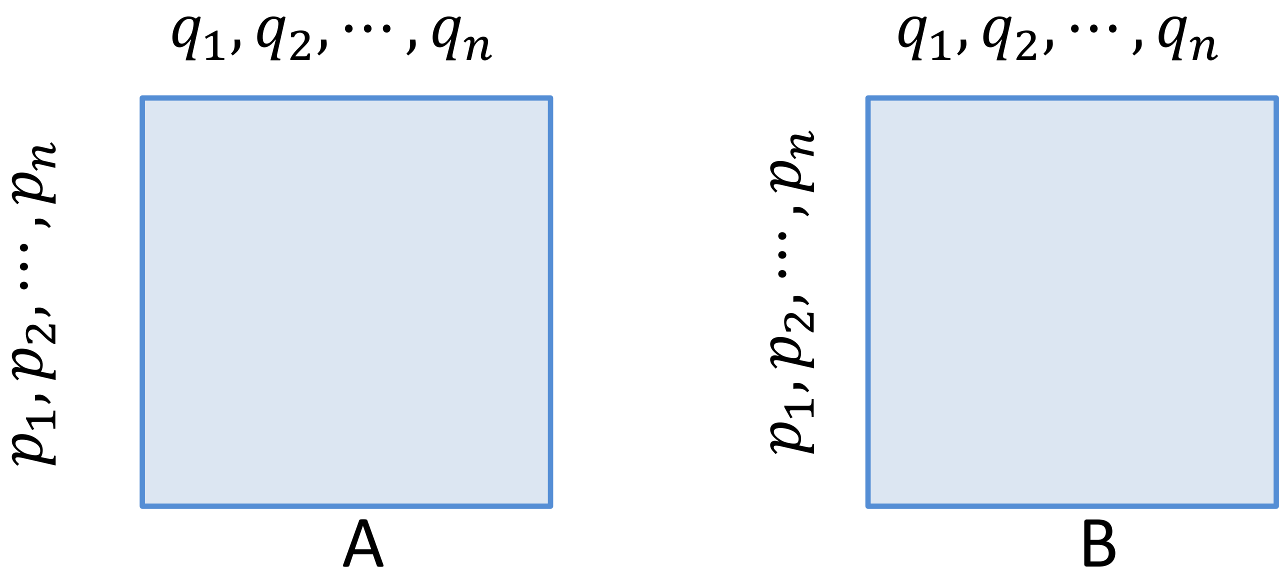
i is a best response to j

j is a best response to i

It needn't exist

	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

Mixed Strategies and Equilibrium



Expected utility of A = $\sum_{i,j} p_i A_{ij} q_j = p^\top A q$

Expected utility of B = $\sum_{i,j} p_i B_{ij} q_j = p^\top B q$

$$p^\top A q \geq p'^\top A q \quad \forall p' \qquad p^\top B q \geq p^\top B q' \quad \forall q'$$

J. von Nuemann's Minimax Theorem

Theorem (1928)

If A+B=0, then equilibrium mixed strategies (p,q) exist.

$$\max_p \min_q p^\top A q = \min_q \max_p p^\top A q$$

"As far as I can see, there could be no theory of games … without that theorem … I thought there was nothing worth publishing until the Minimax Theorem was proved"

John Nash and Nash Equilibrium

Theorem (1950)
Mixed Equilibrium always exist for finite games.

$\exists\ (\vec{p}, \vec{q})$:

