Popular Matchings

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[Joint work with Chien-Chung Huang]

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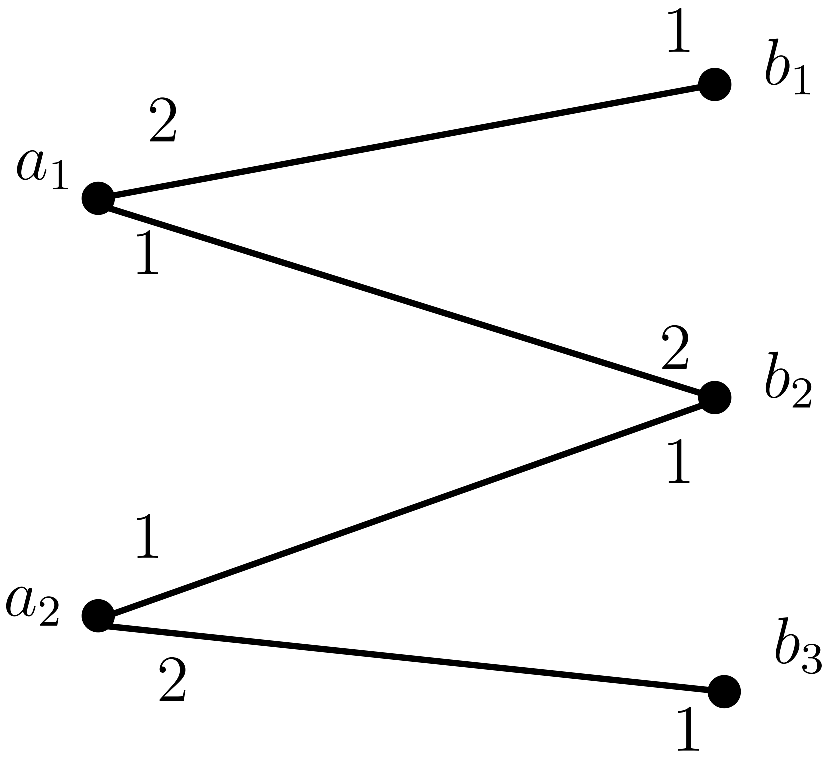

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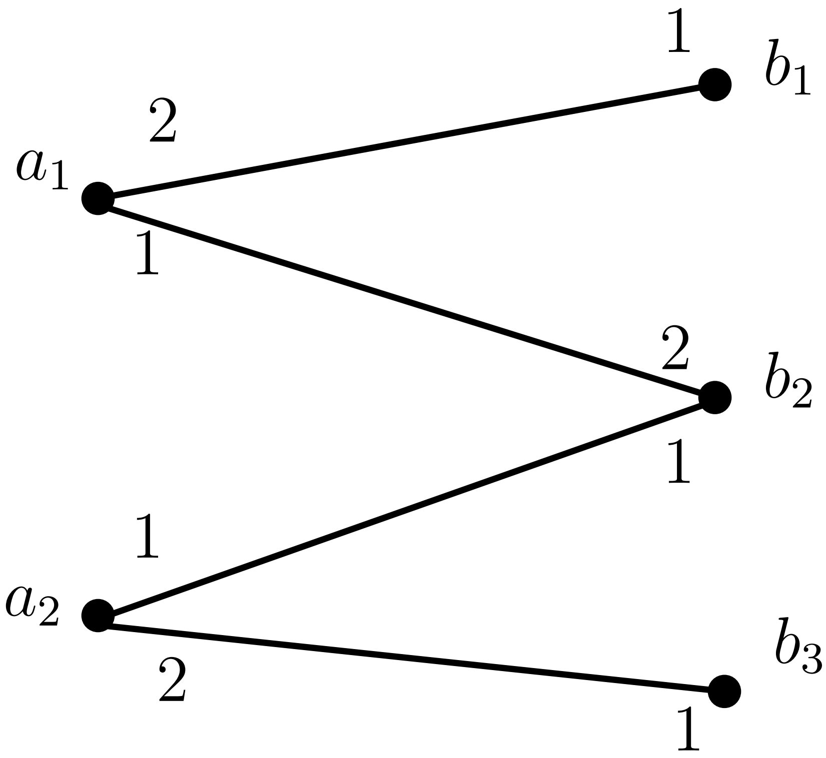

- $\mathcal{A}$: set of students; $\mathcal{B}$: set of advisers.

