Spectral Sparsification and Geometric Applications

Nikhil Srivastava
MSRI

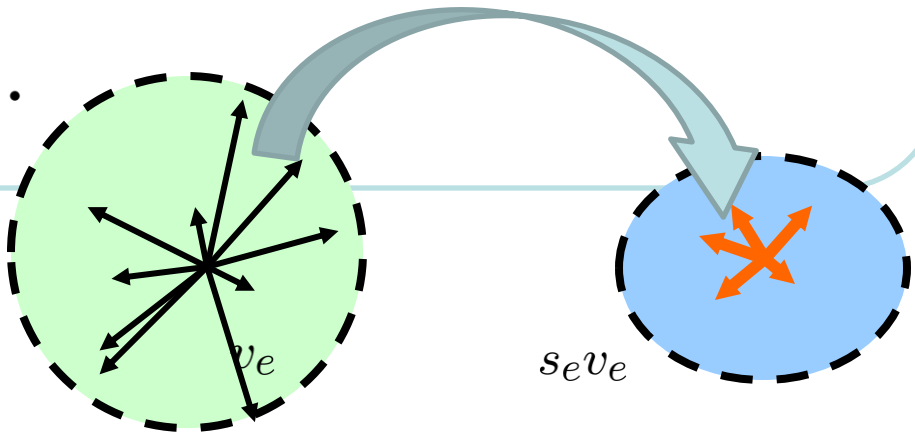
Part II: A Proof

Deterministic Solution [BSS'09]

Spectral Sparsification Theorem:

Given $\sum_{i \le m} v_i v_i^T = I_n$ there are $s_i \geq 0$ with:

- $(1-\epsilon) I \preceq \sum_i s_i v_i v_i^T \preceq (1+\epsilon) I$
- $\mathrm{supp}(s) \leq 4n/\epsilon^2$.



Deterministic Solution [BSS'09]

Spectral Sparsification Theorem:

Given $\sum_{i \le m} v_i v_i^T = V$ there are $s_i \geq 0$ with:

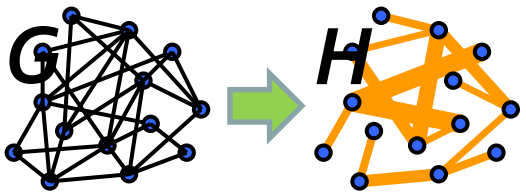
- $(1-\epsilon)V \preceq \sum_i s_i v_i v_i^T \preceq (1+\epsilon)V$
- $\mathrm{supp}(s) \leq 4\mathrm{rank}(V)/\epsilon^2$.

Deterministic Solution [BSS'09]

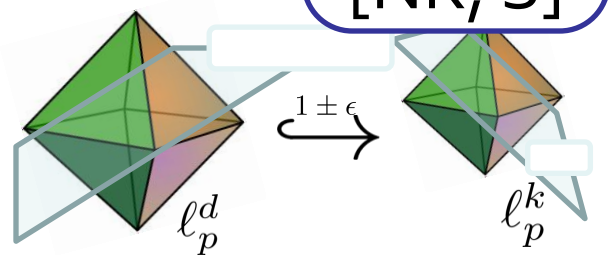
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[NR, S]



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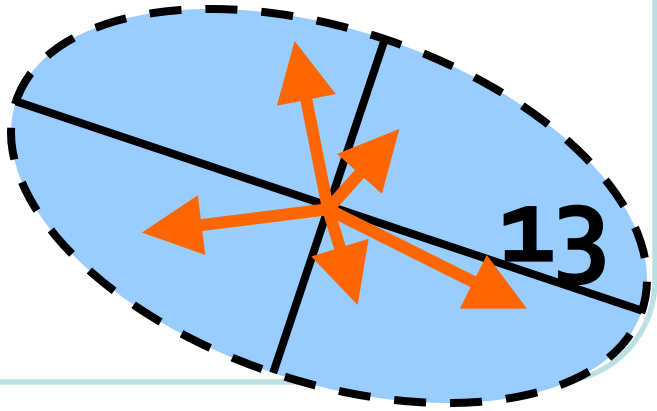
- $I \preceq \sum_i s_i v_i v_i^T \preceq 13 \cdot I$
- $\mathrm{supp}(s) \leq 6n.$

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$$\forall u \in S^{n-1} \qquad \sum_i s_i \langle u, v_i \rangle^2 \in [1, 13]$$

Intuition for the proof

Deterministic Solution [BSS'09]

Spectral Sparsification Theorem:

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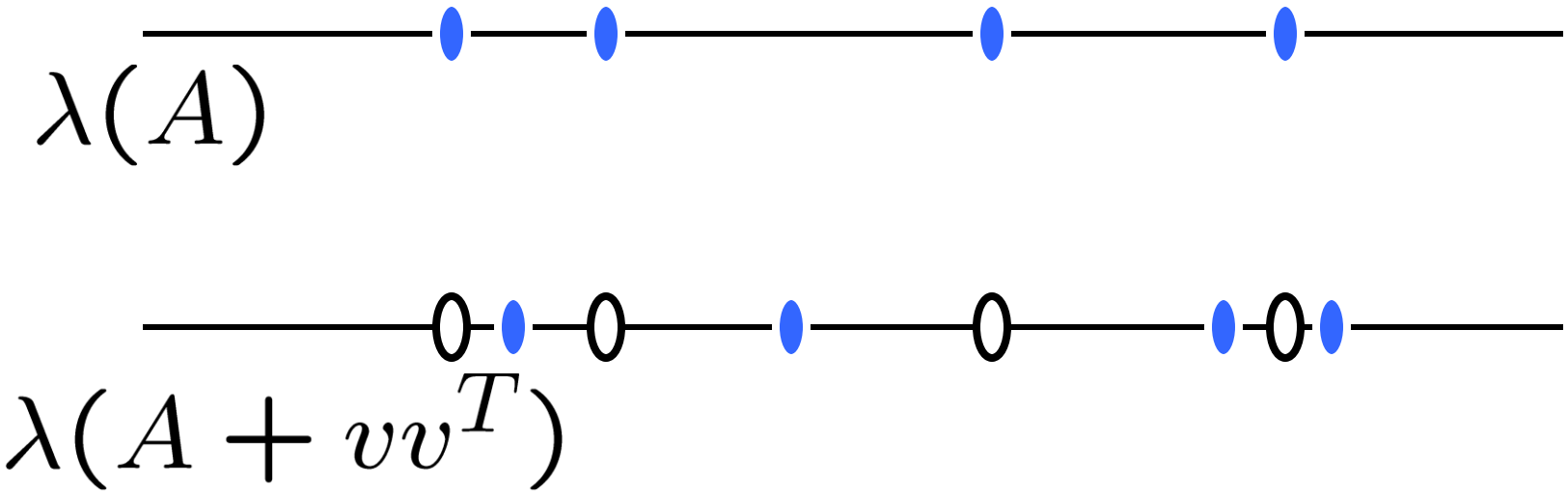
- $I \preceq \sum_i s_i v_i v_i^T \preceq 13 \cdot I$
- $\mathrm{supp}(s) \leq 6n.$

will build this one vector at a time in **6n** steps.

What happens when we add a vector?



Interlacing



More precisely

Characteristic Polynomial:

$$p_A(x) = \det(xI - A)$$

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Matrix-Determinant Lemma:

$$p_{A+vv^T} = p_A\left(1 + \sum_i \frac{\langle v, u_i\rangle^2}{\lambda_i - x}\right)$$

$$\det(xI - A - vv^T) =$$
$$\det(xI - A)\det(I - v^T(xI - A)^{-1}v)$$

Characteristic Polynomi

$$p_A(x) = \det($$ $$I - A)$$

Matrix-Determinant Lem a:

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More precisely

Characteristic Polynomial:

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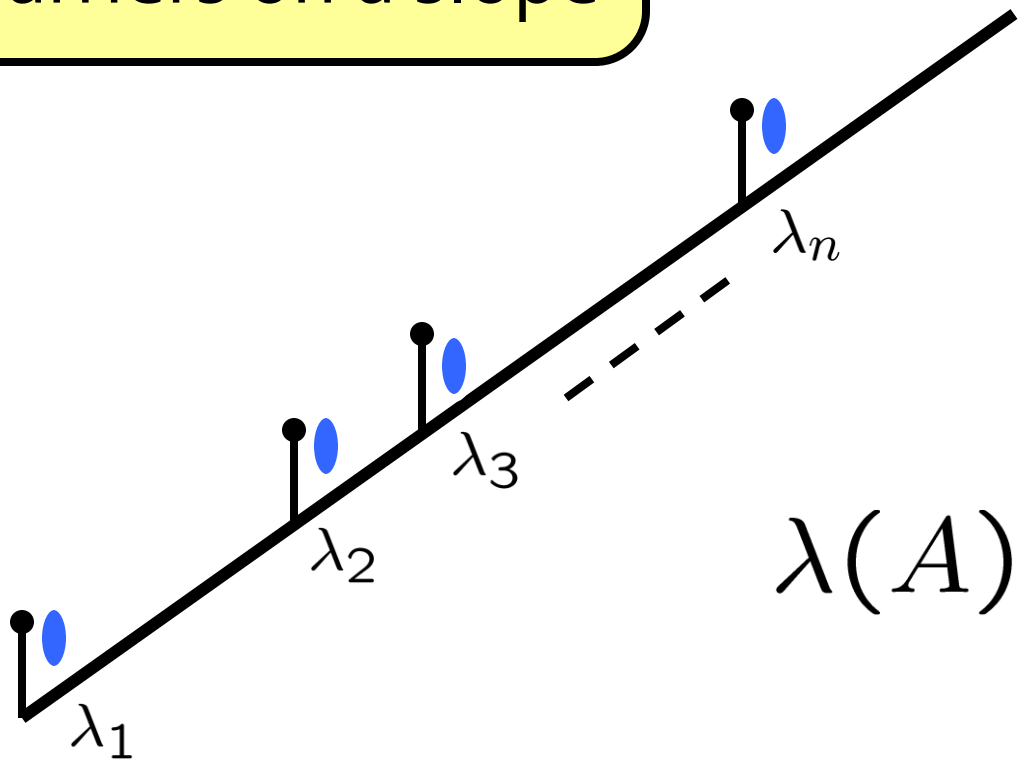
Matrix-Determinant L

$\lambda(A + vv^T)$ are zeros of this.

$$p_{A+vv^T} = p_A\left(1 + \sum_i \frac{\langle v, u_i\rangle^2}{\lambda_i - x}\right)$$

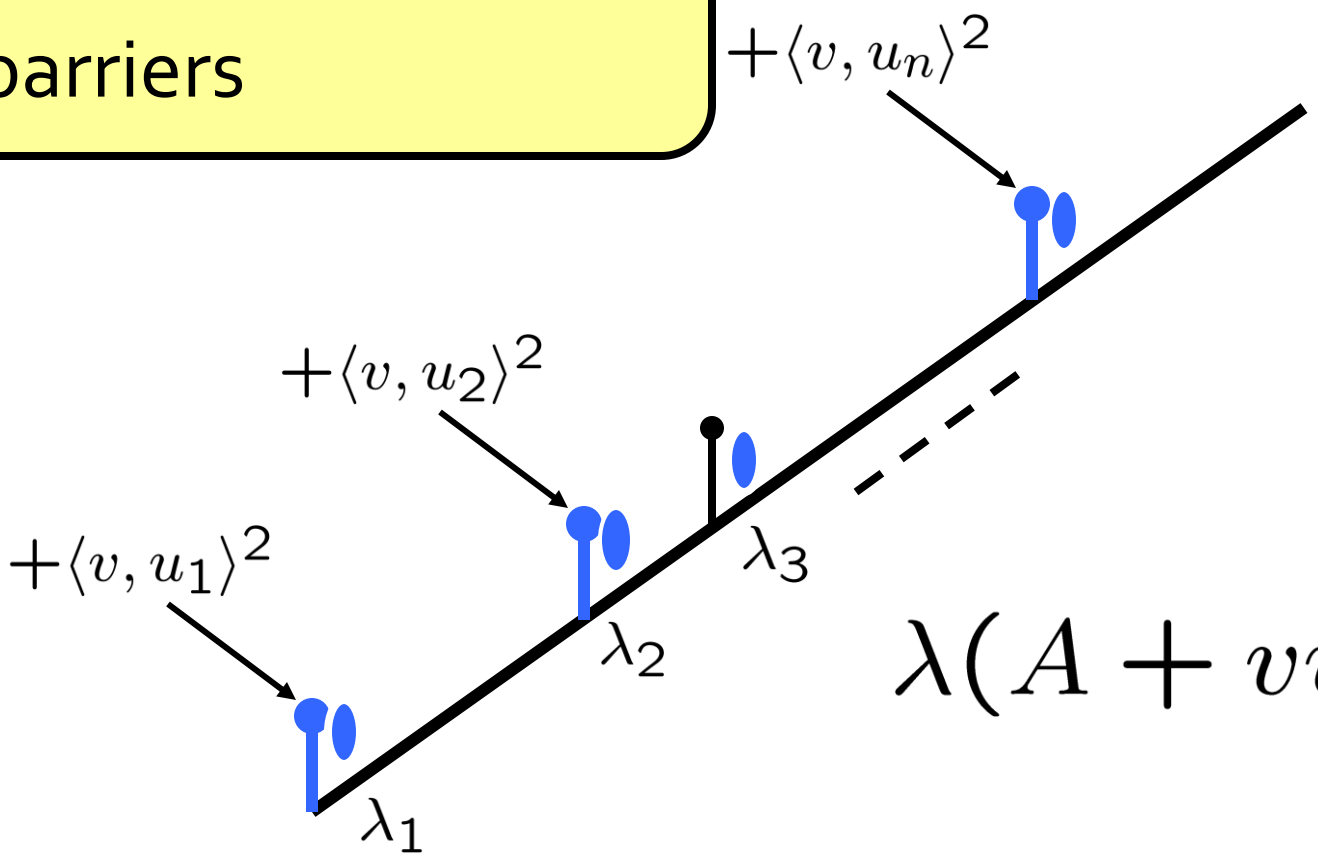
Physical model of interlacing

λ_i = positive unit charges resting at barriers on a slope



Physical model of interlacing

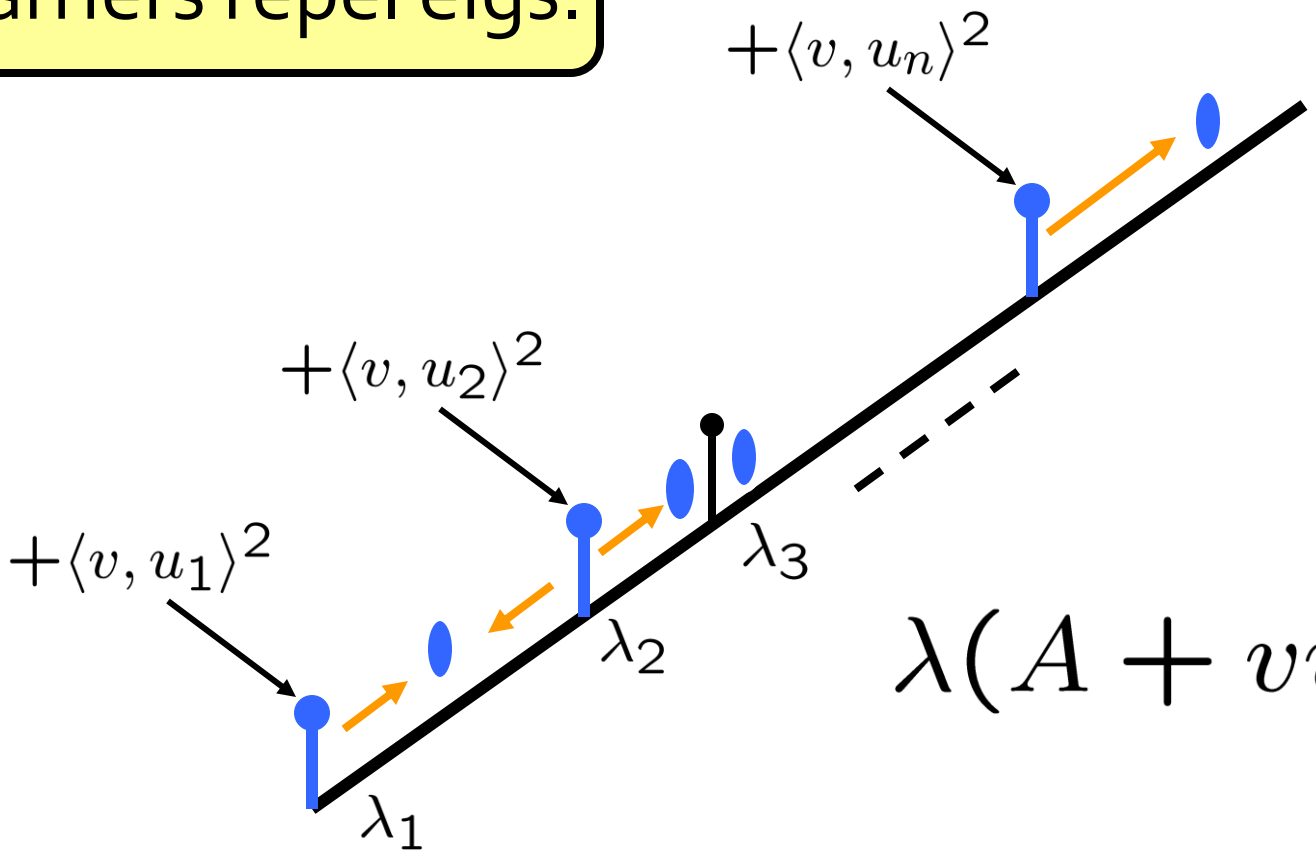
$\langle v, u_i \rangle^2$ =charges added to barriers



$$\lambda(A + vv^T)$$

Physical model of interlacing

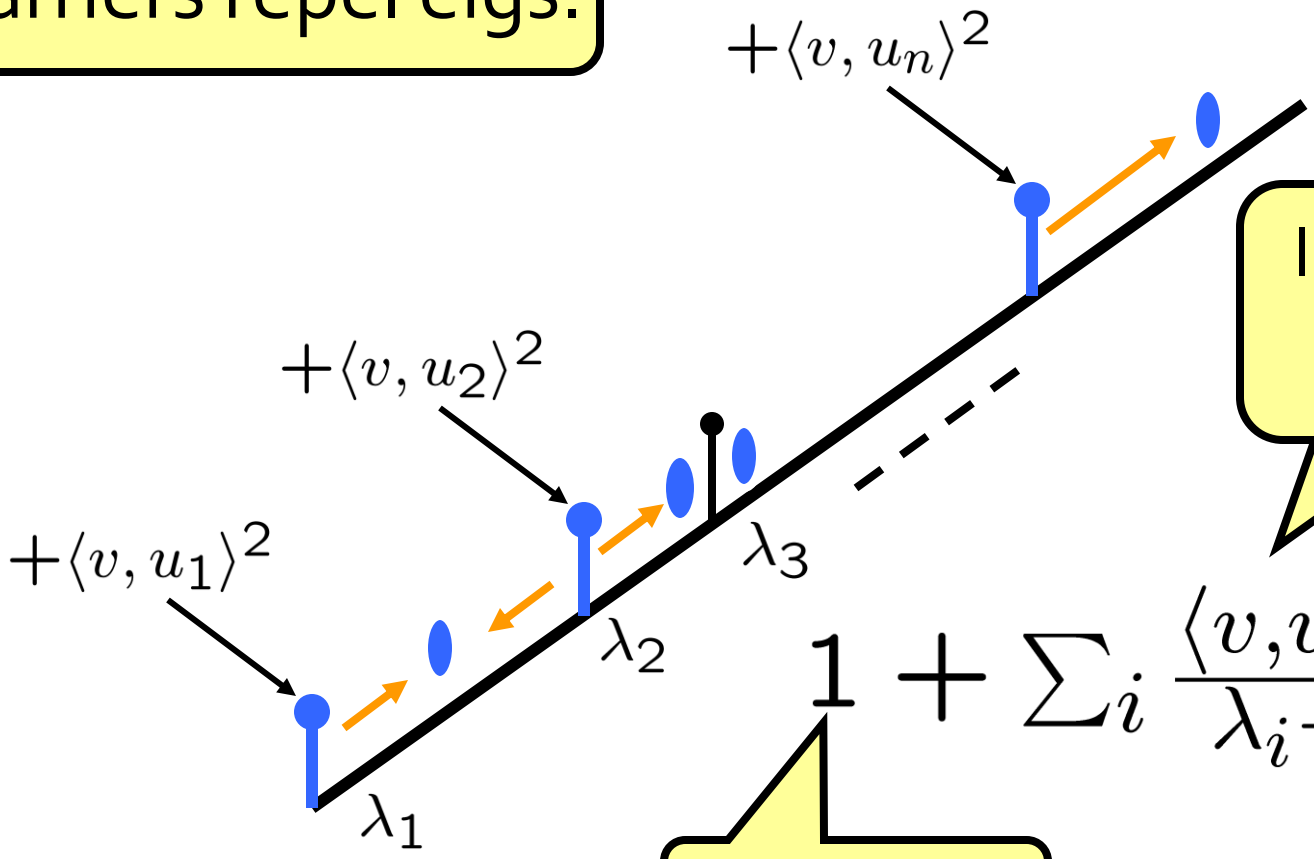
Barriers repel eigs.



$$\lambda(A + vv^T)$$

Physical model of interlacing

Barriers repel eigs.