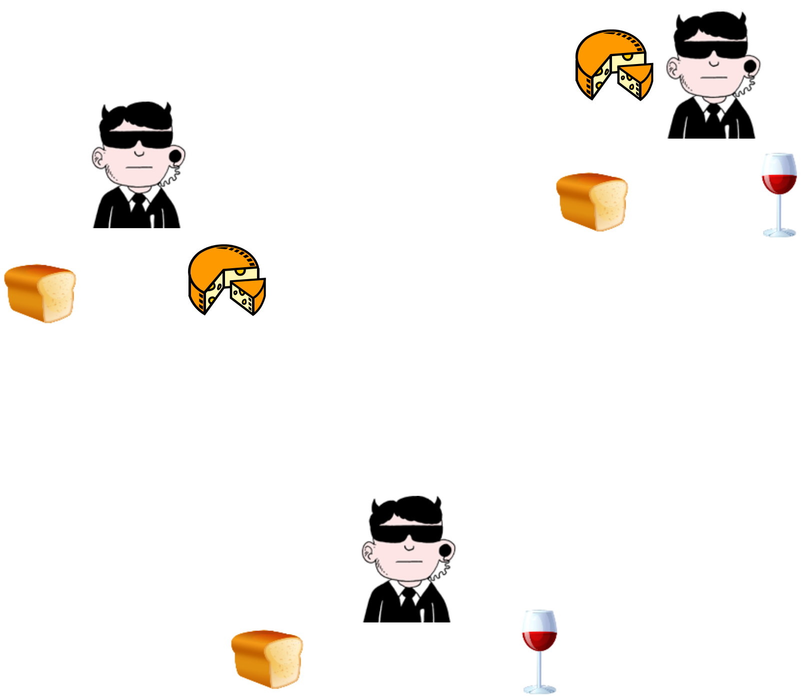
$$p^\top A q \geq p'^\top A q \quad \forall p'$$
$$p^\top B q \geq p^\top B q' \quad \forall q'$$

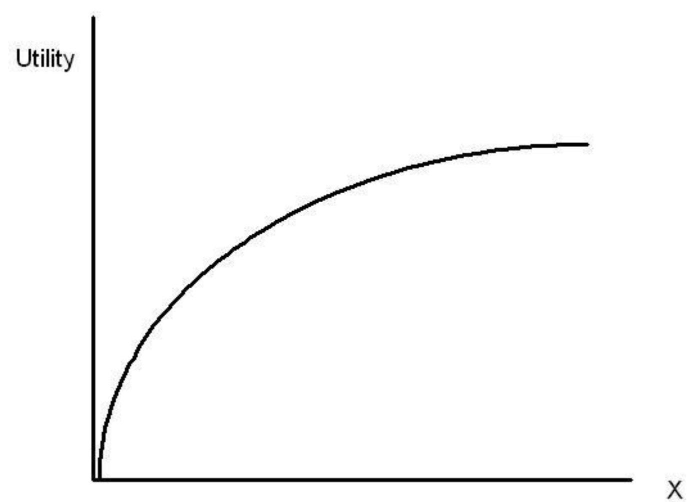
Markets

Walrasian Model: Exchange Economies

- N **goods** {1,2,…,N}
- M **agents** {1,2,…,M}

- $e_i \coloneqq (e_{i1}, e_{i2}, \dots, e_{iN})$
- $U_i(\boldsymbol{x}) = U_i(x_{i1}, \dots, x_{iN})$





Walrasian Model: Exchange Economies

- N **goods** {1,2,…,N}
- M **agents** {1,2,…,M}
- $e_i := (e_{i1}, e_{i2}, \dots, e_{iN})$
- $U_i(\boldsymbol{x}) = U_i(x_{i1}, \dots, x_{iN})$

- Prices $\boldsymbol{p} = (p_1, \dots, p_N) \Rightarrow$ Demand $(D_1, \dots, D_M)$

$$D_i(\boldsymbol{p}) = \arg\max \{ U_i(\boldsymbol{x}) : \boldsymbol{p} \cdot \boldsymbol{x} \leq \boldsymbol{p} \cdot \boldsymbol{e_i} \}$$

- $(\boldsymbol{p}, \boldsymbol{x^1}, \dots, \boldsymbol{x^M})$ are a **Walrasian equilibrium** if
  - $\boldsymbol{x^i} \in D_i(\boldsymbol{p})$ for all $i \in [M]$ Utility Maximization
  - $\sum_i \boldsymbol{x_j^i} = \sum_{\boldsymbol{i}} e_{ij}$ for all j $\in [N]$ Market Clearing