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- Problem: compute a “good” matching in G.
  - every vertex is *selfish*
    - u wants to be matched to its best ranked neighbor who is willing to be matched to u.

Optimal matchings

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  - The following property should hold for every u:
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  - Such a matching M is *stable*.

Stable Matchings

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    - v is unmatched or prefers u to $M(v)$.

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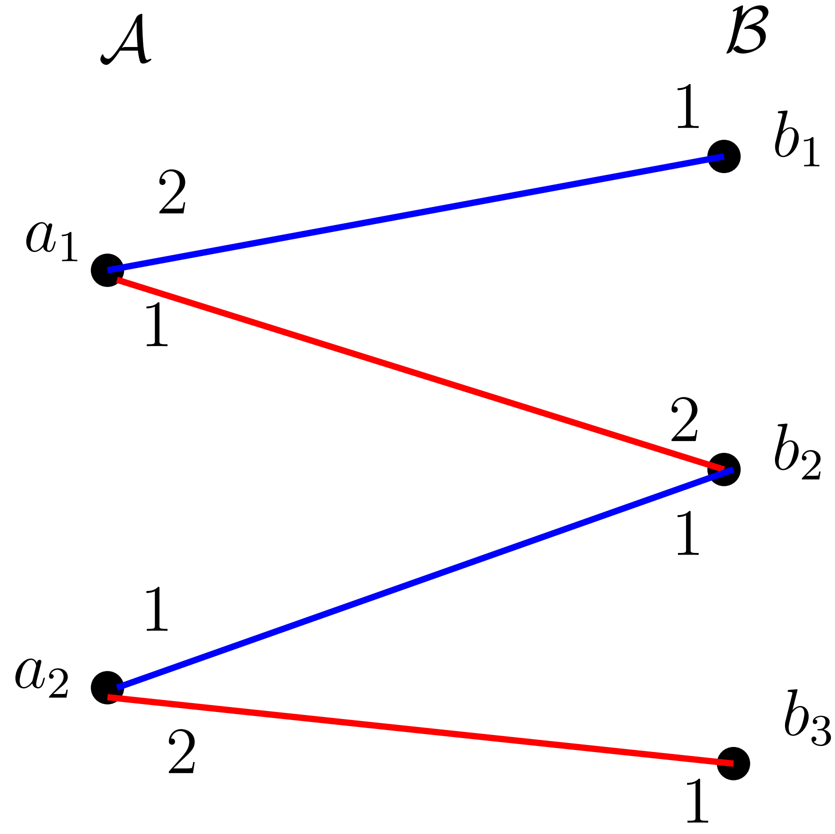