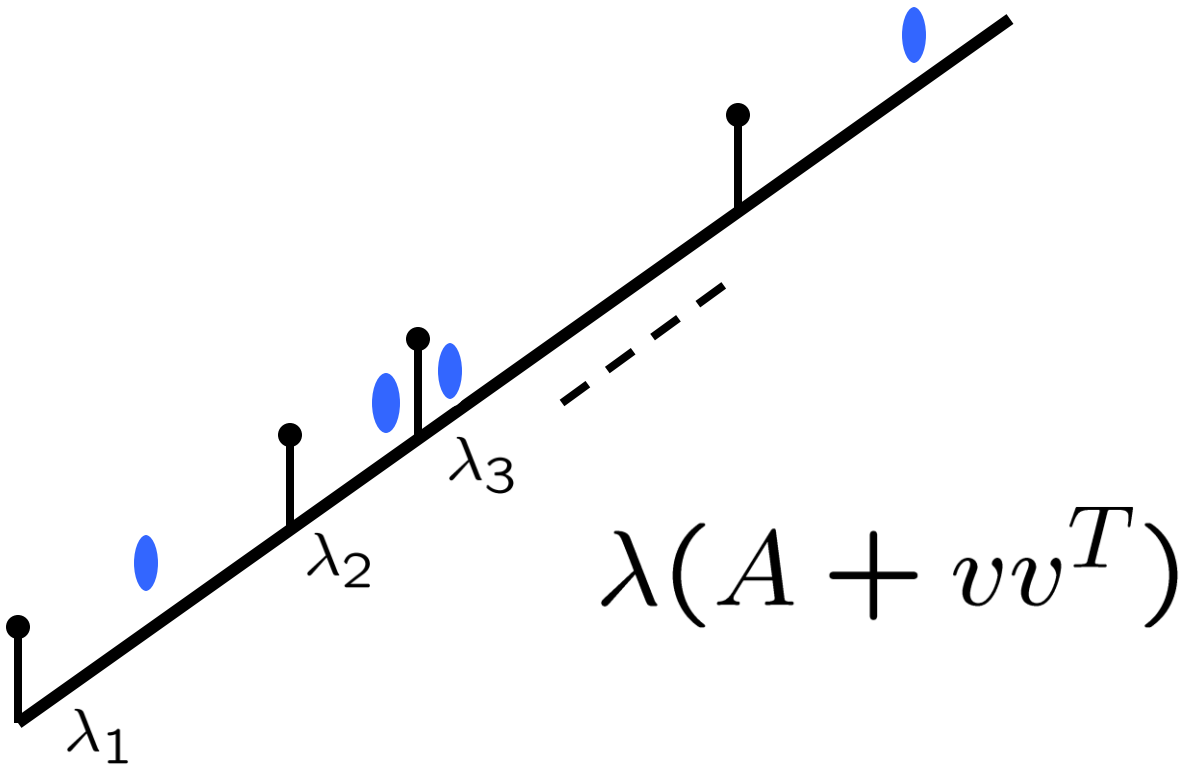
Inverse law repulsion

$$1 + \sum_i \frac{\langle v, u_i \rangle^2}{\lambda_i - x} = 0$$

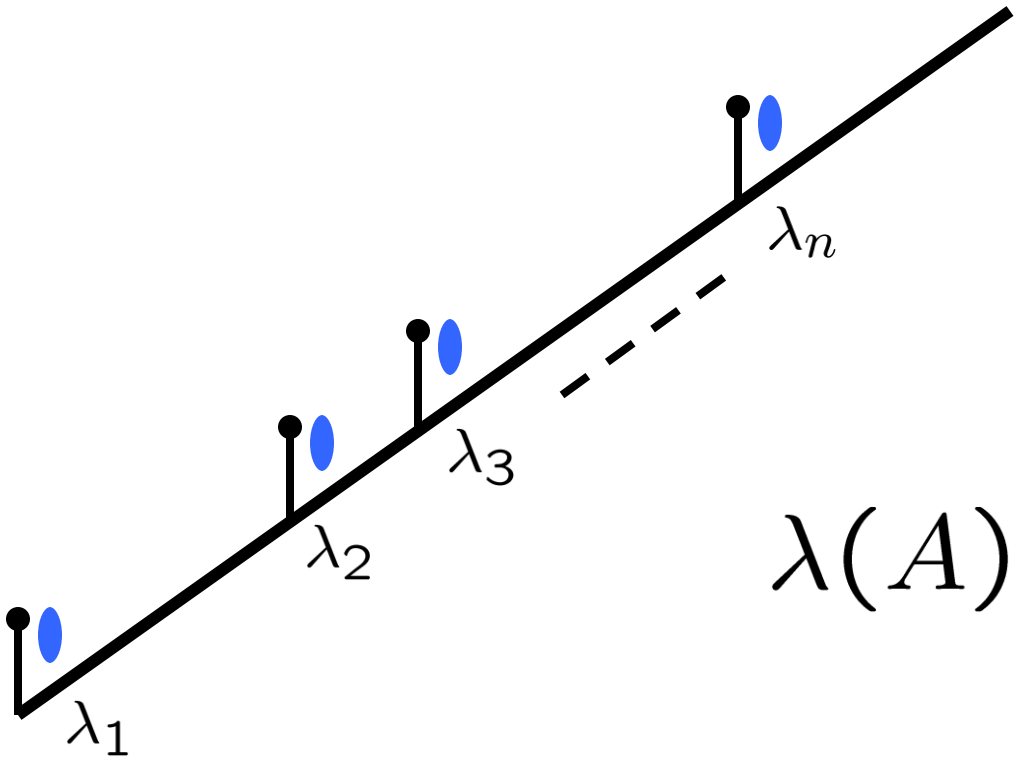
gravity

Physical model of interlacing

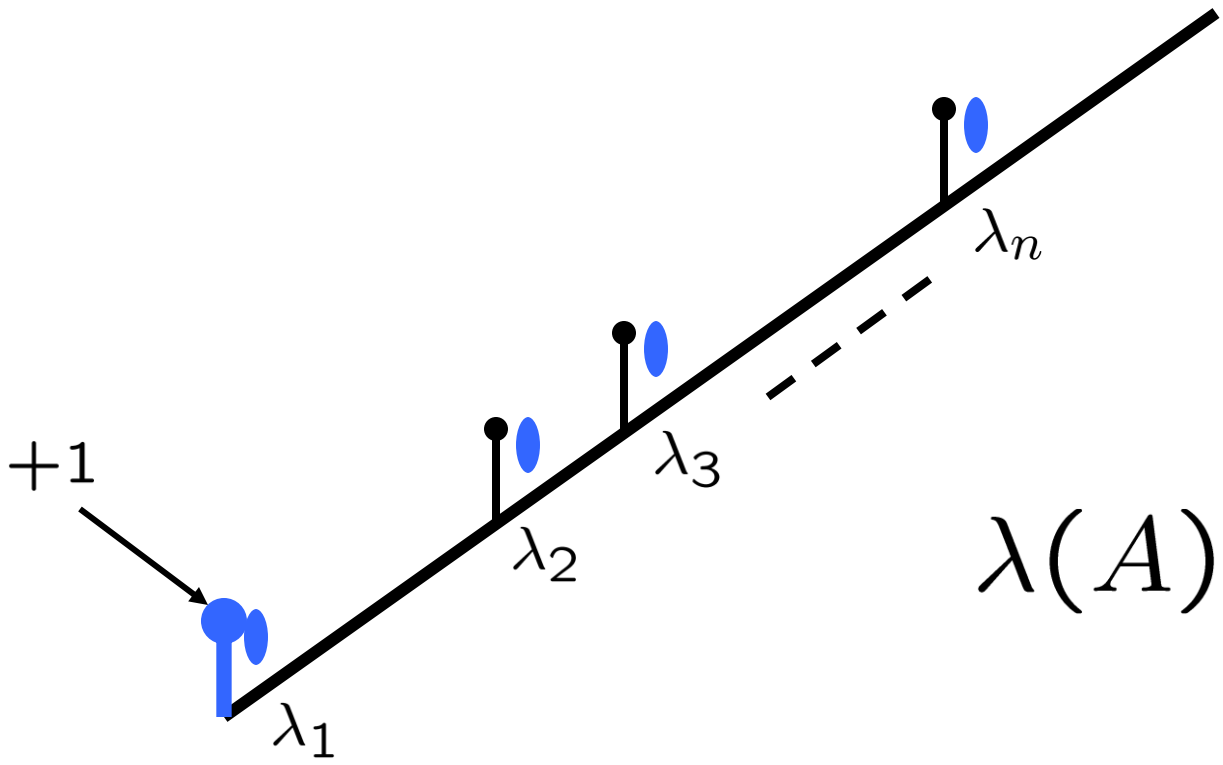
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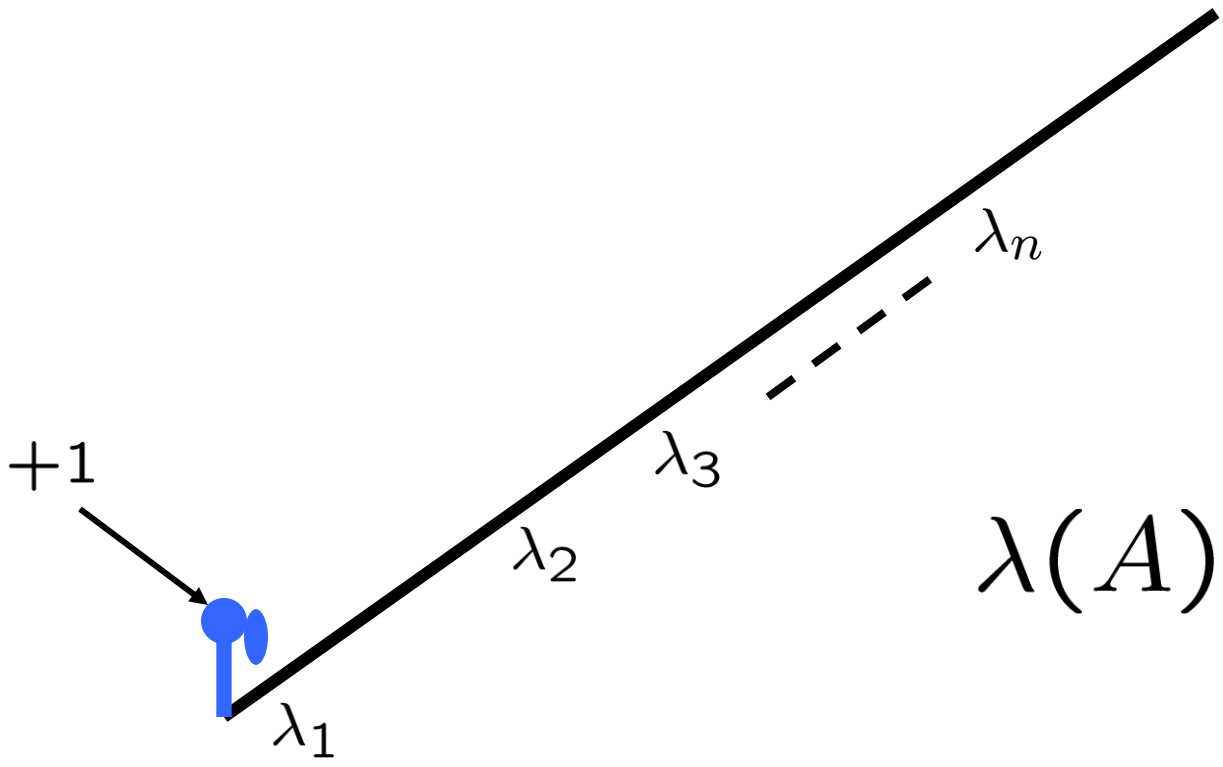
Examples



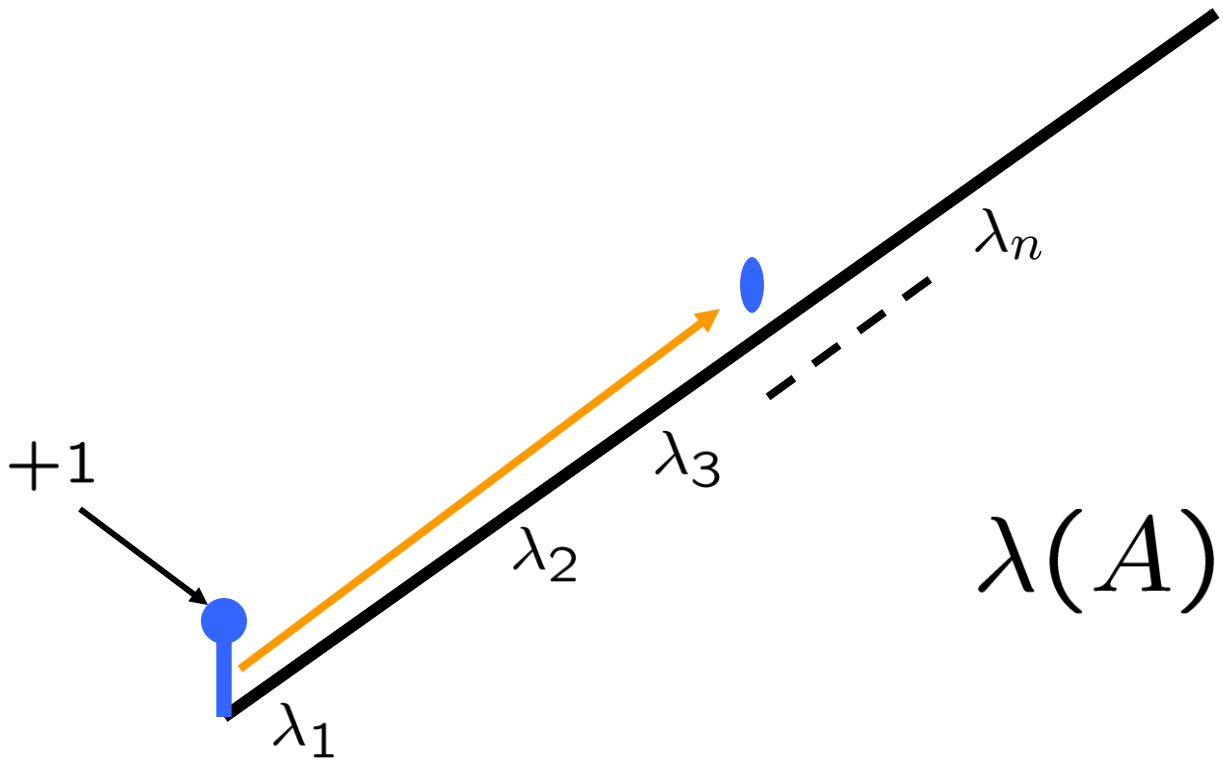
Ex1: All weight on u_1



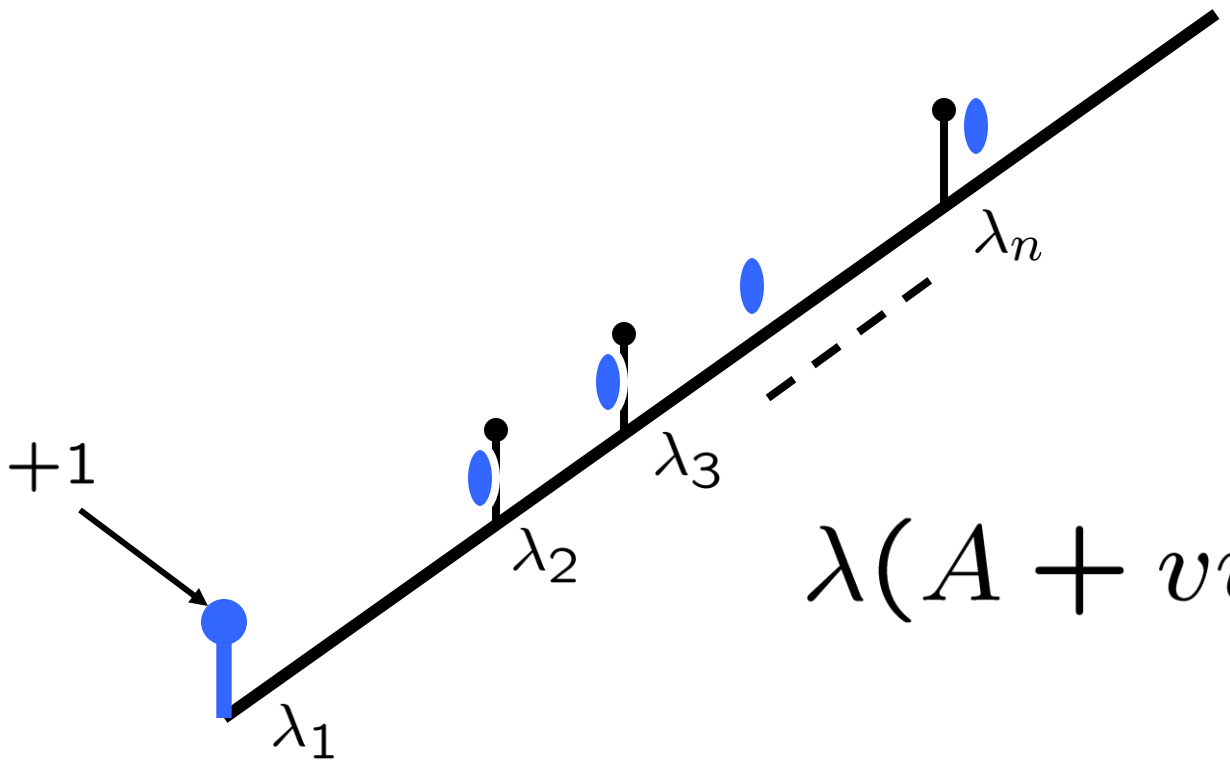
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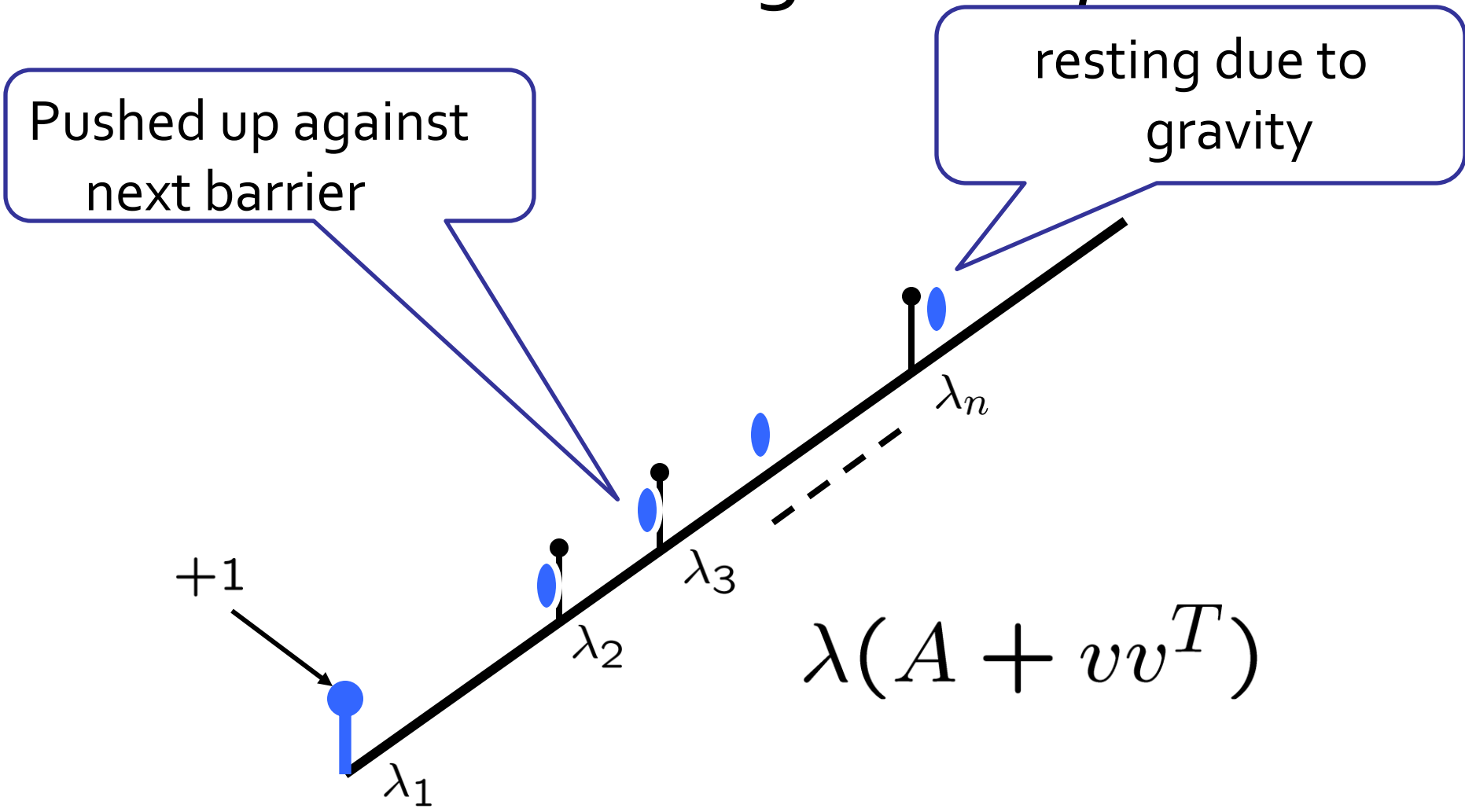