Leon Walras and the tatonnement

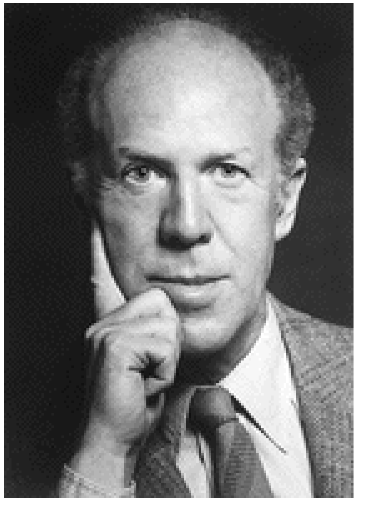
- Given price **p**, calculate demand for each good i.
- If demand exceeds supply, raise price.
- If demand is less than supply, decrease price.

1874

Does this process converge?
Does equilibrium exist?

Arrow and Debreu

Theorem (1954).
If utilities of agents are continuous, and strictly quasiconcave, then Walrasian equilibrium exists.



How does one prove such theorems?

Nash. $\exists\ (\vec{p}, \vec{q})$: $$p^\top A q \geq p'^\top A q \quad \forall p'$$ $$p^\top B q \geq p^\top B q' \quad \forall q'$$

Define: $\Phi(p,q) \;=\; (p', q')$

- p' is a best-response to q
- q' is a best-response to p
- Mapping is continuous.

$\phi(p,q) = (p,q)$
$\Rightarrow$ Equilibrium.

Brouwers Fixed Point Theorem.
Every continuous mapping from a convex, compact set to itself has a **fixed point.**

EQUILIBRIUM COMPUTATION

Support Enumeration Algorithms

Nash. $\exists\ (\vec{p}, \vec{q})$: $p^\top A q \geq p'^\top A q \quad \forall p'$; $p^\top B q \geq p^\top B q' \ \forall q'$

Suppose knew the *support* S and T of (p,q).

$(\vec{p}, \vec{q})$:

$$e_i^\top A q \geq e_{i'}^\top A q; \quad \forall i \in S, \forall i'$$
$$p^\top B e_j \geq p^\top B e_{j'} \quad \forall j \in T, \forall j'$$
$$p_i = 0; q_j = 0 \quad \forall i \notin S, \forall j \notin T$$

Every point above is a Nash equilibrium.

Lipton-Markakis-Mehta

Lemma. There exists ϵ-approximate NE with support size at most $K = O\left(\frac{\log n}{\epsilon^2}\right)$.

***ϵ*-Nash** $(\vec{p}, \vec{q})$: $p^\top A q \geq p'^\top A q - \epsilon \;\; \forall p'$;

$p^\top B q \geq p^\top B q' - \epsilon \;\; \forall q'$

Values in [0,1]

Given **true** NE (p,q), *sample* P,Q i.i.d. K times.
Uniform distribution over P,Q is ϵ-Nash.

Chernoff Bounds

Approximate Nash Equilibrium

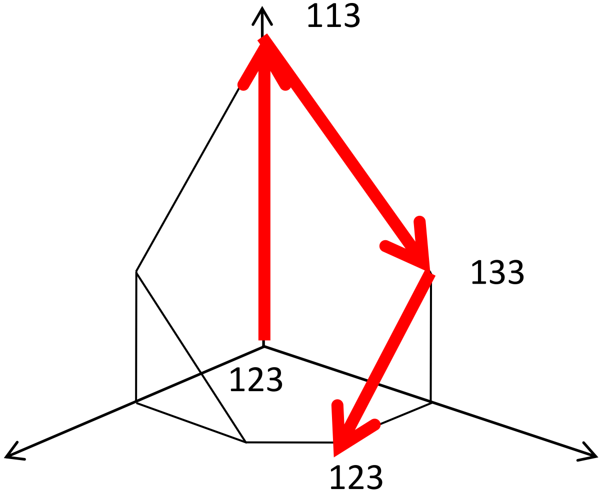
Theorem (2003) [LMM]
For any bimatrix game with at most n strategies, an ϵ-approximate Nash equilibrium can be computed in time $n^{O\left(\log n / \epsilon^2\right)}$

Theorem (2009) [Daskalakis, Papadimitriou]
Any *oblivious* algorithm for Nash equilibrium runs in expected time $\Omega(n^{[0.8-0.34\epsilon]\log n})$.