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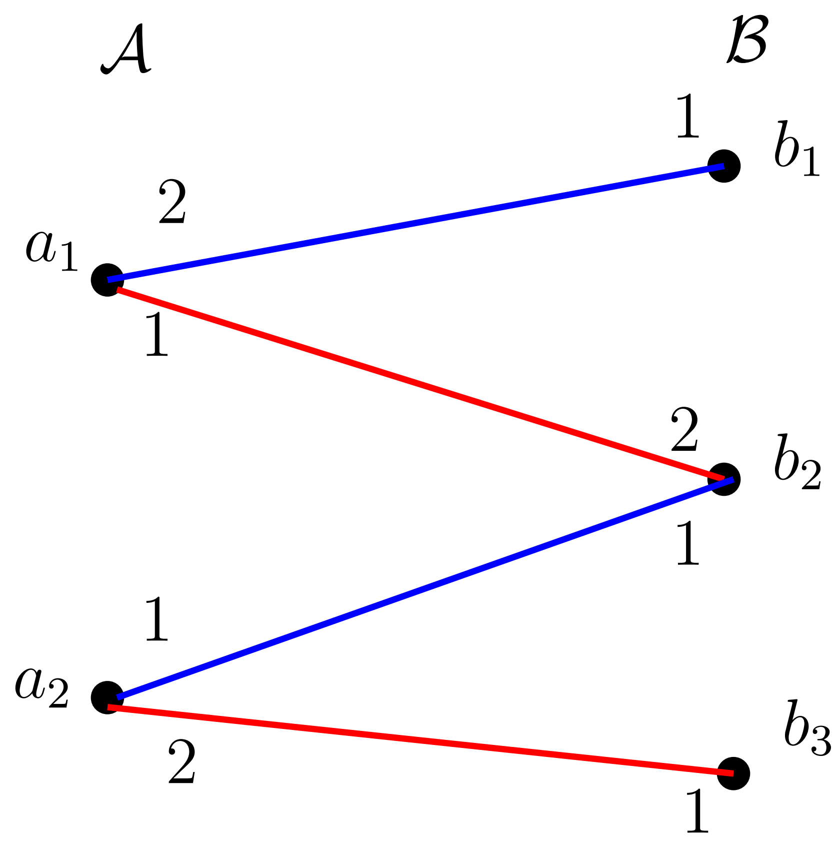

- The blue matching is stable while the red is not.

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- *Gale-Shapley algorithm*: Men (vertices of A) propose and Women (those in B) dispose.

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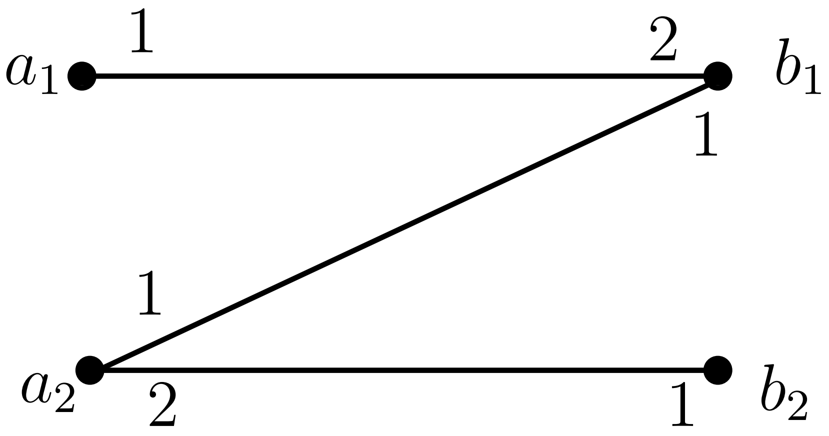