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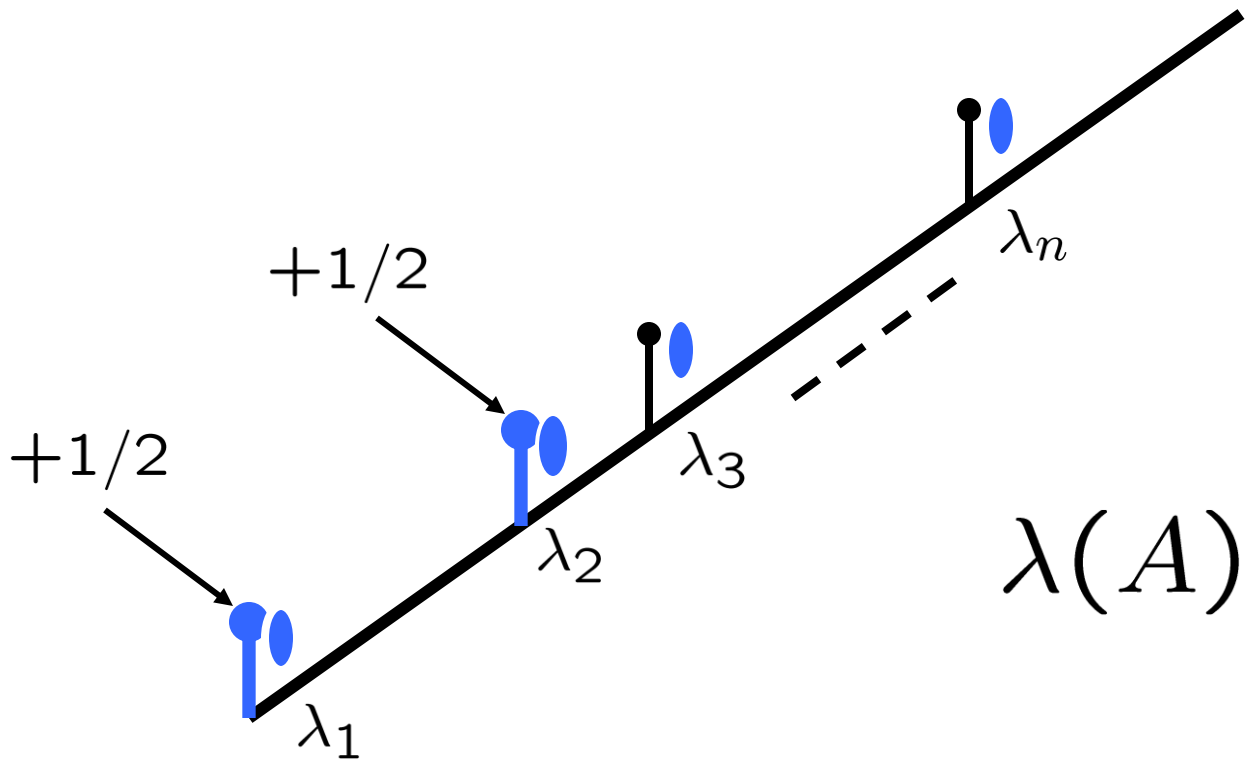


$$\lambda(A + vv^T)$$

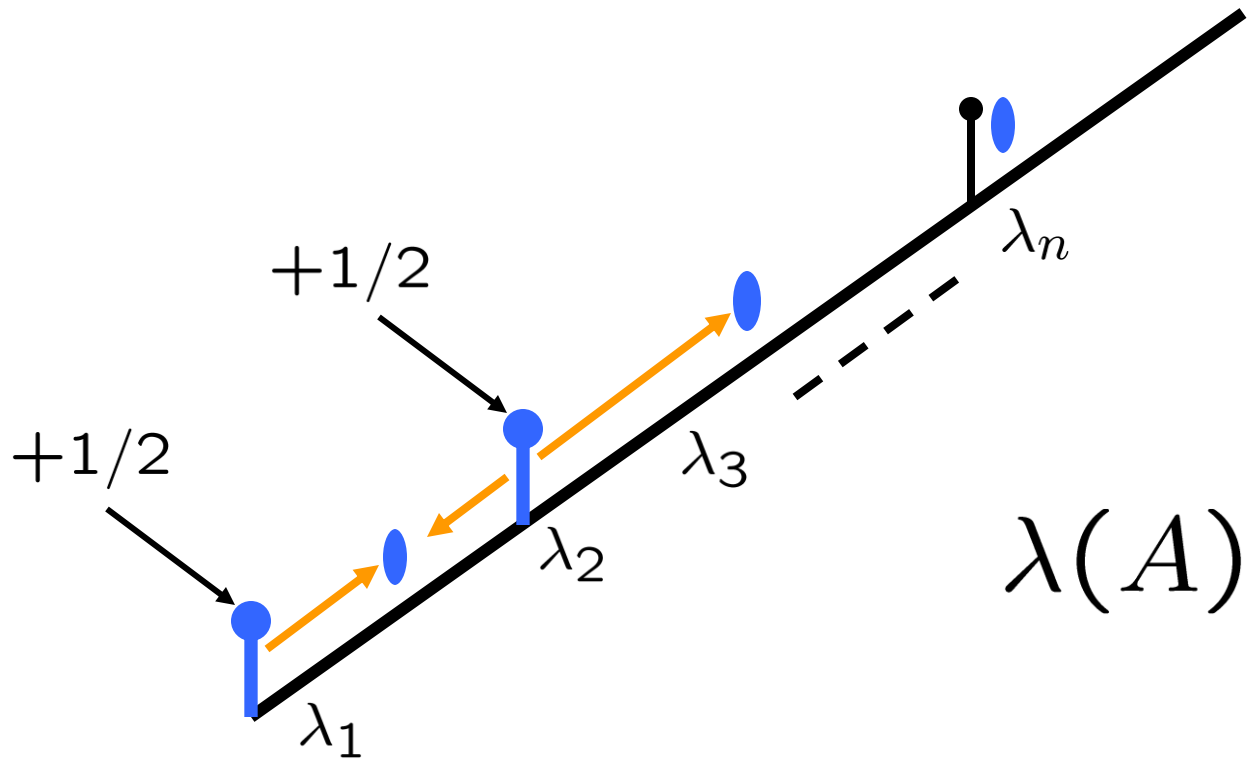
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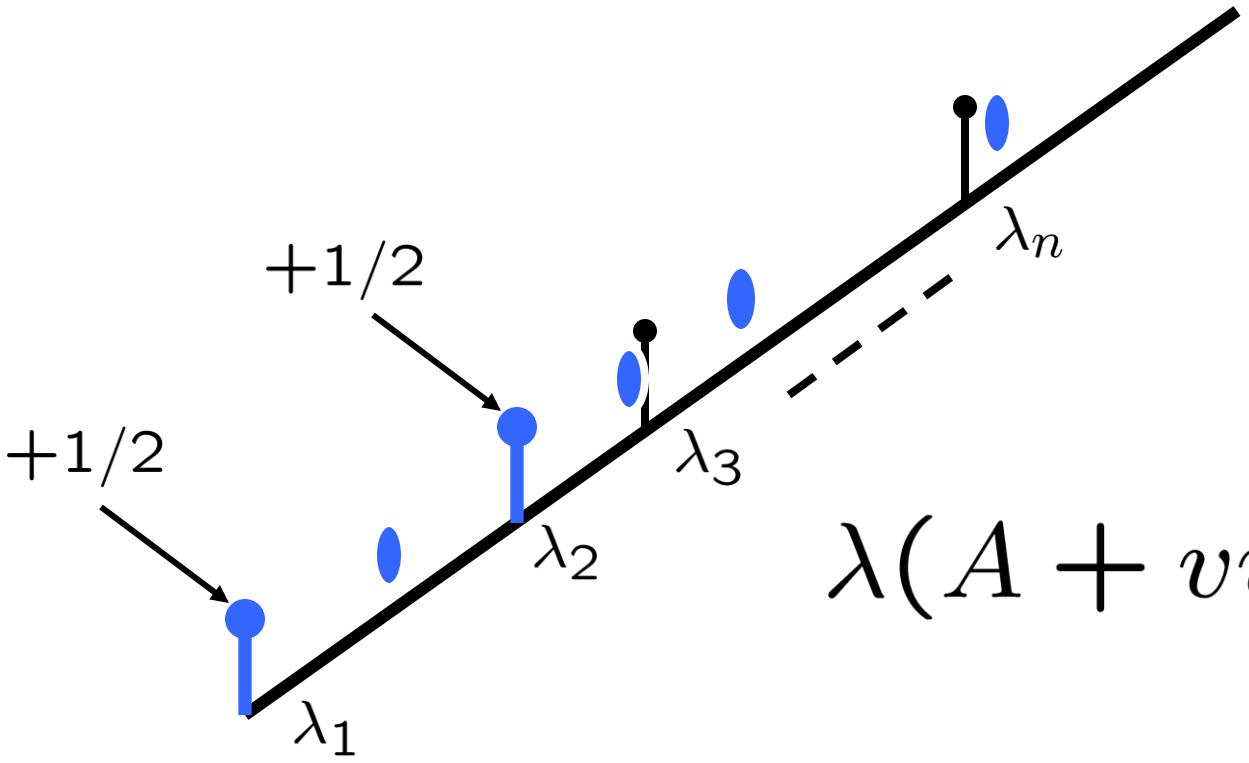
Ex2: Equal weight on u_1, u_2



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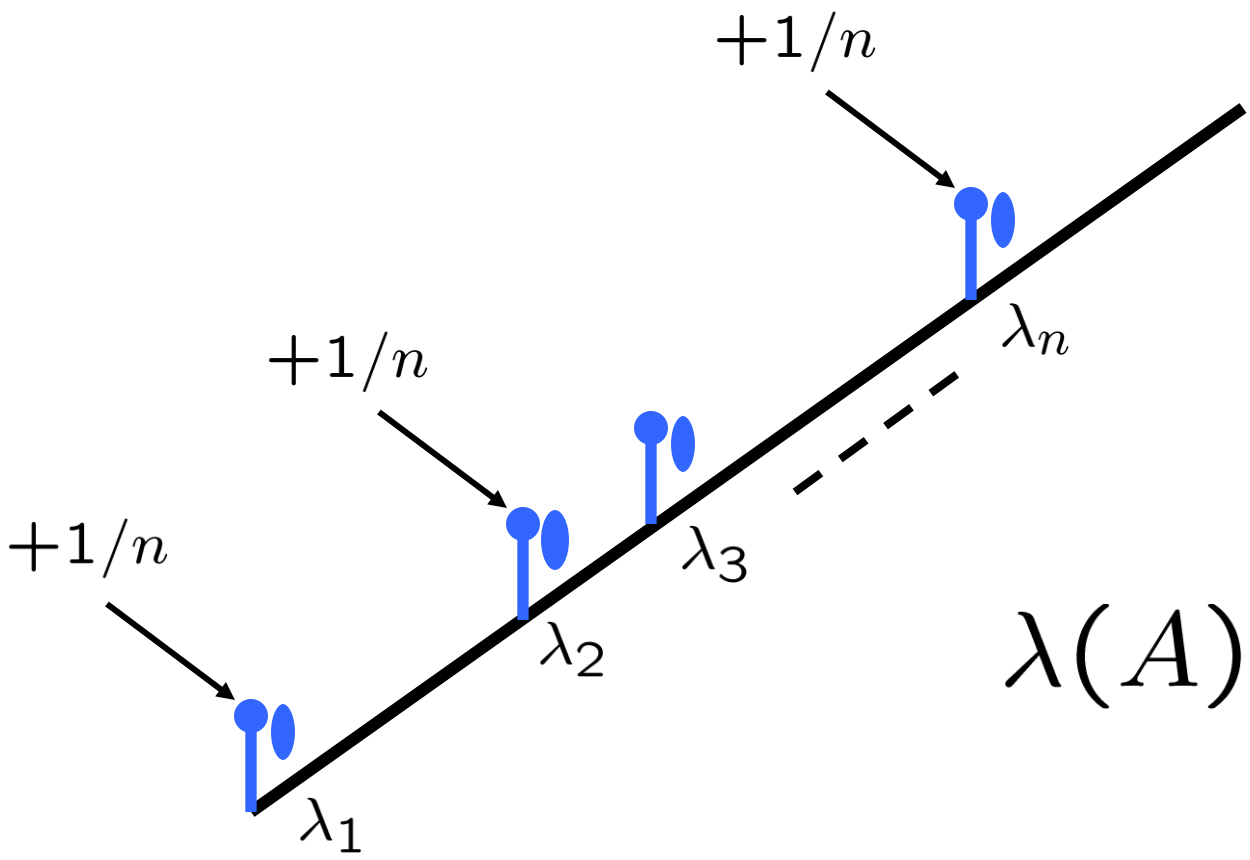


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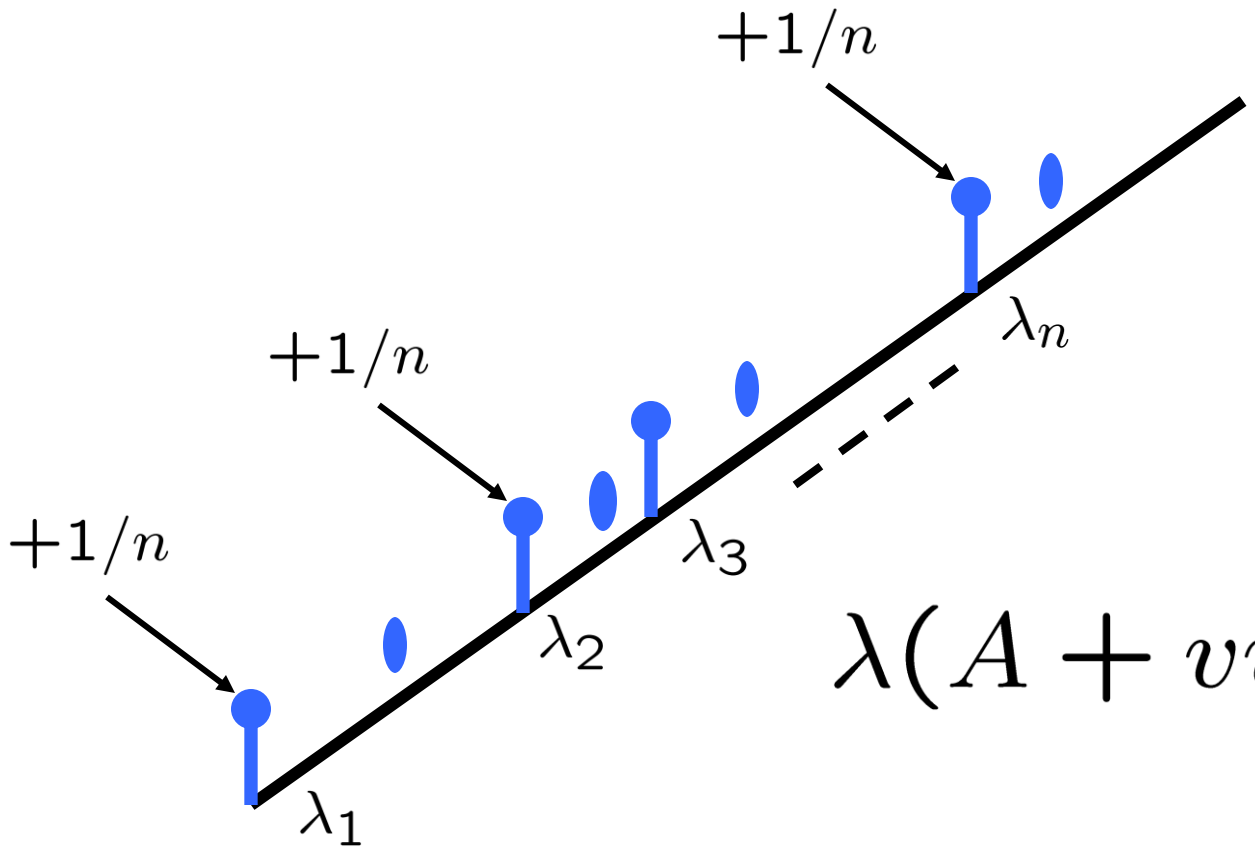


$$\lambda(A + vv^T)$$

Ex3: Equal weight on all $u_1, u_2, \ldots u_n$



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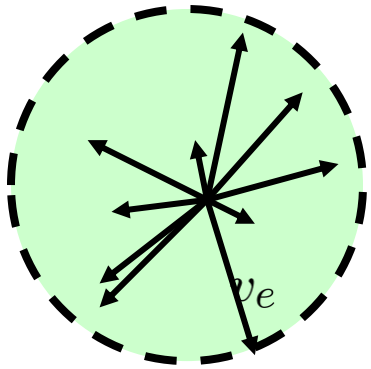
$$\lambda(A + vv^T)$$

Adding a balanced vector

$$
\begin{aligned}
p_{A+vv^t} &= p_A \left(1 + \sum_i \frac{\langle v, u_i \rangle^2}{\lambda_i - x}\right) \\
&= p_A \left(1 + \sum_i \frac{1}{\lambda_i - x}\right) \\
&= p_A - p'_A
\end{aligned}
$$

Consider a random vector

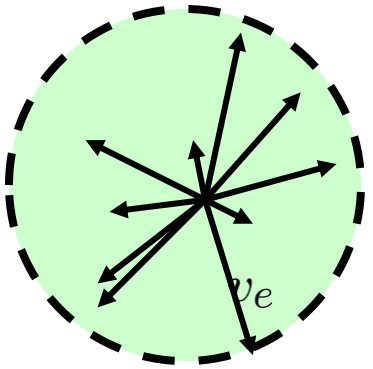
If $\sum_e v_e v_e^T = I$



For every u_i : $\sum_e \langle v_e, u_i \rangle^2 = 1.$

Consider a random vector

If $\sum_e v_e v_e^T = I$



For every u_i : $\sum_e \langle v_e, u_i \rangle^2 = 1.$

thus a 'random' vector has the same expected projection in *every* direction ***i*** :

$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

Ideal proof



$$A^{(0)} = 0$$

$$p^{(0)} = x^n$$

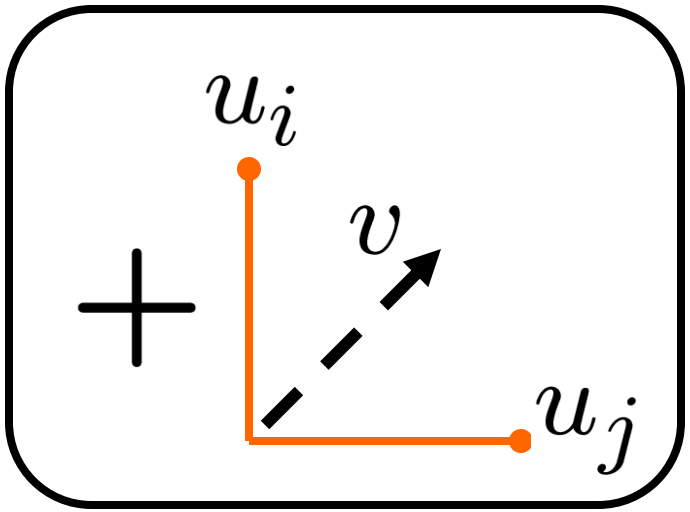
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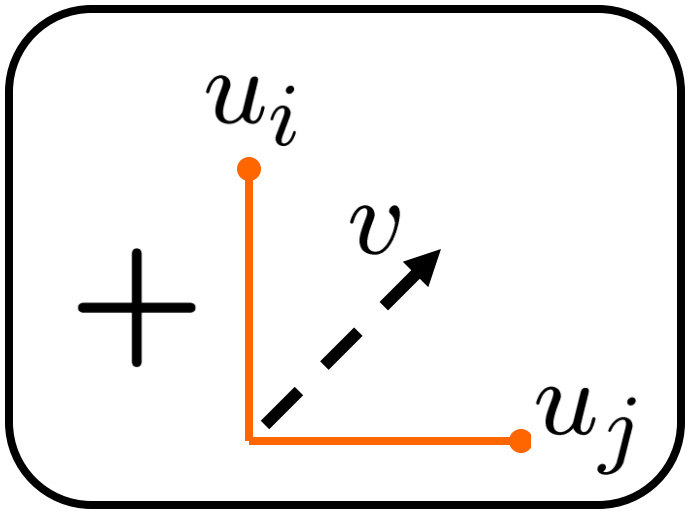