Lemke-Howson 1964

$$Az \leq \mathbf{1}$$
$$z \geq 0$$

$$\forall i: \;\; z_i = 0 \;\; \text{OR}$$
$$(Az)_i = 1$$



- Nonzero solution $\approx$ Nash equilibrium.
- Every vertex has one “escape” and “entry”
- There must be a sink $\equiv$ Non zero solution.

How many steps?

- n strategies imply at most $O(2^n)$ steps.

- **Theorem (2004).[Savani, von Stengel]** Lemke-Howson can take $\Omega(c^n)$ steps, for some $c > 1$.

Hardness of NASH

Theorem (2005) **[Daskalakis, Papadimitriou, Goldberg]**

Computing Nash equilibrium in a 4-player game is PPAD hard.

Theorem (2006) **[Chen, Deng]**

Computing Nash equilibrium in a 2-player game is PPAD hard.

I’ve heard of NP. What’s this PPAD?

- Subclass of “search” problems whose solutions are guaranteed to exist (more formally, **TFNP**)
- **PPAD** captures problems where existence is proved via a parity argument in a digraph.
- **End of Line.** Given G = ($\{0,1\}^n, A$), in/out-deg ≤ 1 out-deg(0^n)=1, oracle for in/out(v); find v with in-deg(v)=1 and out-deg(v) = 0.
- **PPAD** is problems reducible to **EOL.**

Summary of Nash equilibria

- Exact calculation is **PPAD**-hard. Even getting an FPTAS is **PPAD**-hard (Chen,Deng,Teng ‘07)
  Quasi-PTAS exists.

- Some special cases of games have had more success
  – Rank 1 games. **(Ruta’s talk!)**
  – PTAS for constant rank (Kannan, Theobald ’07)
    sparse games (D+P ‘09)
    small probability games (D+P ’09)

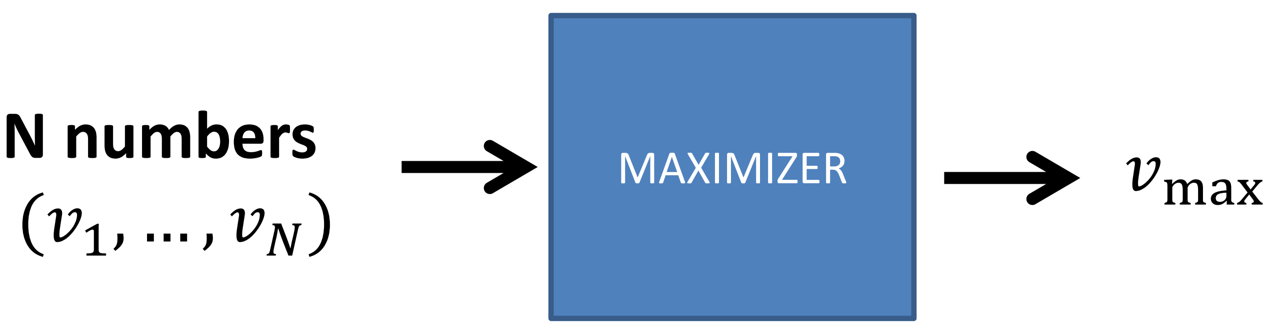
- **OPEN:** Is there a PTAS to compute NE?