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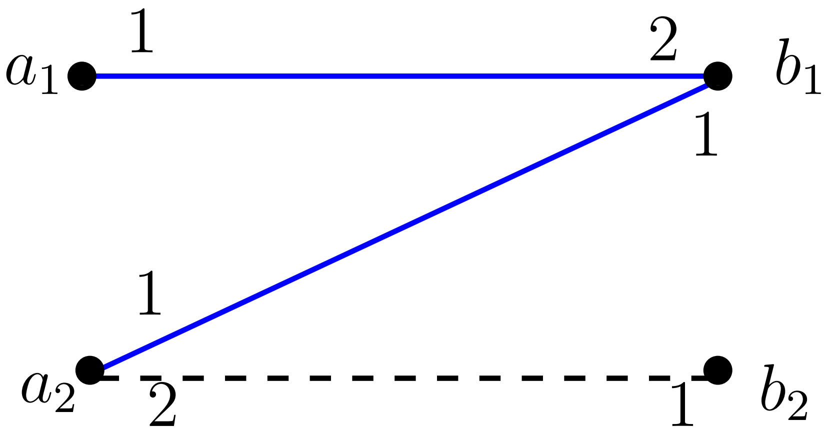
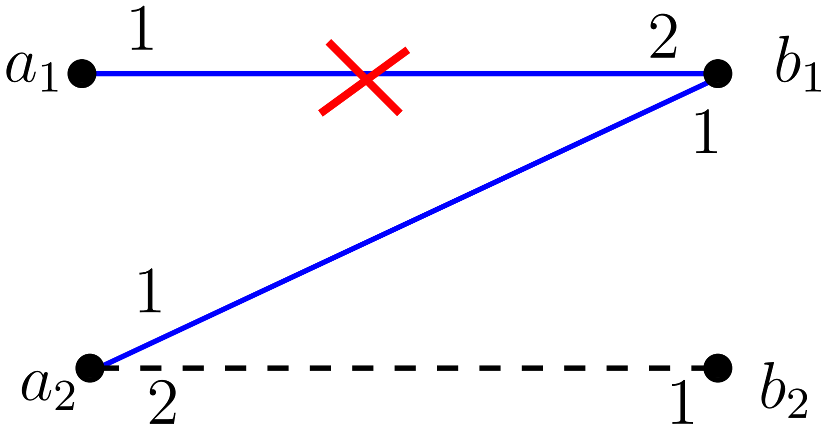


- a_1 proposes to his top neighbor b_1; so does a_2.

Gale-Shapley algorithm for stable matchings

- b_1 rejects a_1 and accepts a_2.



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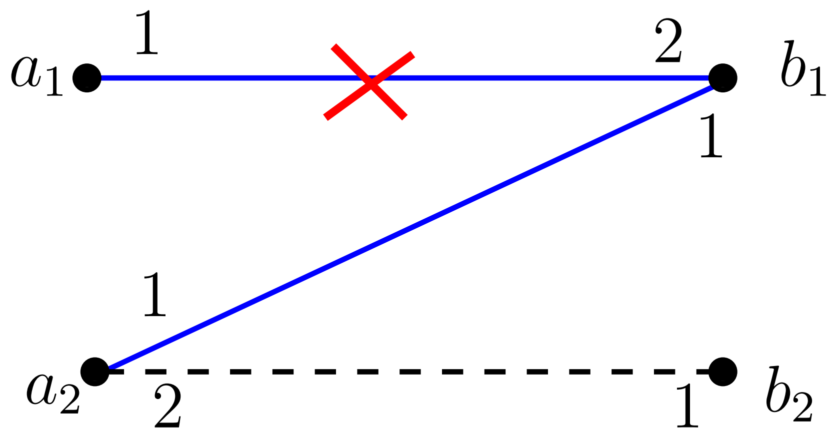

- The algorithm terminates when every man is either rejected by all his nbrs or gets matched to some nbr.

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  - |stable matching| could be as low as $|M_{max}|/2$.

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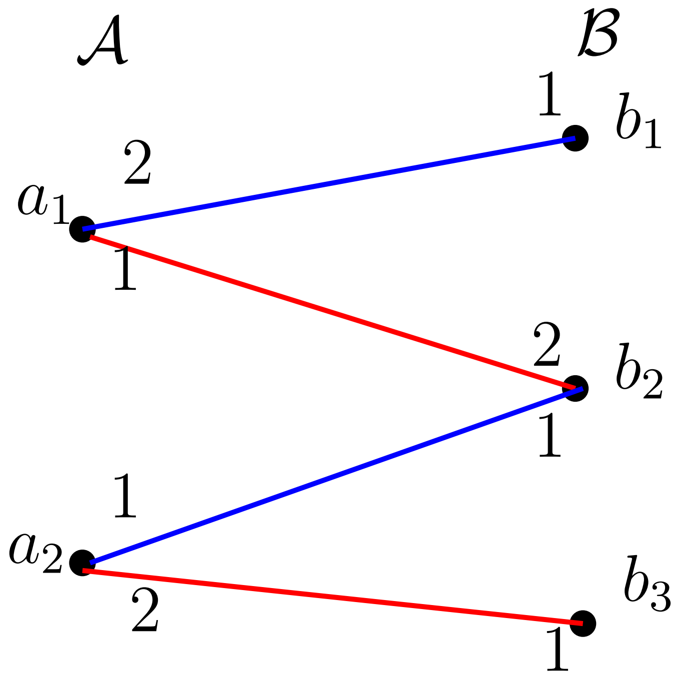
- A notion based on *popularity*:

Matching M_1 is *more popular* than matching M_2 if $|\{$vertices that prefer $M_1\}| > |\{$vertices that prefer $M_2\}|$.

An example



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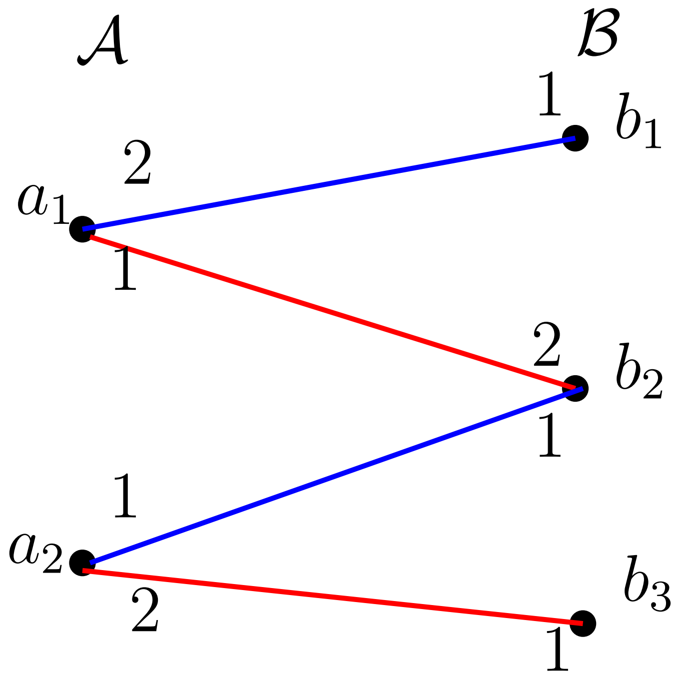

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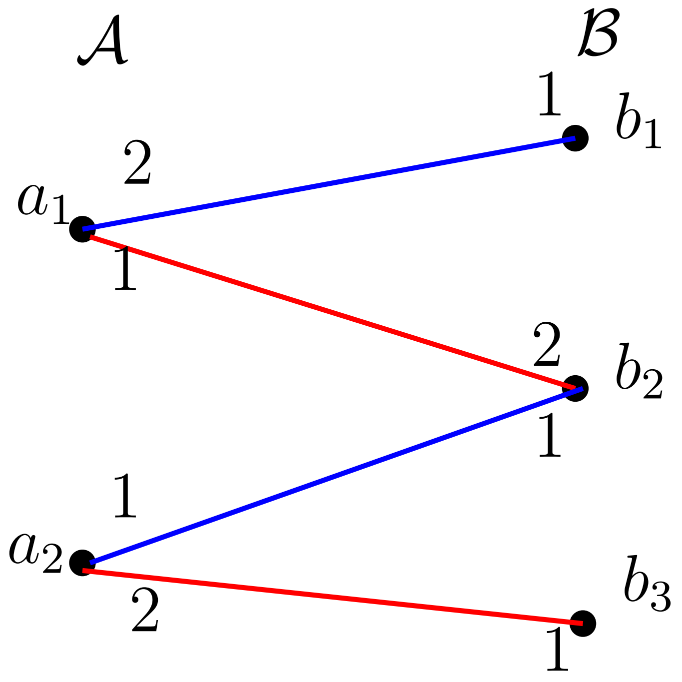