$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

$$A^{(1)} = 0 + vv^T$$

$$p^{(1)} = x^n - nx^{n-1}$$

Ideal proof

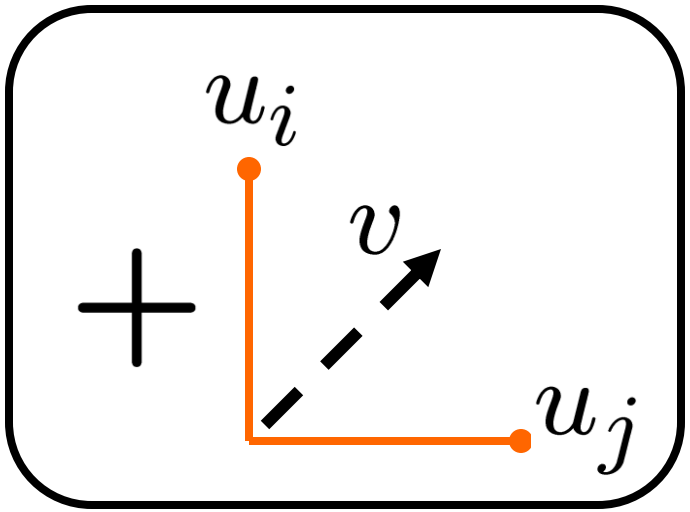


$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

$$A^{(2)} = A^{(1)} + vv^T$$

$$p^{(2)} = x^n - 2nx^{n-1} + n(n-1)x^{n-2}$$

Ideal proof

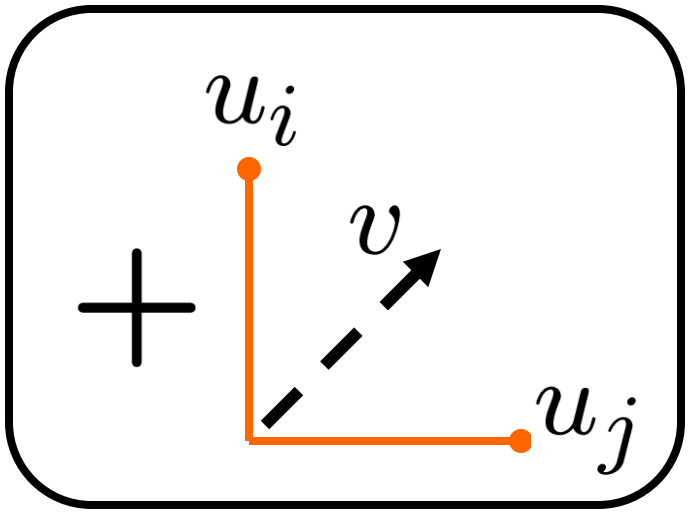


$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

$$A^{(3)} = A^{(2)} + vv^T$$

$$p^{(3)} = p^{(2)} - p^{(2)'}$$

Ideal proof

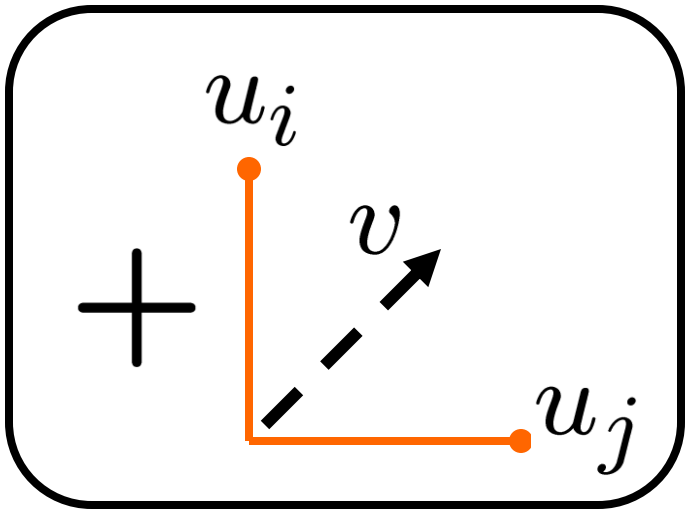


$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

$$A^{(i+1)} = A^{(i)} + vv^T$$

$$p^{(i+1)} = p^{(i)} - p^{(i)\prime}$$

Ideal proof

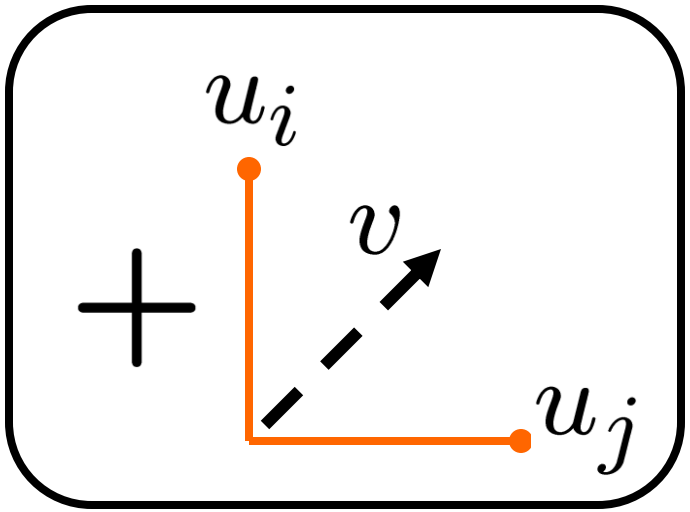


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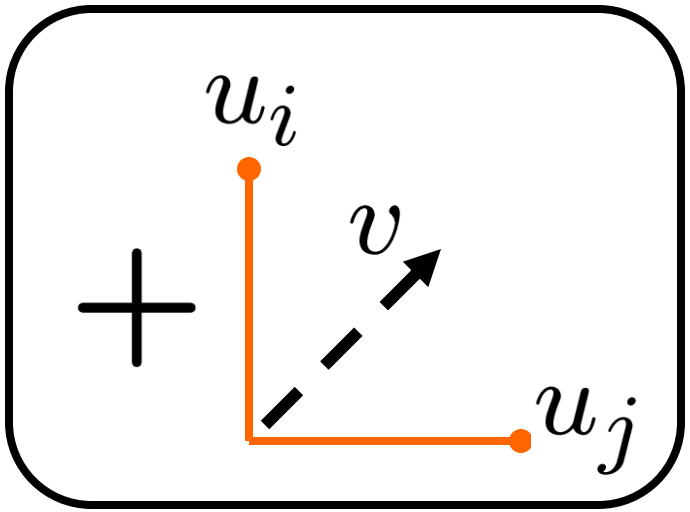


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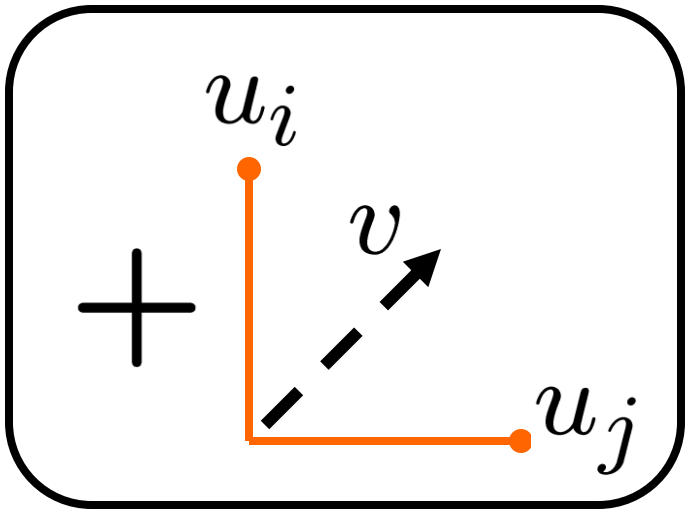


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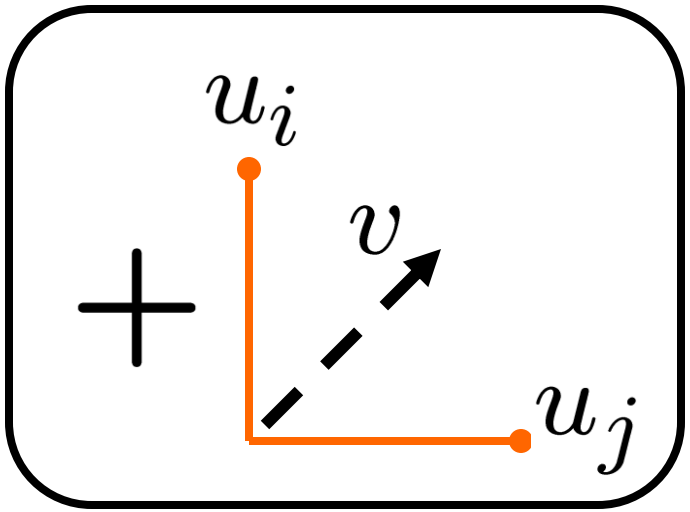


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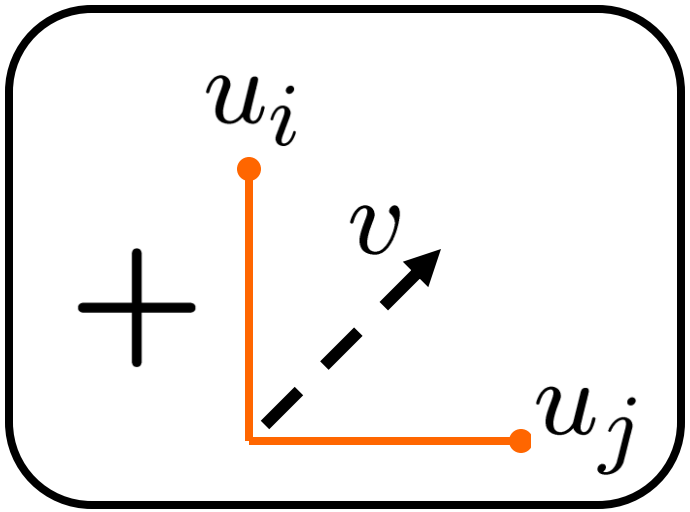


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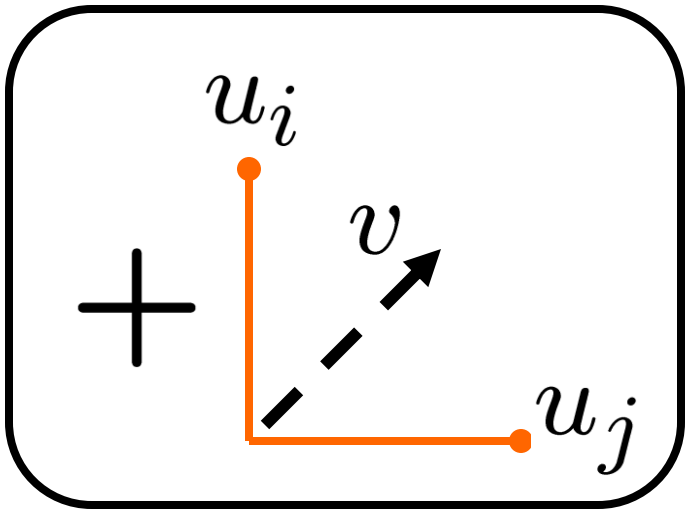
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.....

Ideal proof



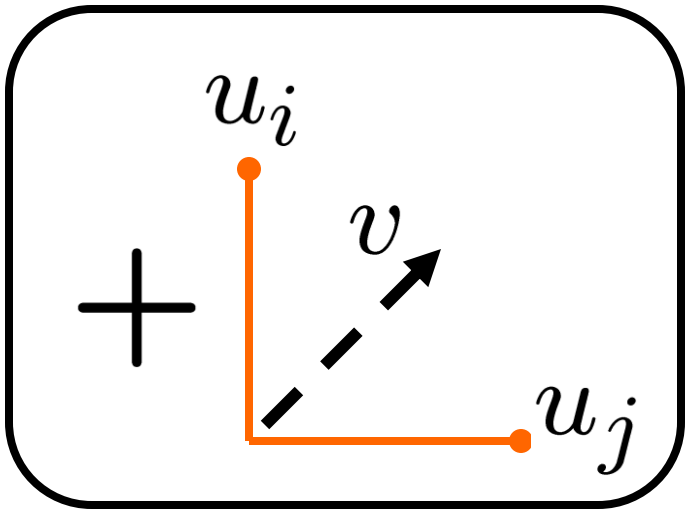
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∞

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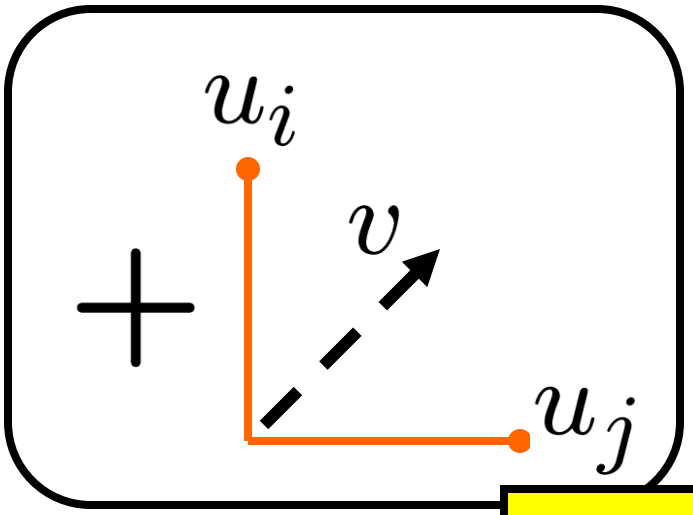
∞

$$A^{(i+1)} = A^{(i)} + vv^T$$

$$p^{(i+1)} = p^{(i)} - p^{(i)\prime}$$

$$\frac{\lambda_n(A)}{\lambda_1(A)} \leq \kappa?$$

Punch Line



$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

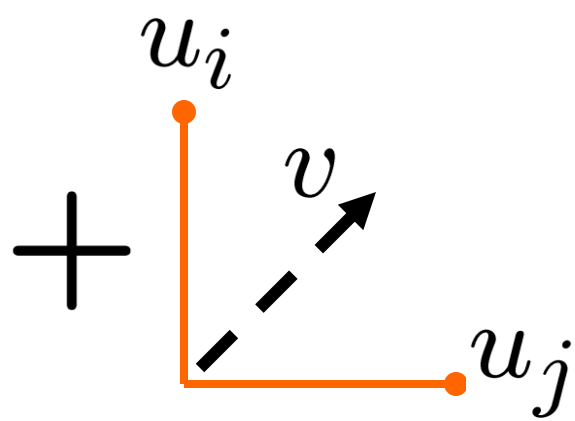
$$p^{(i)} = \text{Laguerre}^{(i)}$$

$$A^{(i+1)} =$$

$$p^{(i+1)} = p^{(i)} - p^{(i)\prime}$$

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Punch Line



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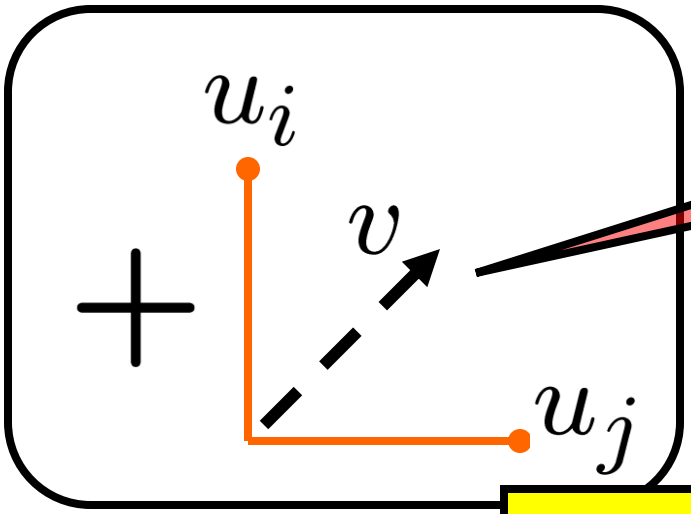
$$p^{(i+1)} = p^{(i)} - p^{(i)\prime}$$

$$\lambda_n(A)$$

In $O(n/\epsilon^2)$ steps: $\frac{\lambda_n(A)}{\lambda_1(A)} \leq 1 + \epsilon$

Punch Line

find actual vectors that realize this ideal behavior.



$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

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$$p^{(i+1)} = p^{(i)} - p^{(i)\prime}$$

$$\lambda_n(A)$$

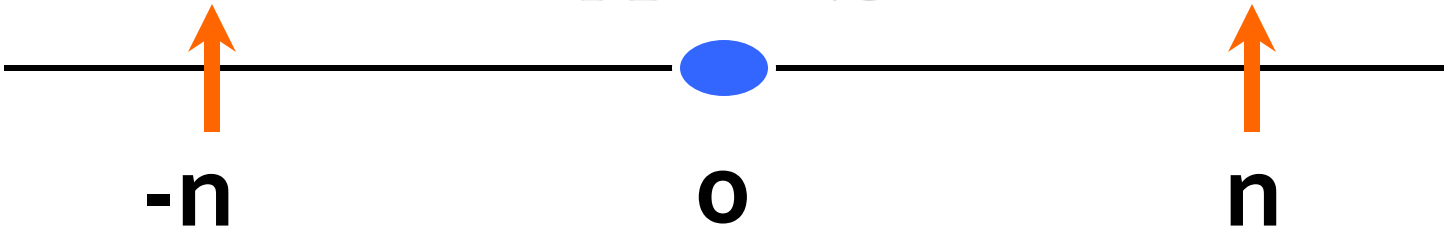
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Step 3: Actual Proof

(for 6n vectors, 13-approx)

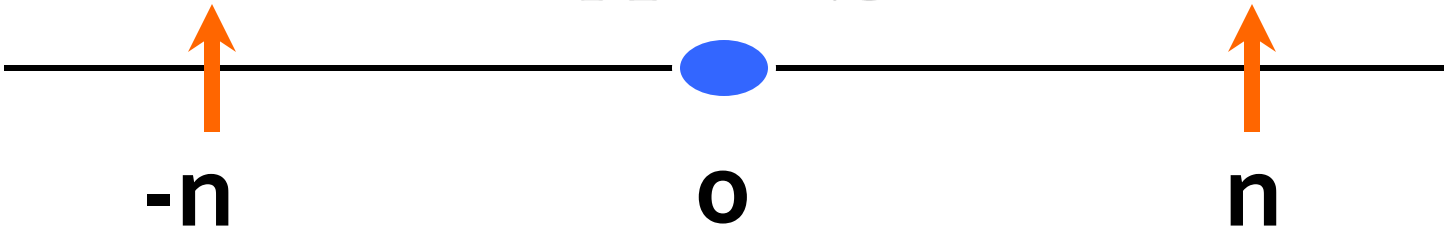
Steady progress by moving barriers

$$A = \varnothing$$



Step 1

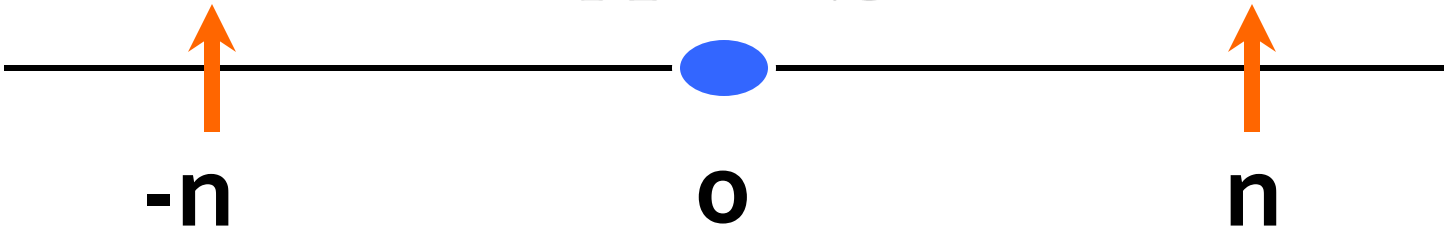
$$A = \varnothing$$



$$+\mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\}$$

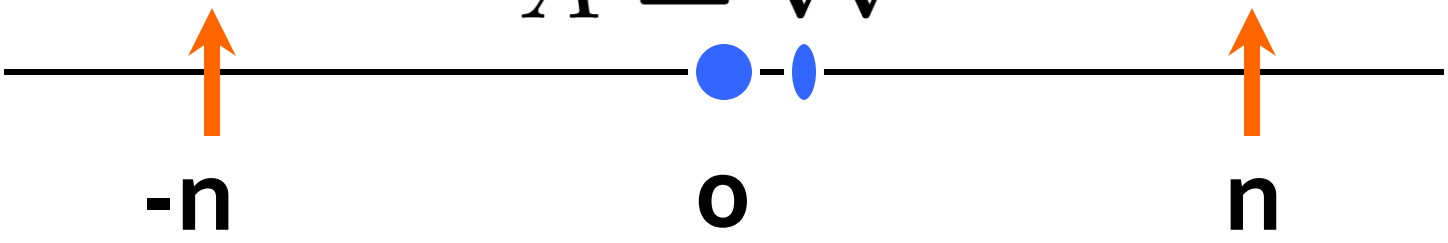
Step 1

$$A = \varnothing$$



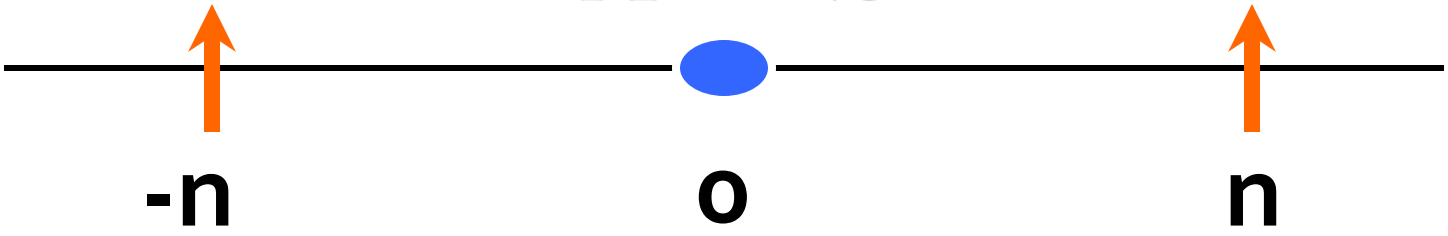
$$+\mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\}$$