General Equilibrium Story

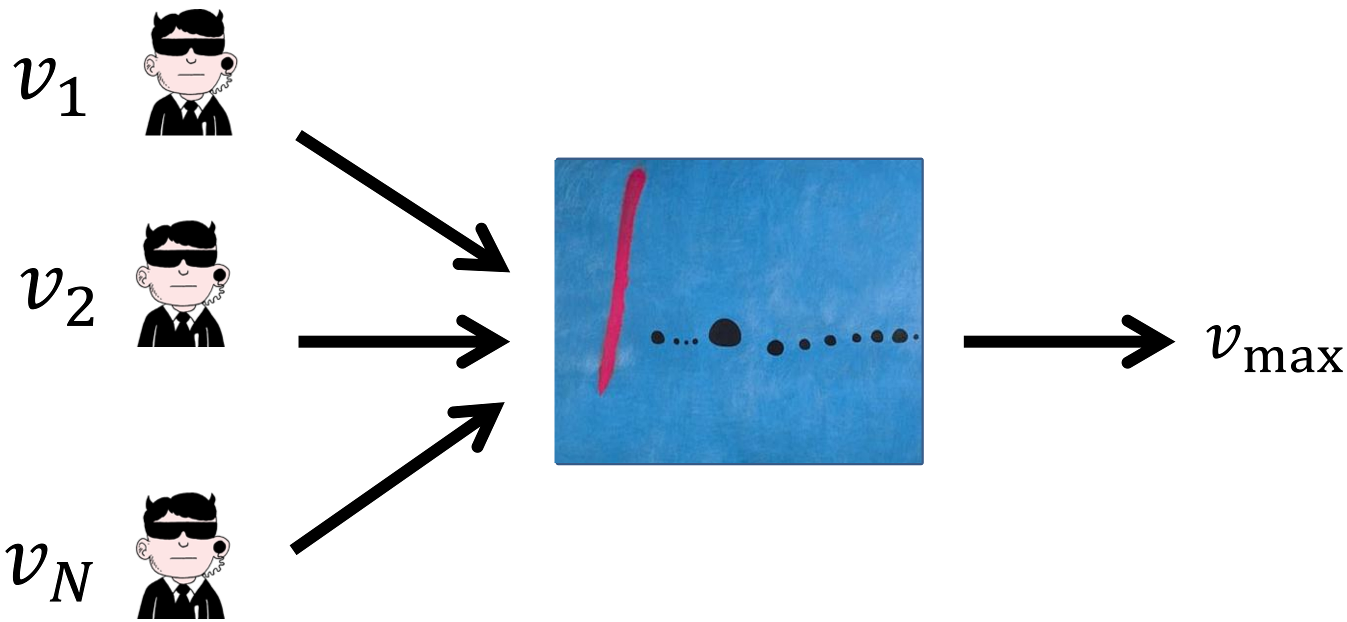
- Computing General Equilibria is **PPAD-**hard. (Reduction to Bimatrix games)
- Linear case and generalizations can be solved via convex programming techniques. Combinatorial techniques.
- “Substitutability” makes tatonnement work.

MECHANISM DESIGN

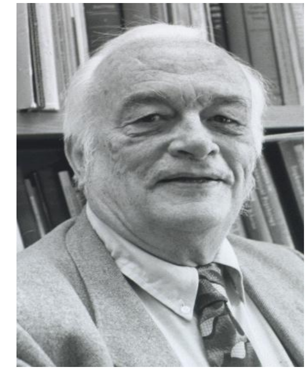
Traditional Algorithms



Auctions



Vickrey Auction



- **Setting**
  - Solicit **bids** from agents.
  - Allocate item to **one** agent.
  - Charge an agent (no more than bid.)
- **Second Price Auction**
  - Assign item to the highest bidder.
  - Charge the bid of the second highest bidder.

Theorem (1961).
No player has an incentive to misrepresent bids in the second price auction.

Mechanism Design Framework

- **Feasible Solution Space.** $F \subseteq \boldsymbol{R}^N$
- **Agents.** Valuations $v_i : F \mapsto \boldsymbol{R}$