$$A = \mathbf{v}\mathbf{v}^T$$

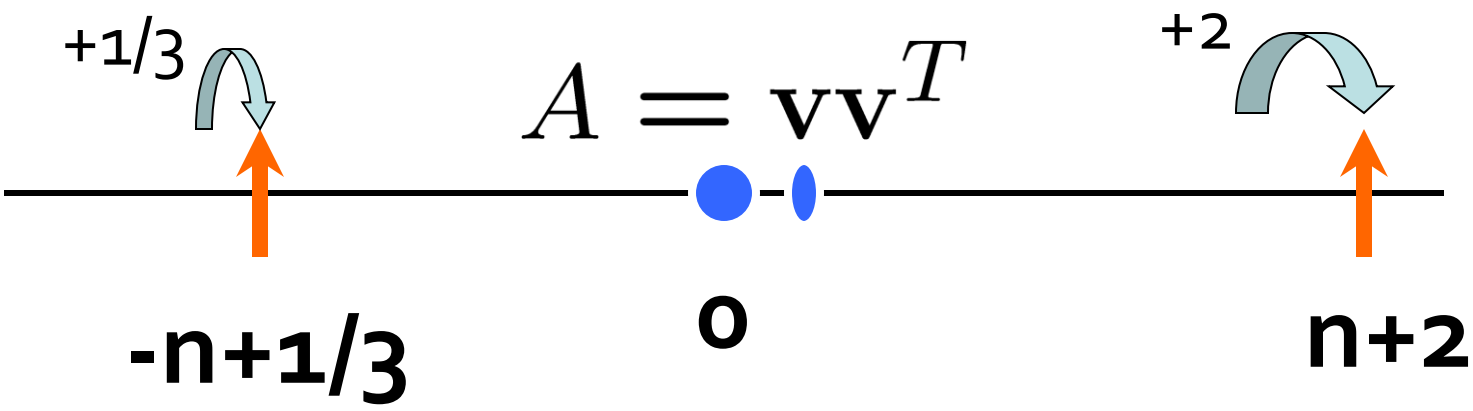


Step 1

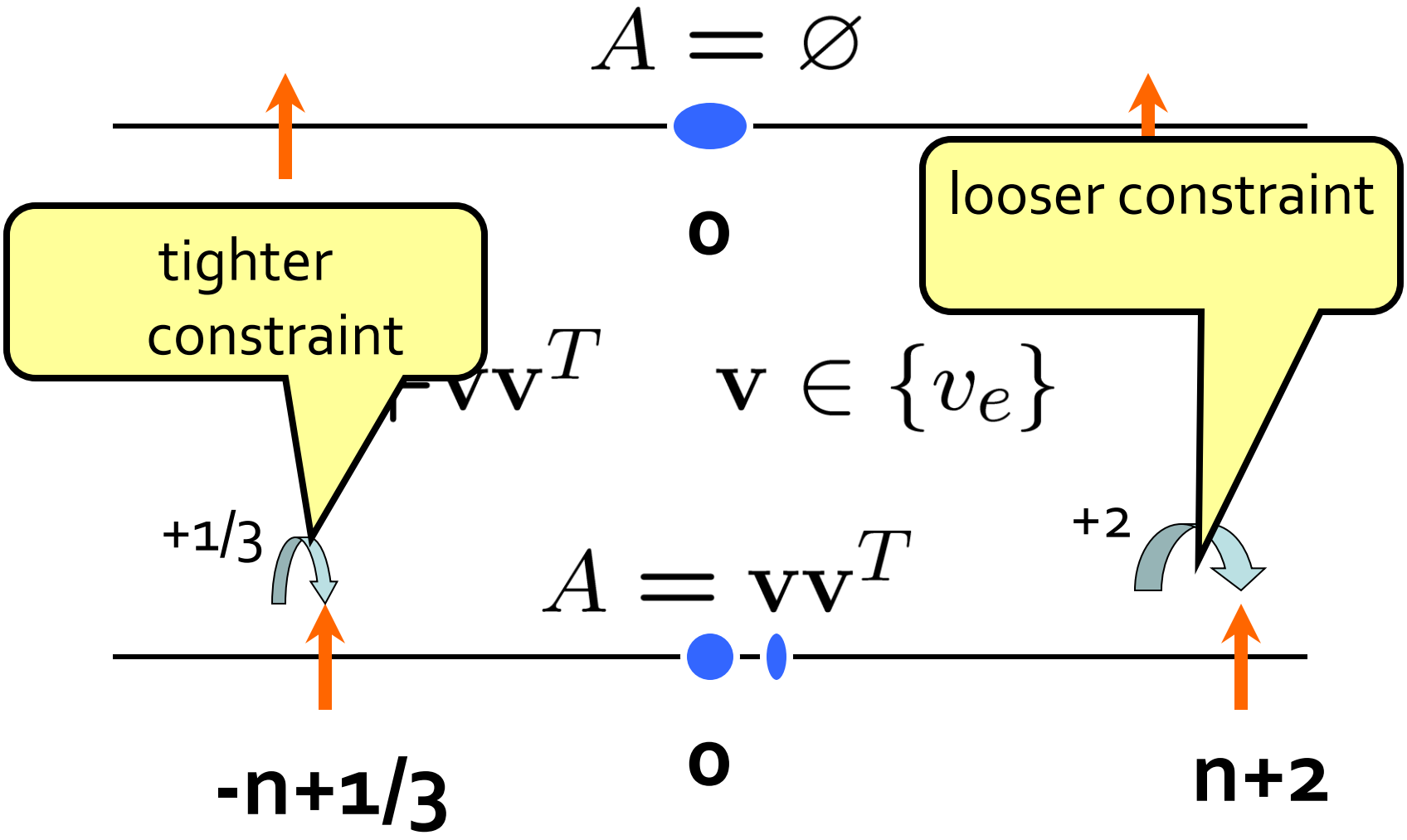
$$A = \varnothing$$



$$+\mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\}$$

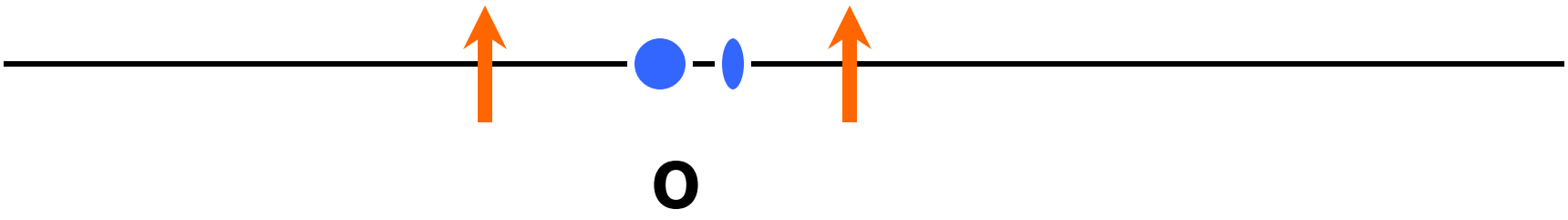


Step 1


$A = \varnothing$
looser constraint
tighter constraint
0
$-\mathbf{v}\mathbf{v}^T$
$\mathbf{v} \in \{v_e\}$
+1/3
+2
$A = \mathbf{v}\mathbf{v}^T$
-n+1/3
0
n+2

Step i+1

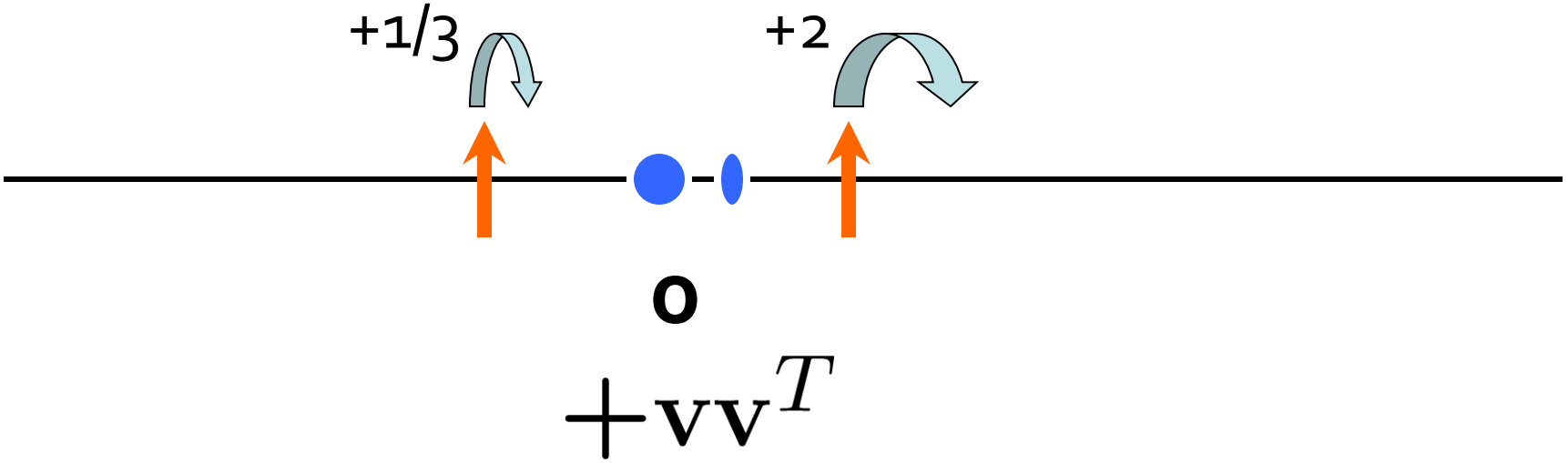
$A^{(i)}$



$$\uparrow \leq \lambda_i \leq \uparrow$$

Step i+1

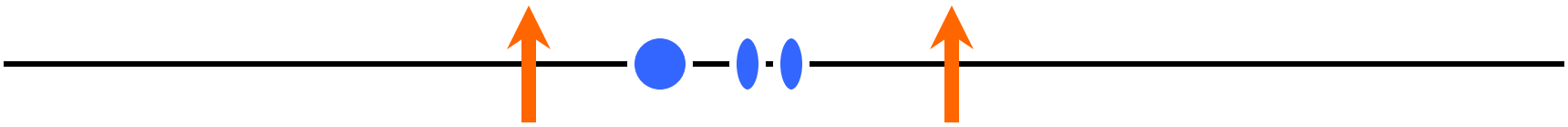
$A^{(i)}$



$+\mathbf{v}\mathbf{v}^T$

Step i+1

$A^{(i)}, A^{(i+1)}$

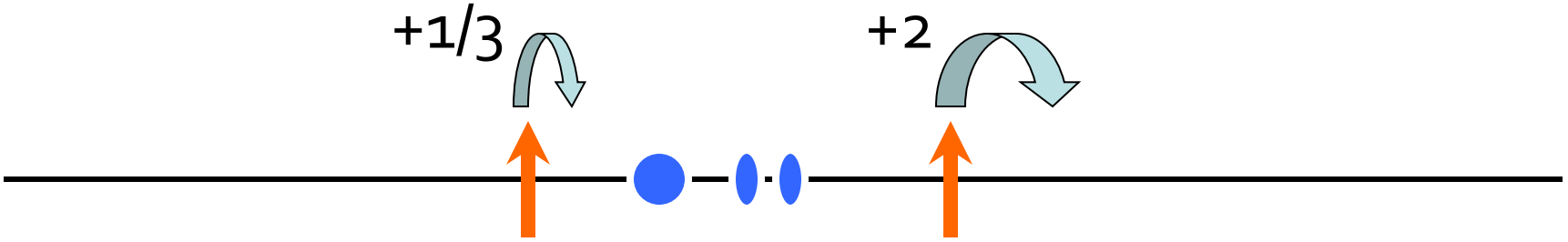


o

$$\uparrow \leq \lambda_i \leq \uparrow$$

Step i+1

$A^{(i)}, A^{(i+1)}$

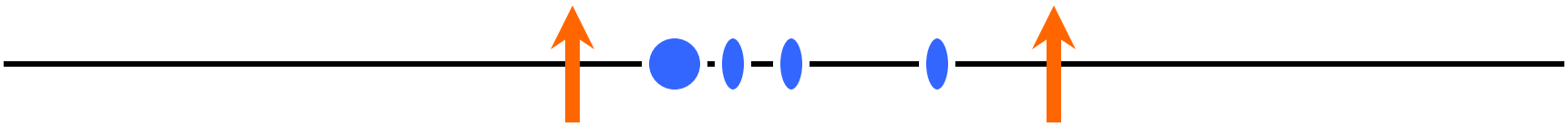


0

$+\mathbf{v}\mathbf{v}^T$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}$$

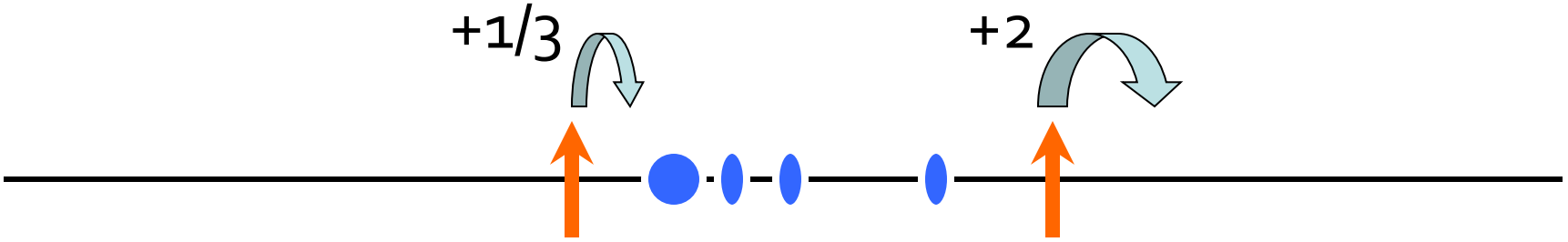


o

$$\uparrow \leq \lambda_i \leq \uparrow$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}$$

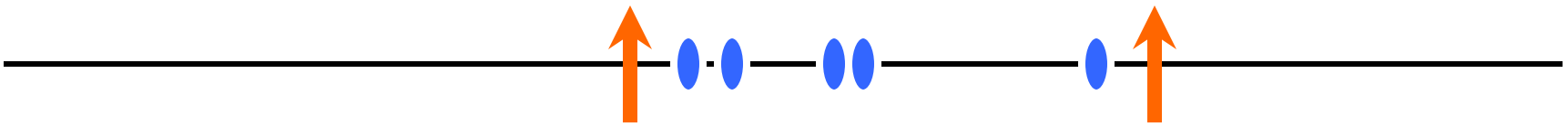


o

$$+\mathbf{v}\mathbf{v}^T$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$$

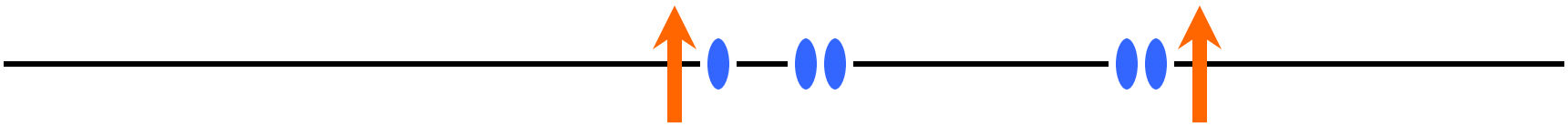


o

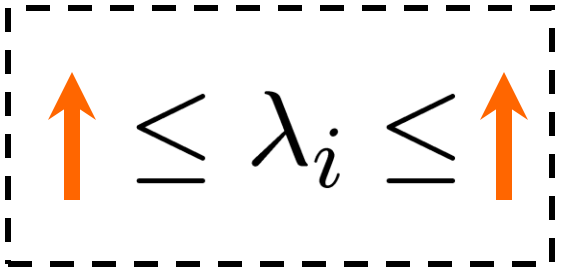
$$\uparrow \leq \lambda_i \leq \uparrow$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

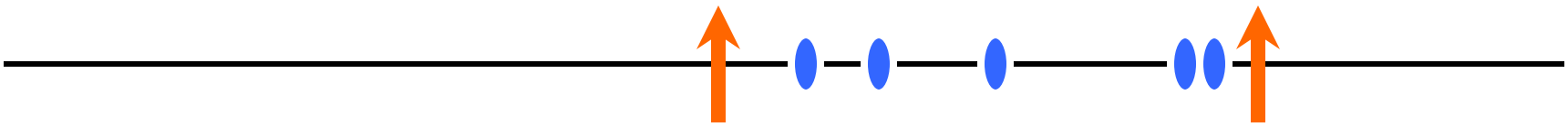


o

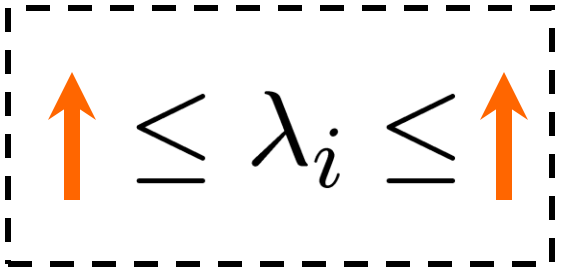


Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

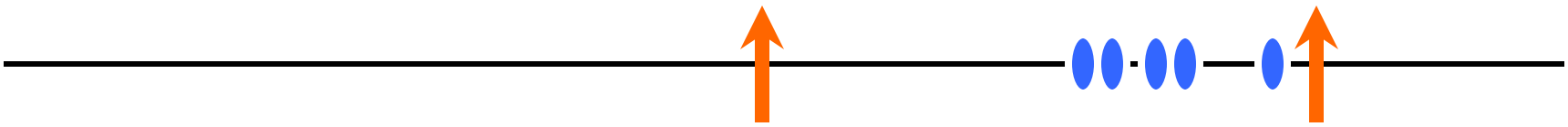


0

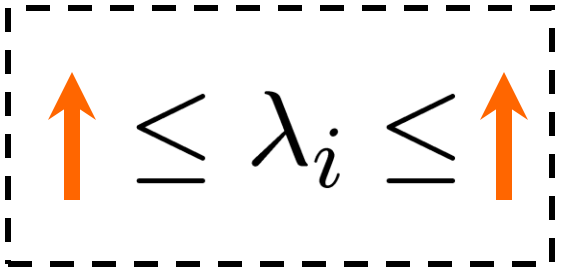


Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$



0



Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$



$$\uparrow \leq \lambda_i \leq \uparrow$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$



13-approximation with 6n vectors.

□

Problem

need to show that an appropriate

$$v_e v_e^T$$

always exists.

Problem

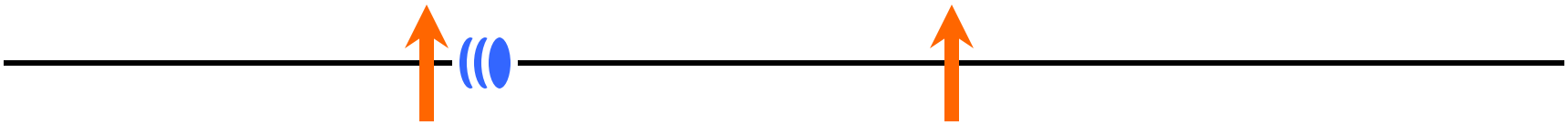
need to show that an appropriate

$$v_e v_e^T$$

always exists.