  Reports type/bid $b_i : F \mapsto \boldsymbol{R}$
- **Mechanism.**
  - *Allocation:* $(x_1, x_2, \dots, x_N) \in F$
  - *Prices:* $(p_1, p_2, \dots p_N)$
  - *Individual Rationality:* $p_i \leq b_i(x_i)$
  - *Incentive Compatibility:*

    $$v_i(x_i(v_i)) - p(v_i) \geq v_i(x_i(b_i)) - p_i(b_i)$$
- **Goal.**

Welfare Maximization: VCG

Goal. Maximize $\sum_i v_i(x_i)$: $(x_1, \ldots, x_N) \in F$

VCG Mechanism

- Find $\boldsymbol{x}^*$ which maximizes welfare.
- For agent a calculate
  - $\tilde{\boldsymbol{x}}$ maximizing $\sum_{\{i \neq a\}} v_i(x_i)$: $\boldsymbol{x} \in F$
  - Charge $p_a = \sum_{i \neq a} v_i(\tilde{x}_i) - \sum_{i \neq a} v_i(x_i^*)$
  - Utility $= \sum_i v_i(x_i^*) - \sum_{i \neq a} v_i(\tilde{x}_i)$
- For single item $\mapsto$ second price auction.

Combinatorial Auctions

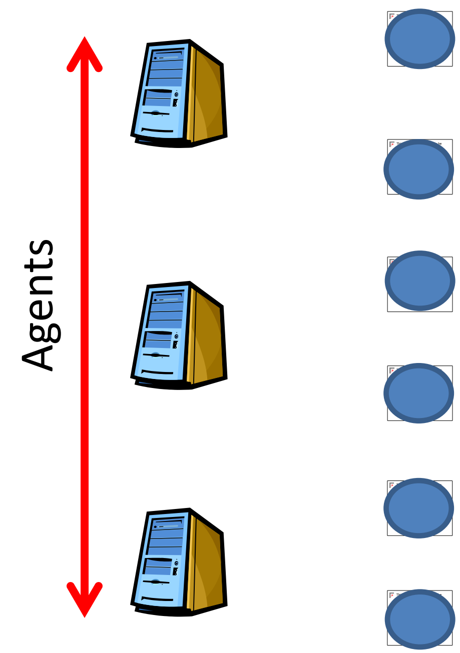
- N agents, M **indivisible** items. Utilities U_i.
- Hierarchies of utilities: subadditive, submodular, budgeted additive, ….
- Problem with VCG: maximization **NP hard.**
- Approximation and VCG don't mix: even a PTAS doesn't imply truthfulness.
- Algorithmic Mechanism Design

Nisan, Ronen 1999.

Minimax Objective: Load balancing

Goal. $\text{minimize} \max_{\mathrm{i}} \; v_i(x_i)\text{:} \;\; (x_1, \ldots, x_N) \in F$

- VCG only works for SUM
- $O(1)$-approximation known for only special cases.
- General case = $\Omega(n)$?
- Characterization (?)



Single Dimensional Settings

- Each agent controls **one** parameter privately.
- **Monotone** allocation algorithm $\Rightarrow$ Truthful. (x_i should increase with v_i)
- Is there a setting where monotonicity constraint degrades performance?

Techniques in AMD

- **Maximal in Range Algorithms.**
  Restrict range a priori s.t. maximization in **P**.
  Argue restriction doesn't hurt much.
  Nisan-Ronen 99, Dobzinski-Nisan-Schapira '05, Dobzinski-Nisan '07, Buchfuhrer et al 2010….

- **Randomized mechanisms.**
  Linear programming techniques **Lavi-Swamy 2005.**
  Maximal in *Distributional* Range **Dobzinski-Dughmi 2009.**
  Convex programming **Dughmi-Roughgarden-Yan 2011.**

- **Lower Bounds.**
  Via characterizations. **Dobzinski-Nisan 2007**
  Communication complexity. **Nisan Segal 2001**

Summary of Mechanism Design

- AMD opens up a whole suite of algorithm questions.
- Lower bounds to performance. Are we asking the right question?
- Can characterizations developed by economists exploited algorithmically?

WHAT I COULDN’T COVER


networks
optimal
complexity
maximization
search
dynamic
revenue
price
games
Social
information
privacy
allocation
sponsored
nash
display
Efficiency
Computing
analysis
auctions
Approximate
matching
prediction
Prediction
scheduling
Computational
mechanism
equilibria
Mechanism
markets
truthful
social
ad
Optimal
online
market
advertising
algorithms
externalities
network
voting
pricing
learning
equilibrium
incentives
Matching
sequential
strategies
Information
game

THANK YOU