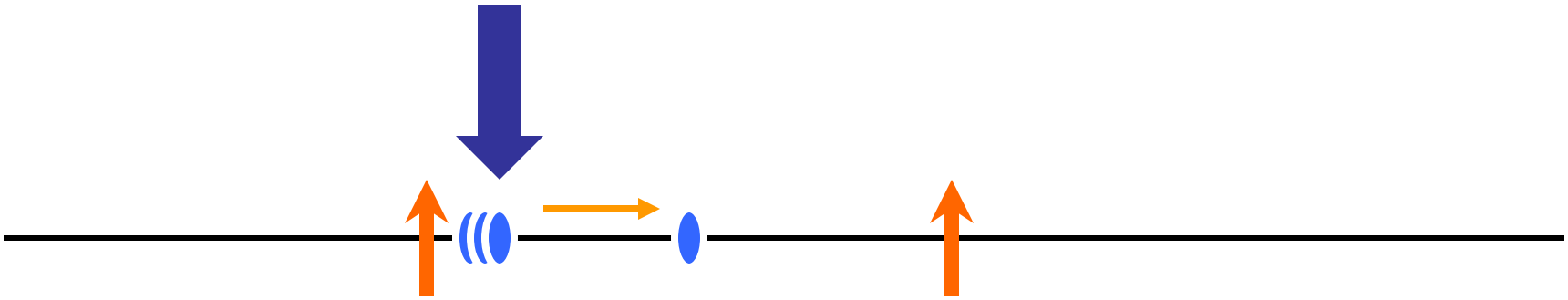
Hope: vectors are well-spread: there must be one which is well-behaved.

Bad: Accumulation of Eigenvalues

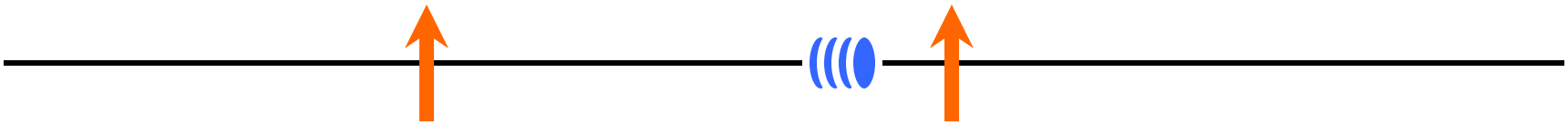


Bad: Accumulation of Eigenvalues

$$+\mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\}$$

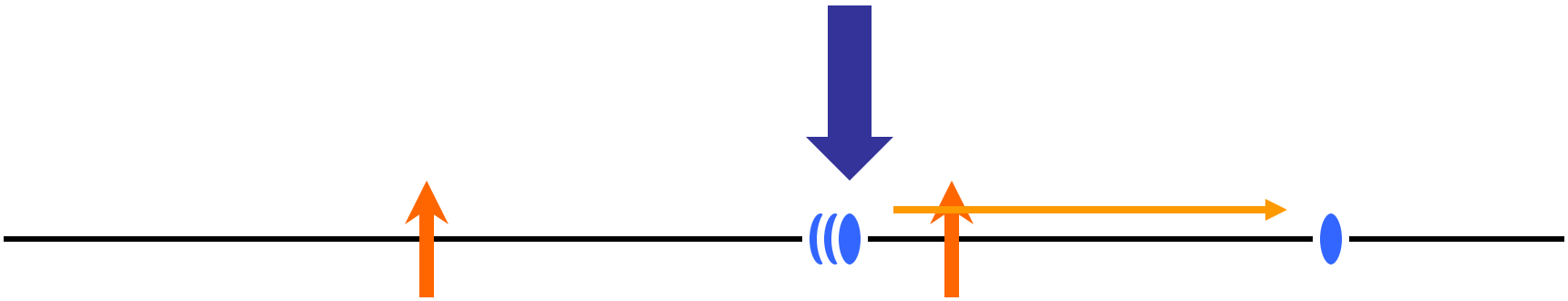


Bad: Accumulation of Eigenvalues



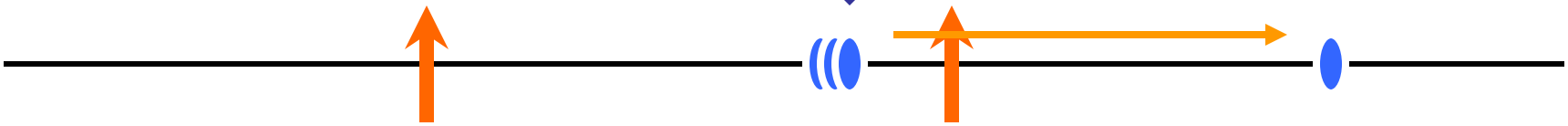
Bad: Accumulation of Eigenvalues

$$+\mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\}$$



Bad: Accumulation of Eigenvalues

$$+\mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\}$$

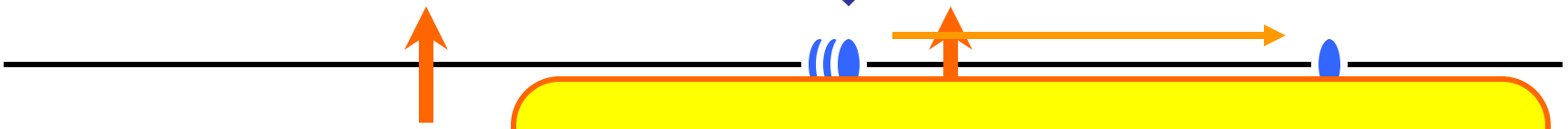


$$\uparrow \leq \lambda_i \leq \uparrow$$

is not strong enough to do the induction.

Bad: Accumulation of Eigenvalues

$$+\mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\}$$



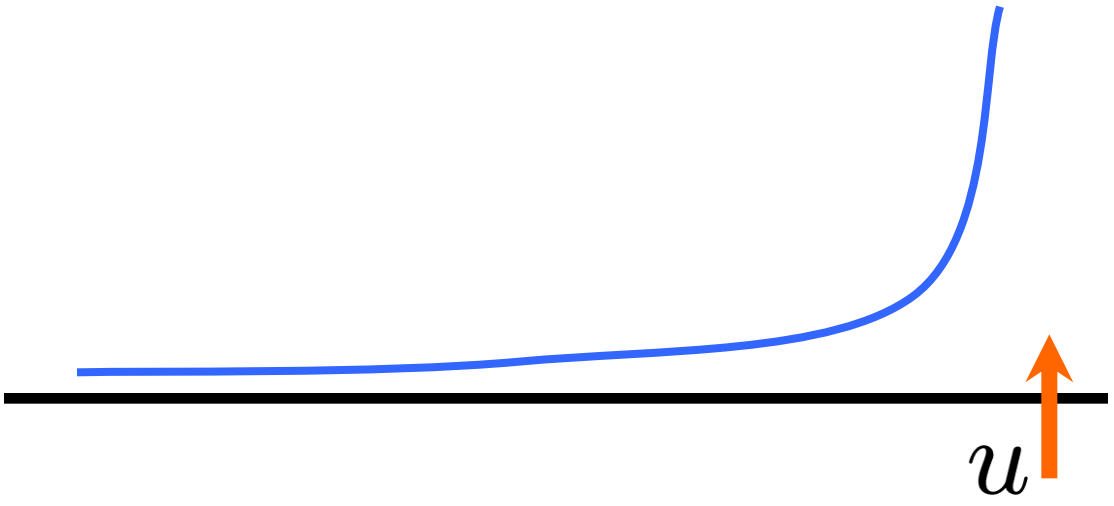
need a better way to measure quality of eigenvalues.

$\uparrow \leq \lambda_i \leq \uparrow$

is not strong enough to do the induction.

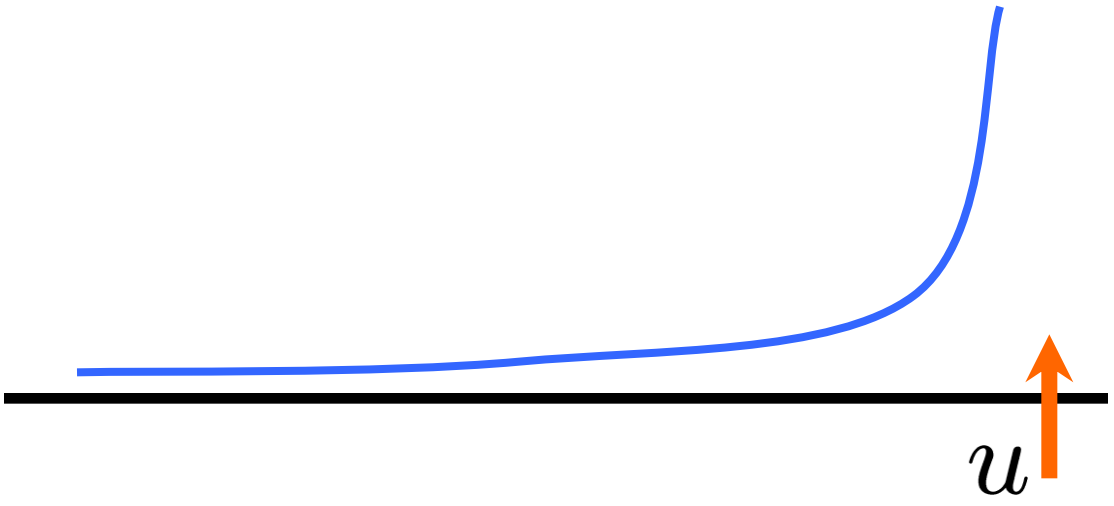
The Upper Barrier

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} = \sum_i \frac{1}{u-\lambda_i}$$



The Upper Barrier

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} = \sum_i \frac{1}{u - \lambda_i}$$



$$\Phi^u(A) \leq 1 \Rightarrow \lambda_{\max}(A) \ll u$$

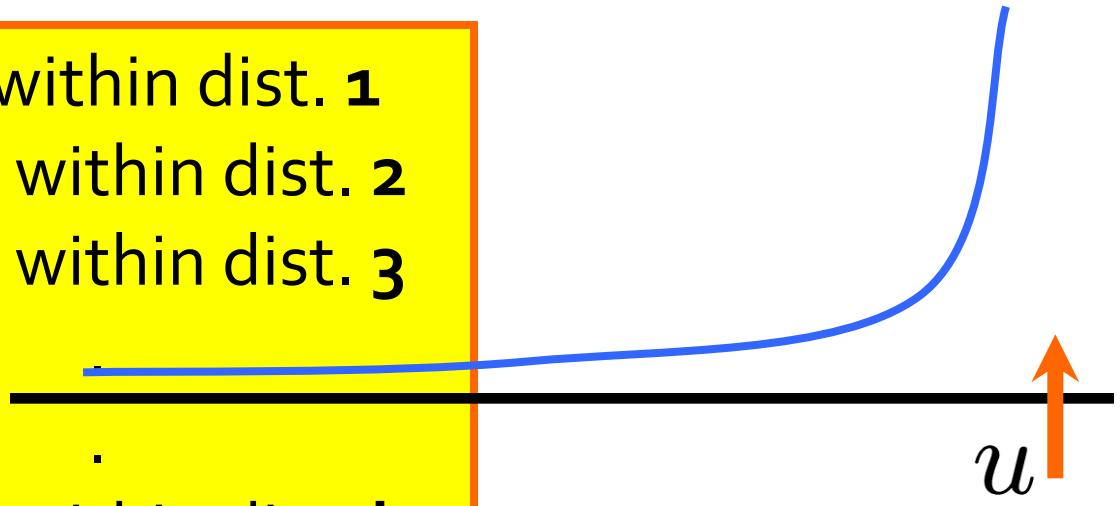
The Upper Barrier

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} = \sum_i \frac{1}{u-\lambda_i}$$

No λ_i within dist. **1**
No **2** λ_i within dist. **2**
No **3** λ_i within dist. **3**

.

No **k** λ_i within dist. **k**



$$\Phi^u(A) \leq 1 \Rightarrow \lambda_{\max}(A) \ll u$$

'Total repulsion' in physical model

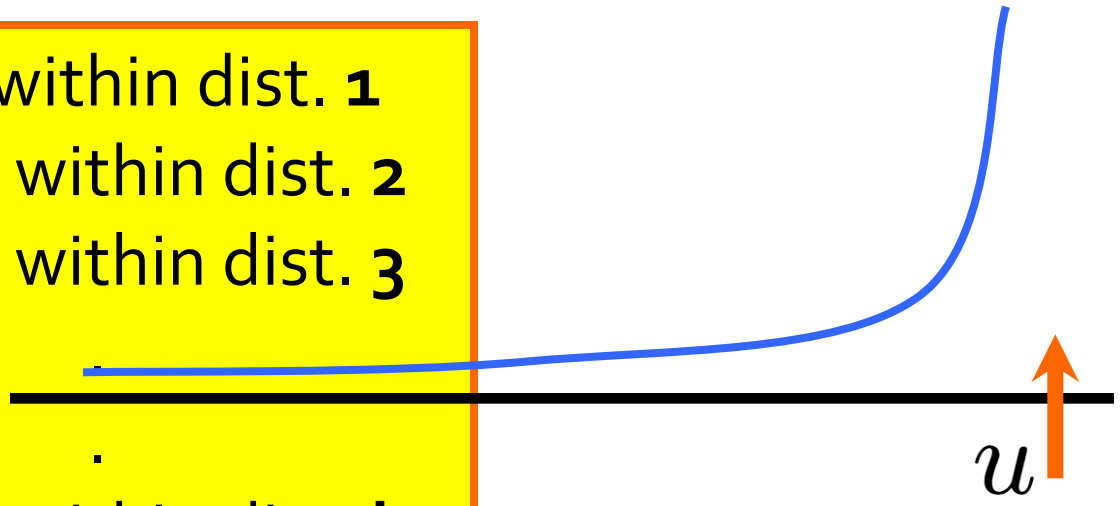
per Barrier

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} = \sum_i \frac{1}{u-\lambda_i}$$

No λ_i within dist. **1**
No **2** λ_i within dist. **2**
No **3** λ_i within dist. **3**

.

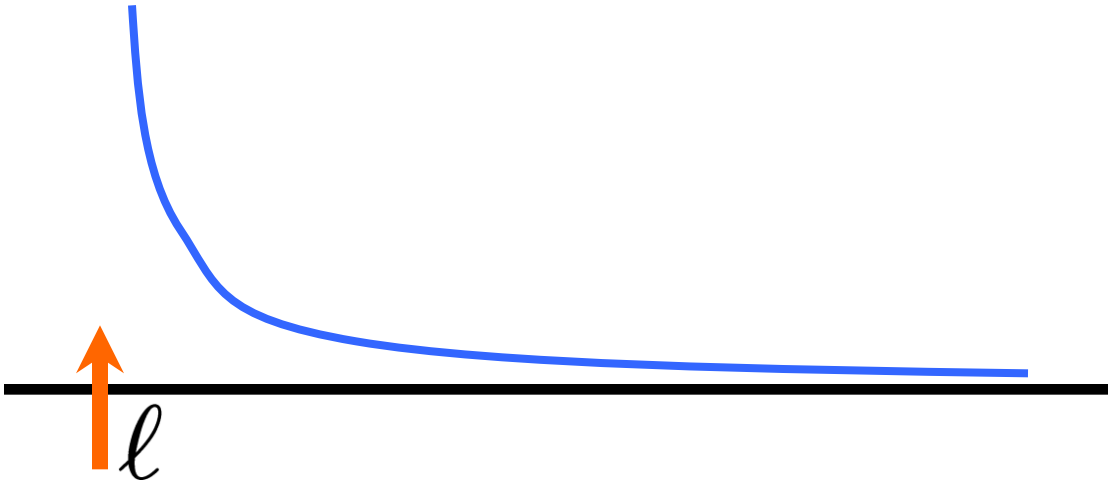
No **k** λ_i within dist. **k**



$$\Phi^u(A) \leq 1 \Rightarrow \lambda_{\max}(A) \ll u$$

The Lower Barrier

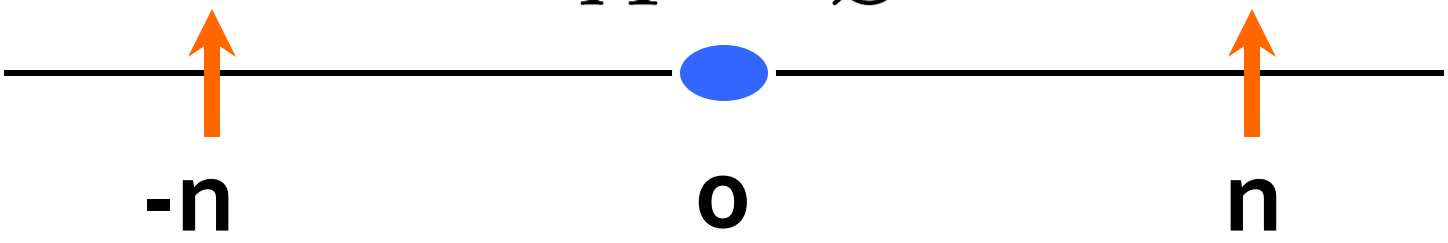
$$\Phi_\ell(A) = \mathrm{Tr}(A - \ell I)^{-1} = \sum_i \frac{1}{\lambda_i - \ell}$$



$$\Phi_\ell(A) \leq 1 \Rightarrow \lambda_{\min}(A) \gg \ell$$

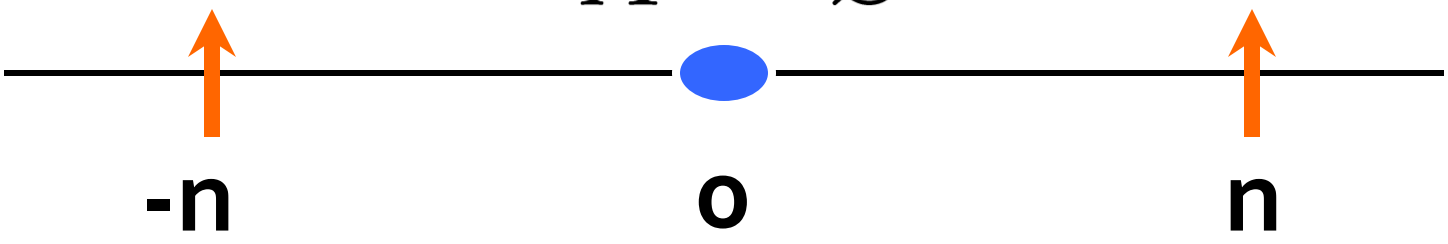
The Beginning

$$A = \varnothing$$



The Beginning

$$A = \varnothing$$

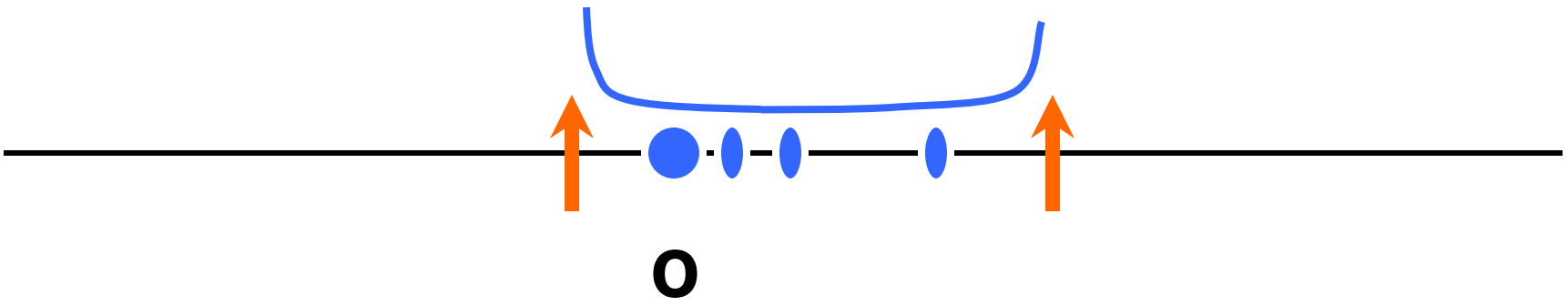


$$\Phi^n(\varnothing) = \mathrm{Tr}(nI)^{-1} = 1$$

$$\Phi_{-n}(\varnothing) = \mathrm{Tr}(nI)^{-1} = 1.$$

Step i+1

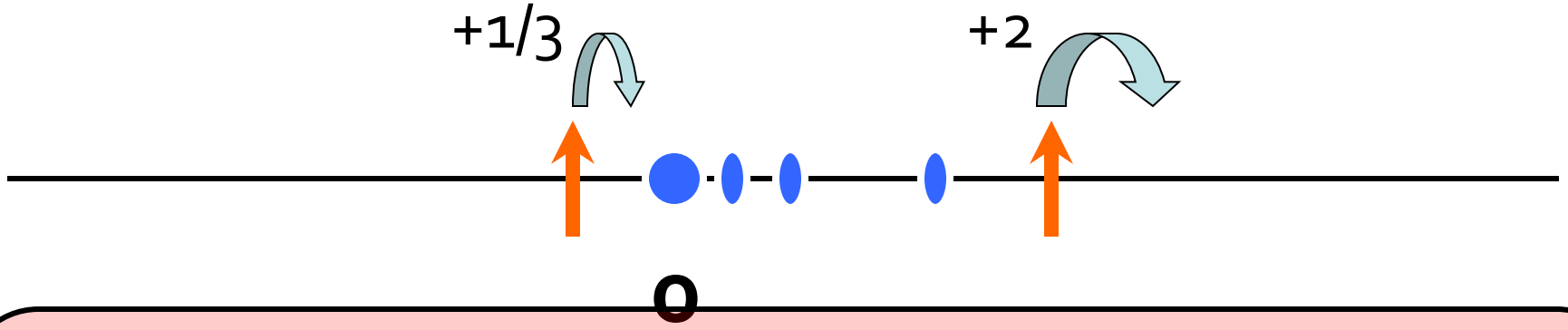
$A^{(i)}, A^{(i+1)}, A^{(i+2)}$



$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}$$



Lemma.

can always choose $+s\mathbf{v}\mathbf{v}^T$

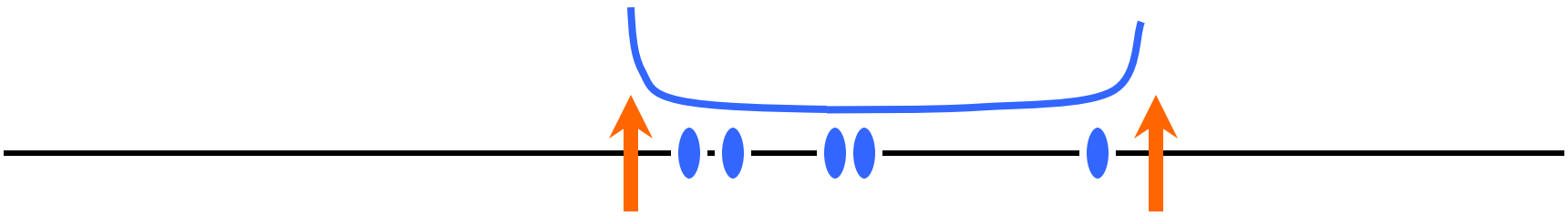
so that potentials do not increase

$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$$



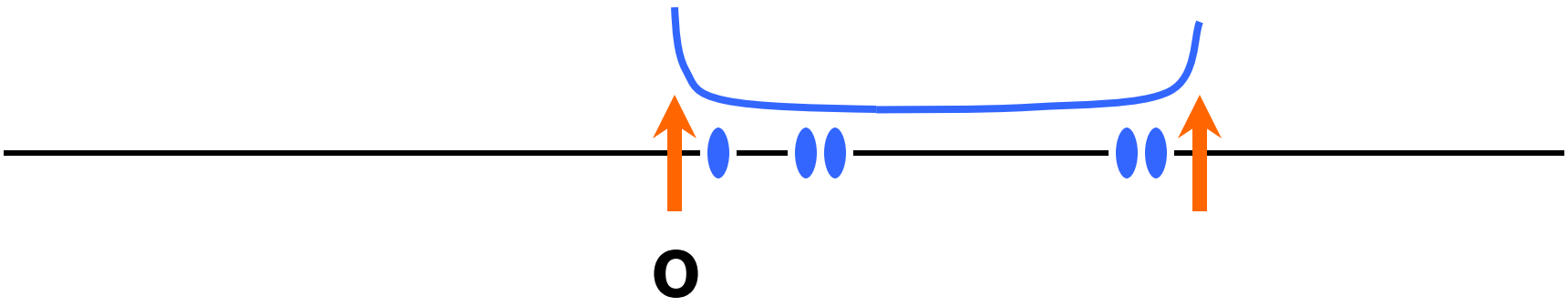
o

$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

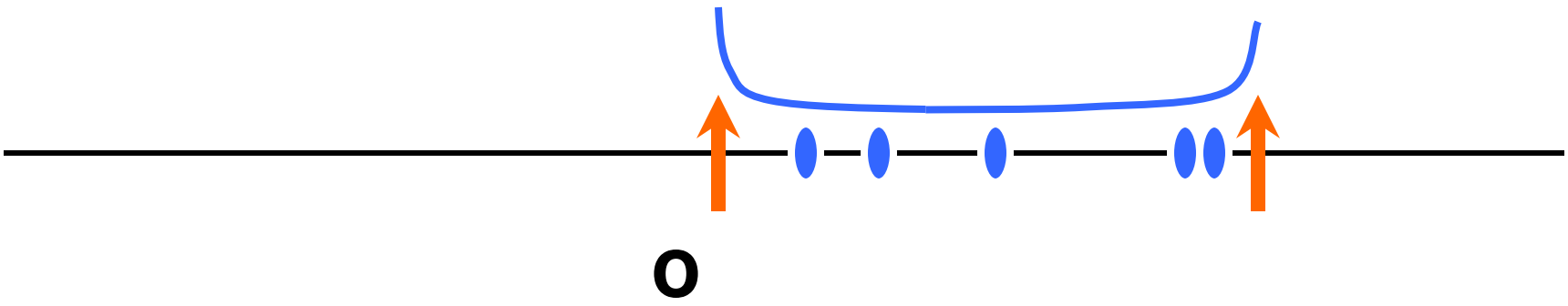


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

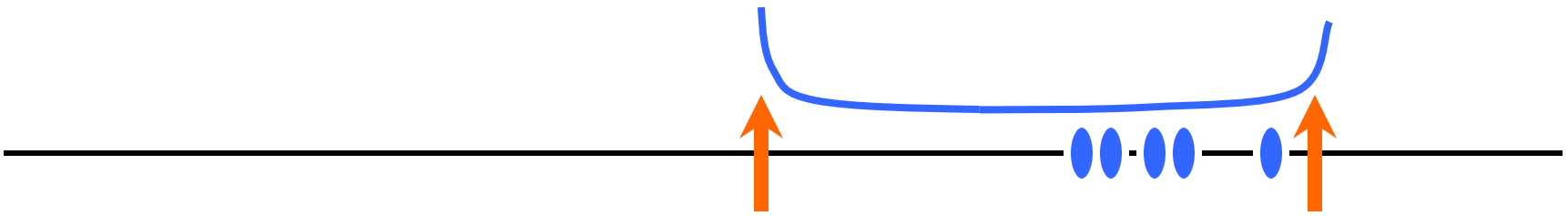


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$



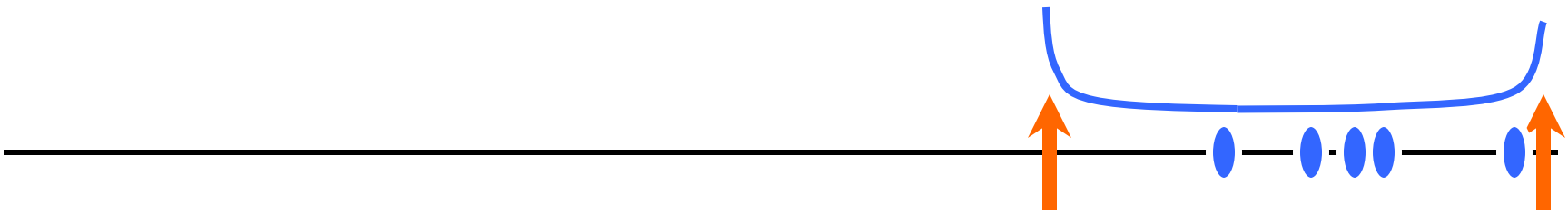
o

$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$



$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$



13-approximation with 6n vectors.

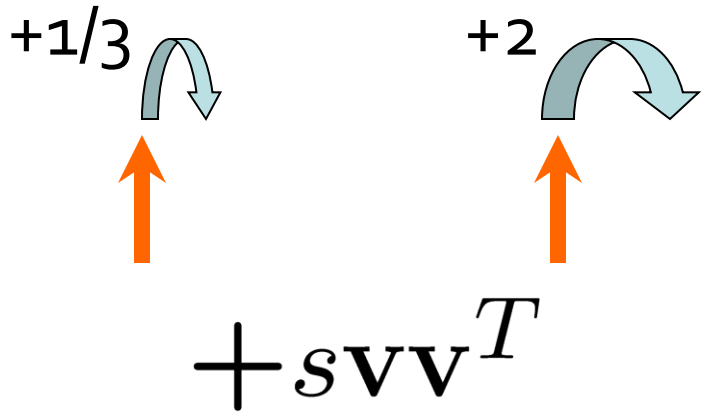
□

Goal

Lemma.

can always choose $+s\mathbf{v}\mathbf{v}^T$ so that *both* potentials do not increase.

$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$



The Right Question

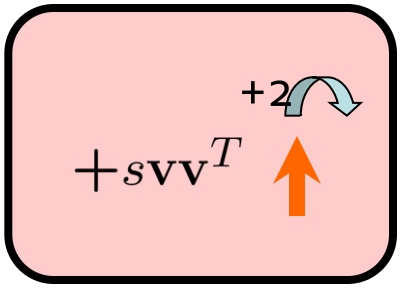
"Which vector should we add?"

The Right Question

~~"Which vector should we add?"~~

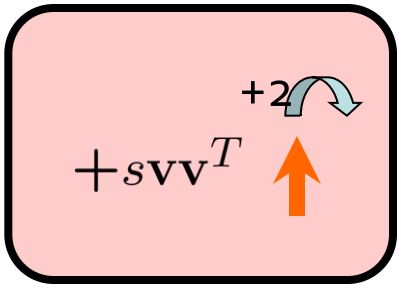
"Given a vector, how much of it can we add?"

Upper Barrier Update



Add $s\mathbf{v}\mathbf{v}^T$ & ***set*** $u' \leftarrow u + 2.$

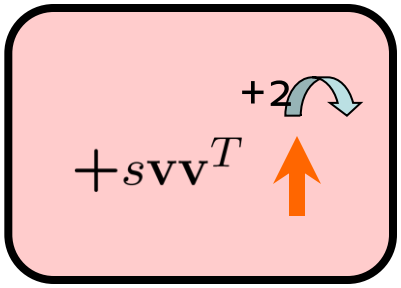
Upper Barrier Update



Add $s\mathbf{v}\mathbf{v}^T$ & ***set*** $u' \leftarrow u + 2.$

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T)$$
$$= \mathsf{Tr}(u'I - A - s\mathbf{v}\mathbf{v}^T)^{-1}$$

Upper Barrier Update



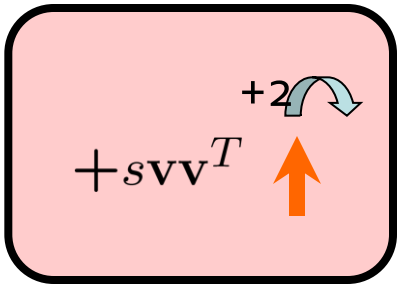
Add $s\mathbf{v}\mathbf{v}^T$ & ***set*** $u' \leftarrow u + 2.$

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T)$$
$$= \mathsf{Tr}(u'I - A - s\mathbf{v}\mathbf{v}^T)^{-1}$$

$$\mathsf{Tr}(A + vv^T)^{-1} = \mathsf{Tr}A^{-1} - \frac{v^T A^{-2} v}{1 + v^T A^{-1} v}$$

Sherman-Morrisson

Upper Barrier Update



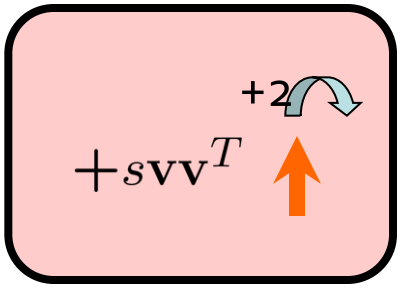
Add $s\mathbf{v}\mathbf{v}^T$ & ***set*** $u' \leftarrow u + 2.$

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T)$$

$$= \mathsf{Tr}(u'I - A - s\mathbf{v}\mathbf{v}^T)^{-1}$$

$$= \Phi^{u'}(A) + \frac{\mathbf{v}^T(u'I - A)^{-2}\mathbf{v}}{1/s - \mathbf{v}^T(u'I - A)^{-1}\mathbf{v}}$$

Upper Barrier Update



Add $s\mathbf{v}\mathbf{v}^T$ & ***set*** $u' \leftarrow u + 2.$

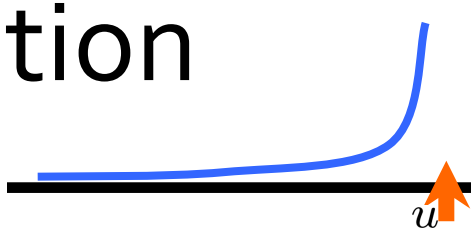
$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T)$$

$$= \mathsf{Tr}(u'I - A - s\mathbf{v}\mathbf{v}^T)^{-1}$$

$$= \Phi^{u'}(A) + \frac{\mathbf{v}^T(u'I - A)^{-2}\mathbf{v}}{1/s - \mathbf{v}^T(u'I - A)^{-1}\mathbf{v}}$$

want $\leq \Phi^u(A).$

Upper feasibility condition



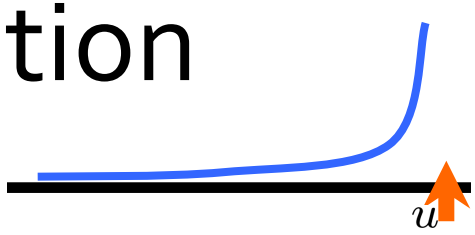
Rearranging:

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^{u}(A)$$

$$\Longleftrightarrow$$

$$\frac{1}{s} \geq \mathbf{v}^T \left(\frac{(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + (u'I - A)^{-1} \right) \mathbf{v}$$

Upper feasibility condition



Rearranging:

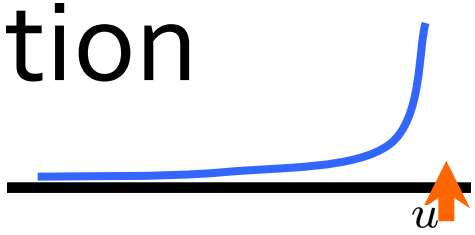
$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^{u}(A)$$

$$\Longleftrightarrow$$

$$\frac{1}{s} \geq \mathbf{v}^T \left(\frac{(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + (u'I - A)^{-1} \right) \mathbf{v}$$

$$\boxed{\frac{1}{s} \geq U_A \bullet \mathbf{v}\mathbf{v}^T}$$

Upper feasibility condition



Rearranging:

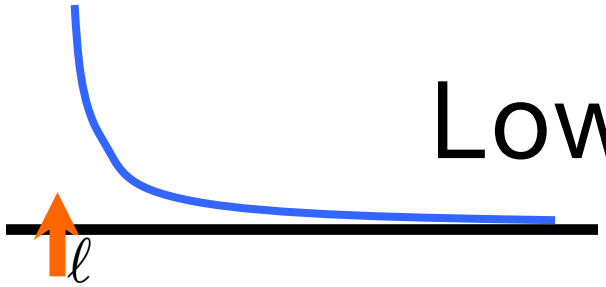
$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^{u}(A)$$

s=0 always feasible

$$\Longleftrightarrow$$

$$\frac{1}{s} \geq \mathbf{v}^T \left(\frac{(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + (u'I - A)^{-1} \right) \mathbf{v}$$

$$\frac{1}{s} \geq U_A \bullet \mathbf{v}\mathbf{v}^T$$



Lower Feasibility

Similarly:

$$\Phi_{\ell'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi_{\ell}(A)$$

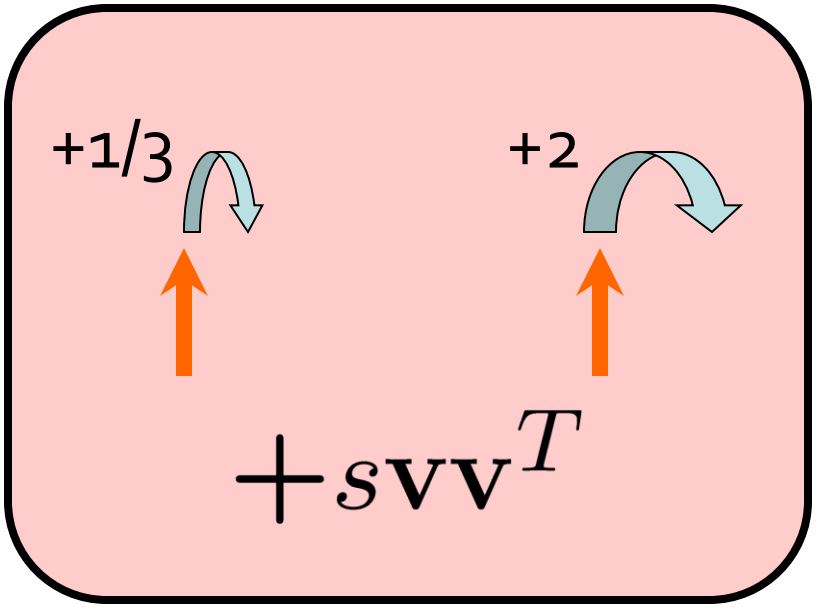
$$\Longleftrightarrow$$

$$\frac{1}{s} \leq \mathbf{v}^T \left(\frac{(A - \ell' I)^{-2}}{\Phi_{\ell'}(A) - \Phi_{\ell}(A)} - (A - \ell' I)^{-1} \right) \mathbf{v}$$

$$\boxed{\frac{1}{s} \leq L_A \bullet \mathbf{v}\mathbf{v}^T}$$

Goal

Show that we can always add some vector while respecting *both* barriers.



Both Barriers

Goal

There is always a vector with

$$U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

Both Barriers

can add

must add

There is always a vector with

$$U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

Both Barriers

can add

must add

There is always a vector with

$$U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

Then, can squeeze scaling factor in between:

$$U_A \bullet \mathbf{v}\mathbf{v}^T \leq \frac{1}{s} \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

Average over all $\boldsymbol{v_e}$

Goal

$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

Average over all $\boldsymbol{v_e}$

Goal

$$\sum_{\mathbf{v}} U_A \bullet \mathbf{v}\mathbf{v}^T \leq \sum_{\mathbf{v}} L_A \bullet \mathbf{v}\mathbf{v}^T$$

Average over all v_e

Goal

$$\sum_{\mathbf{v}} U_A \bullet \mathbf{v}\mathbf{v}^T \leq \sum_{\mathbf{v}} L_A \bullet \mathbf{v}\mathbf{v}^T$$

$$\sum_{\mathbf{v}\in\{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} = U_A \bullet \left(\sum_e v_e v_e^T\right)$$

$$= U_A \bullet I$$

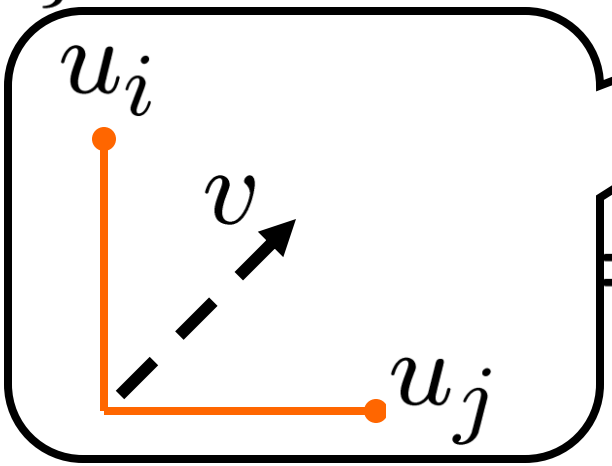
$$= \mathsf{Tr}(U_A).$$

Average over all $\boldsymbol{v_e}$

Goal

$$\sum_{\mathbf{v}} U_A \bullet \mathbf{v}\mathbf{v}^T \leq \sum_{\mathbf{v}} L_A \bullet \mathbf{v}\mathbf{v}^T$$

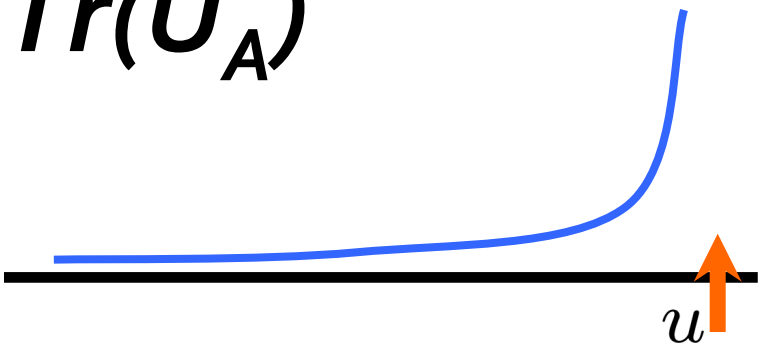
$$\sum_{\mathbf{v}\in\{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} = U_A \bullet \left(\sum_e v_e v_e^T\right)$$



$$= U_A \bullet I$$

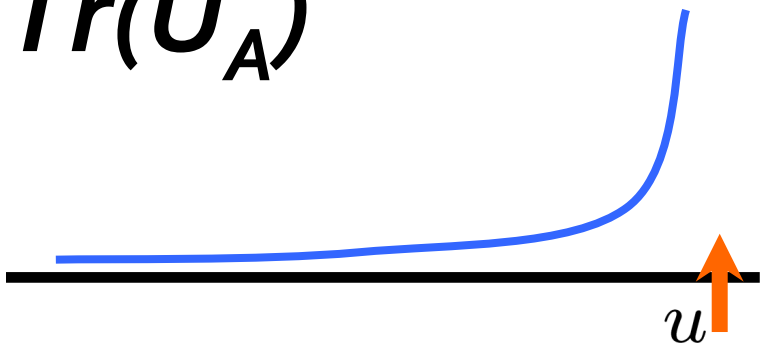
$$= \mathrm{Tr}(U_A).$$

Bounding $Tr(U_A)$



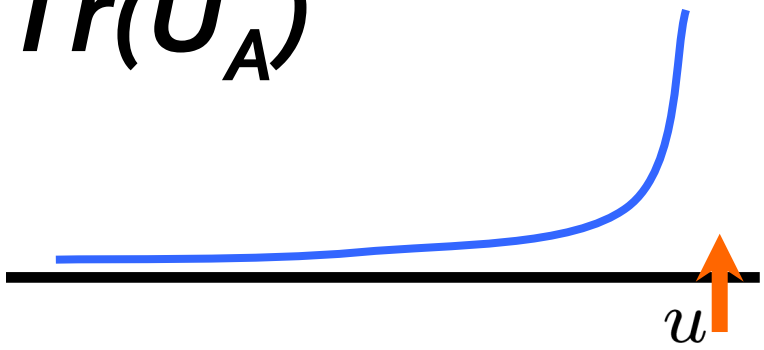
$$\frac{\mathrm{Tr}(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + \mathrm{Tr}(u'I - A)^{-1}$$

Bounding $Tr(U_A)$



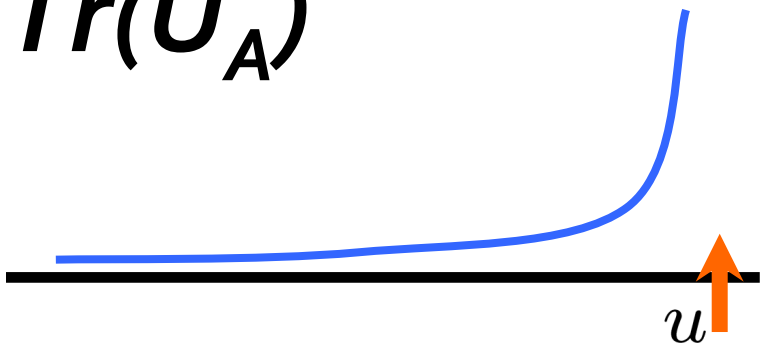
$$\frac{\mathrm{Tr}(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + \Phi^{u'}(A)$$

Bounding $Tr(U_A)$



$$\frac{\mathrm{Tr}(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + \boxed{\leq \Phi^{u}(A)}$$

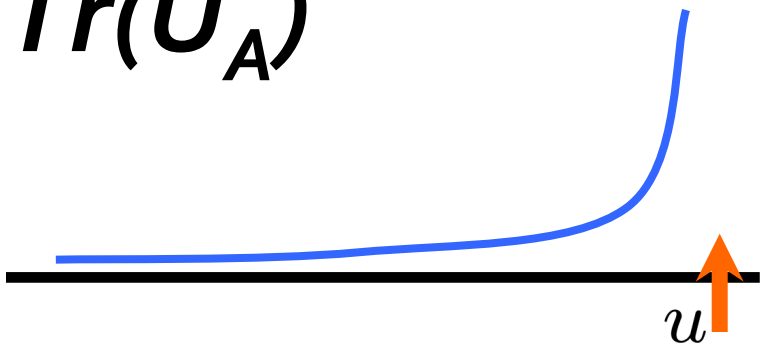
Bounding $Tr(U_A)$



$$\frac{\mathrm{Tr}(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + \leq 1$$

induction

Bounding $Tr(U_A)$

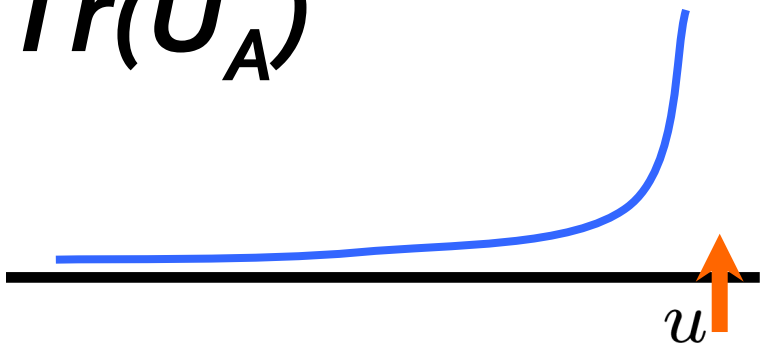


$$\frac{-\frac{\partial}{\partial u'}\Phi^{u'}(A)}{\Phi^{u}(A)-\Phi^{u'}(A)}+\leq 1$$

induction

(Recall $\Phi^{u}(A)=\mathrm{Tr}(uI-A)^{-1}$.)

Bounding $Tr(U_A)$



$$-\frac{\partial}{\partial u'}\Phi^{u'}(A)$$

$$\geq \delta_u\left(-\frac{\partial}{\partial u'}\Phi^{u'}(A)\right)$$

convexity

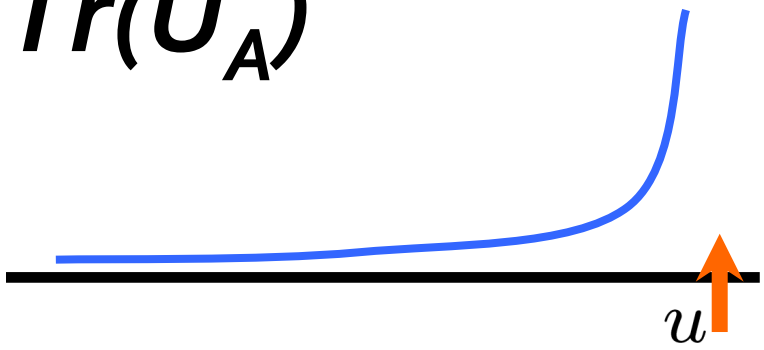
$$+$$

$$\leq 1$$

induction

$$\text{(Recall } \Phi^u(A) = \mathrm{Tr}(uI - A)^{-1}.\text{)}$$

Bounding $Tr(U_A)$



$$-\frac{\partial}{\partial u'}\Phi^{u'}(A)$$

$$\geq \delta_u\left(-\frac{\partial}{\partial u'}\Phi^{u'}(A)\right)$$

convexity

$$+$$

$$\leq 1$$

induction

$$\mathrm{Tr}(U_A) \leq \frac{1}{\delta_u} + 1$$

Taking Averages

Goal

$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

$$\sum_{\mathbf{v} \in \{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \leq \frac{1}{\delta_u} + 1.$$

Taking Averages

$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

$$\sum_{\mathbf{v} \in \{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \leq \frac{1}{\delta_u} + 1.$$

$$\sum_{\mathbf{v} \in \{v_e\}} L_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \geq \frac{1}{\delta_\ell} - 1.$$

Taking Averages

$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

$$\sum_{\mathbf{v} \in \{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \leq \frac{1}{2} + 1.$$

=3/2

$$\sum_{\mathbf{v} \in \{v_e\}} L_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \geq \frac{1}{\delta_\ell} - 1.$$

Taking Averages

$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

$$\sum_{\mathbf{v} \in \{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathrm{T}} \leq \frac{1}{2} + 1.$$

=3/2

$$\sum_{\mathbf{v} \in \{v_e\}} L_A \bullet \mathbf{v}\mathbf{v}^{\mathrm{T}} \geq \frac{1}{1/3} - 1.$$

2

Taking Averages

$$\exists \mathbf{v}, U_A \bullet \mathbf{v}\mathbf{v}^T \leq L_A \bullet \mathbf{v}\mathbf{v}^T$$

✓

$$\sum_{\mathbf{v} \in \{v_e\}} U_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \leq \frac{1}{2} + 1.$$

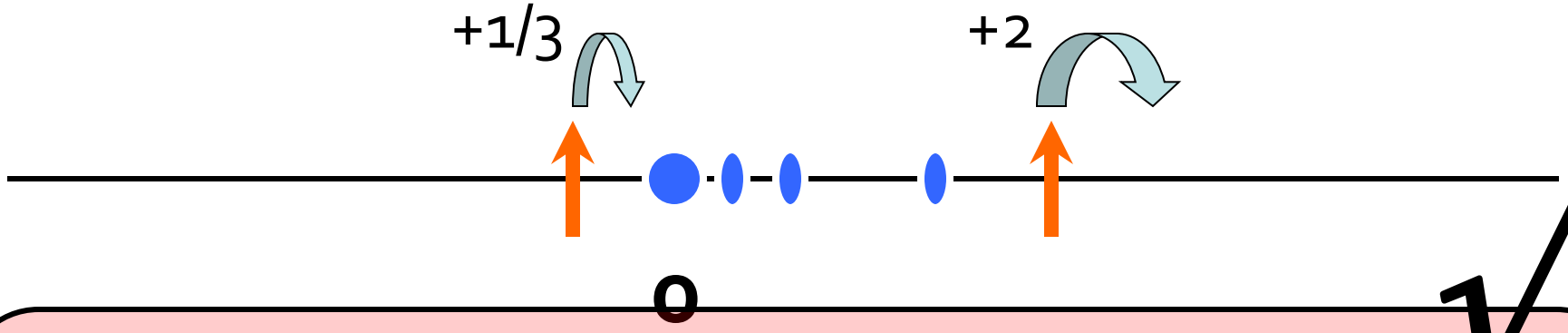
=3/2

$$\sum_{\mathbf{v} \in \{v_e\}} L_A \bullet \mathbf{v}\mathbf{v}^{\mathbf{T}} \geq \frac{1}{1/3} - 1.$$

2

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}$$



Lemma.

can always choose $+s\mathbf{v}\mathbf{v}^T$

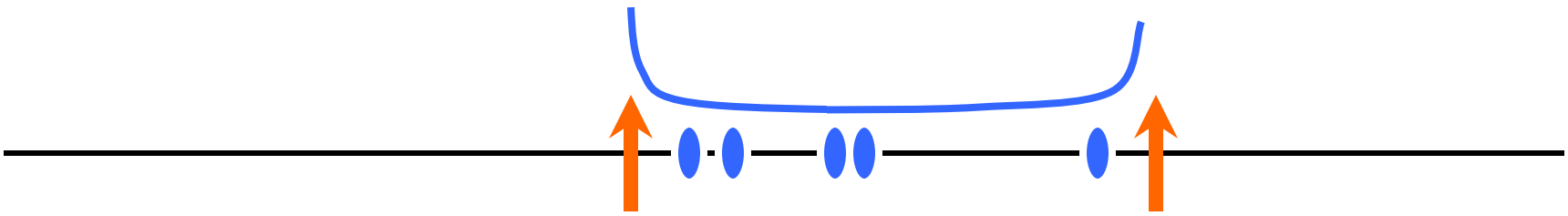
so that potentials do not increase.

$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$$



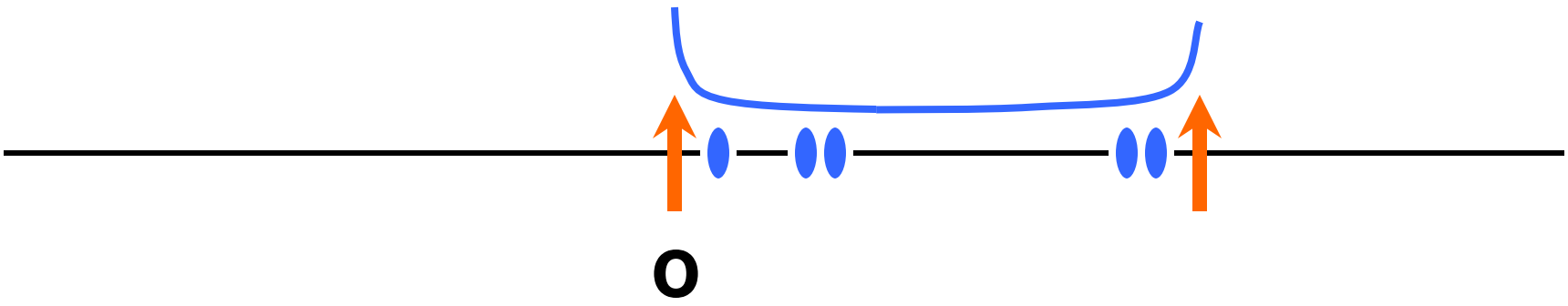
o

$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

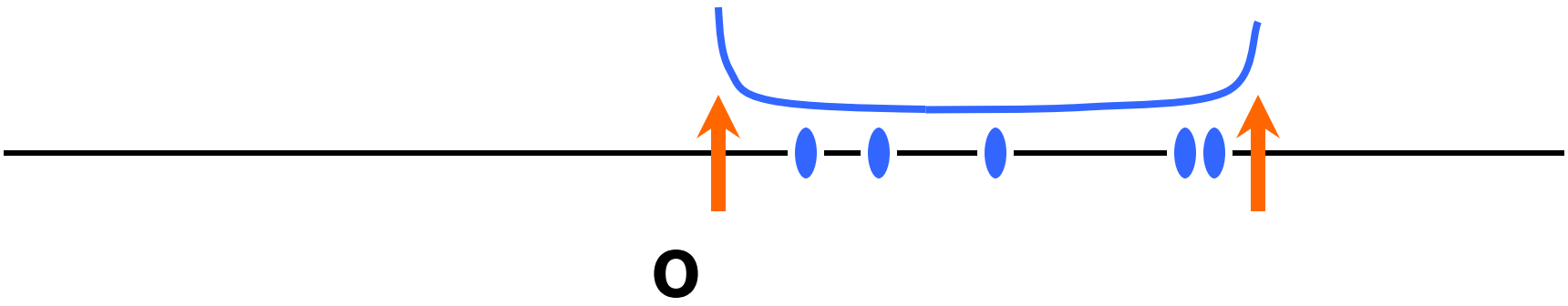


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

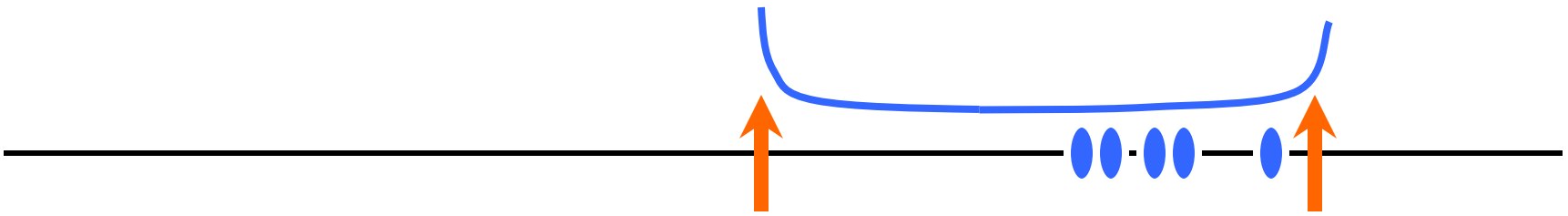


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$



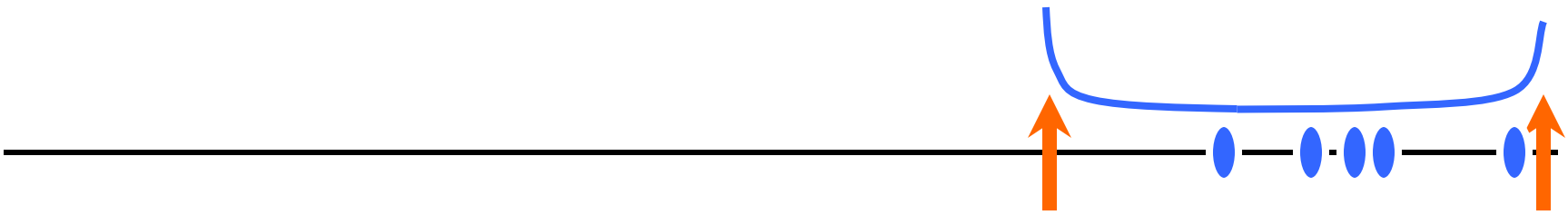
o

$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$



$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$



13-approximation with 6n vectors.

□

Done!

Spectral Sparsification Theorem: $\surd$

Given $\sum_{i \le m} v_i v_i^T = I_n$ there are $s_i \geq 0$ with:

- $I \preceq \sum_i s_i v_i v_i^T \preceq 13 \cdot I$
- $\mathrm{supp}(s) \leq 6n.$

Nearly Optimal bound

Fixing dn steps and tightening parameters gives

$$\kappa = \frac{(\sqrt{d}+1)^2}{(\sqrt{d}-1)^2}$$

(zeros of Laguerre polynomials).

This is within a factor of 2 of the optimal Ramanujan Bound [LPS, Alon-Boppana].



Why does this work?

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} \leq 1$$

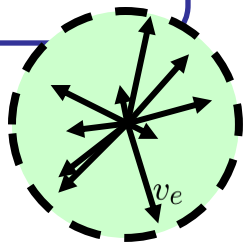
Efficiently bounds $\lambda(A)$

Why does this work?

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} \leq 1$$

Efficiently bounds $\lambda(A)$

$$= \mathbb{E}_{v_e} v_e^T (uI - A)^{-1} v_e$$



$$I = \sum_e v_e v_e^T$$

Why does this work?

$$\Phi^u(A) = \mathrm{Tr}(uI - A)^{-1} \leq 1$$

Efficiently bounds $\lambda(A)$

$$= \mathbb{E}_{v_e} v_e^T (uI - A)^{-1} v_e$$

guarantees good vector

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^u(A)$$

Why does this work?

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guarantees good vector

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^u(A)$$

Major Themes

- Electrical model of **interlacing** is useful
- Can use barrier potential to **iteratively** construct matrices with desired spectra
- Analysis of progress is **greedy / local**
- Requires **fractional weights** on vectors

Extension to Arbitrary Rank

Theorem [de Carli Silva, Harvey, Sato'11]

Given **matrices** $\sum_{i \le m} V_i = V$ there are $s_i \ge 0$ with:

- $(1-\epsilon)V \preceq \sum_i s_i V_i \preceq (1+\epsilon)V$
- $\mathrm{supp}(s) \le 4\mathrm{rank}(V)/\epsilon^2$.

Proof of Higher Rank

Recall upper feasibility condition:

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^{u}(A)$$

$$\Longleftrightarrow$$

$$\frac{1}{s} \geq \frac{(u'I - A)^{-2} \bullet \mathbf{v}\mathbf{v}^T}{\Phi^{u}(A) - \Phi^{u'}(A)} + (u'I - A)^{-1} \bullet \mathbf{v}\mathbf{v}^T$$

Used:

$$(A + \mathbf{v}\mathbf{v}^T)^{-1} = A^{-1} - \frac{A^{-1}\mathbf{v}\mathbf{v}^T A^{-1}}{1 + \mathbf{v}^T A^{-1}\mathbf{v}}$$

Proof of Higher Rank

[deCS,H,S] For positive semidefinite **X**:

$$\Phi^{u'}(A + s\mathbf{X}) \leq \Phi^{u}(A)$$

$$\Longleftrightarrow$$

$$\frac{1}{s} \geq \frac{(u'I - A)^{-2} \bullet \mathbf{X}}{\Phi^{u}(A) - \Phi^{u'}(A)} + (u'I - A)^{-1} \bullet \mathbf{X}$$

Uses:

$$(A + RR^T)^{-1} = A^{-1} - A^{-1}R(I + R^T A^{-1} R)^{-1} RA^{-1}$$

Extension to Arbitrary Rank

Theorem [de Carli Silva, Harvey, Sato'11]

Given **matrices** $\sum_{i \le m} V_i = V$ there are $s_i \ge 0$ with:

- $(1-\epsilon)V \preceq \sum_i s_i V_i \preceq (1+\epsilon)V$
- $\mathrm{supp}(s) \le 4\mathrm{rank}(V)/\epsilon^2$.

Applications: sparse SDP solutions, sparsifiers with costs

Some Other Uses of the Method

Sharp constructive form of Bourgain-Tzafriri Restricted Invertibility [Spielman-S'09].

Counting Contact Points of Convex Bodies [S'09]

Subgraph Sparsification [Kolla-Makarychev-Saberi-Teng'10]

Column Subset Selection [Boutsidis-Drineas-Magdon-Ismail'11]

Future Work

- The Ramanujan bound
- Unweighted sparsifiers for $\mathbf{K_n}$
- A faster algorithm (currently $O(n^5)$)
- Directed graphs? (must be weaker notion)
- The Kadison-Singer Conjecture

Tomorrow

Rerandomizing the proof:

improvement of Rudelson's O(nlogn)

for nice distributions [S-Vershynin'11]