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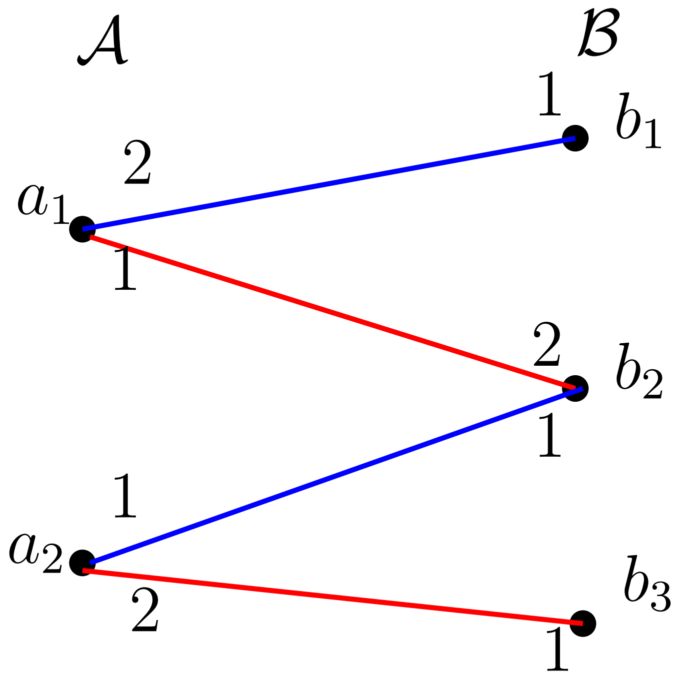

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- b_1, b_2, and a_2 prefer the blue matching
- blue matching is more popular than red matching.

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- The "*more popular than*" relation is not transitive: we can have $M_1 \succ M_2 \succ M_3 \succ M_1$.
- M is popular if there is *no* M' such that $M' \succ M$.

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    - yes, in fact, every stable matching is popular.

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  - So a stable matching is always popular.
    - In fact, it is a *minimum* size popular matching.

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  - we will show that M has to be unpopular.
- $|M| < |S|$, so $M \oplus S$ has an augmenting path p wrt M.
- *Claim:* $M \oplus p \succ M$.
- $(M \oplus p)(u) = S(u)$ if $u \in p$,
  $(M \oplus p)(u) = M(u)$ otherwise.

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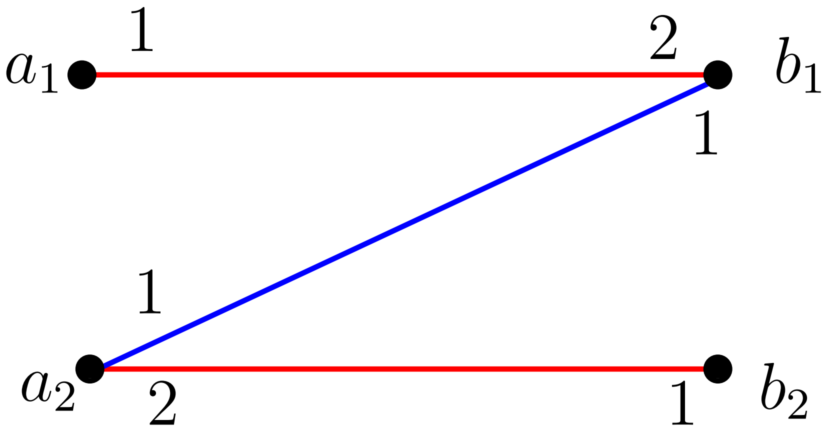
- Thus restricted to p, we have $S \succ M$.
  So $M \oplus p \succ M$.

Min vs max size popular matchings

- The blue matching is a minimum size popular matching.



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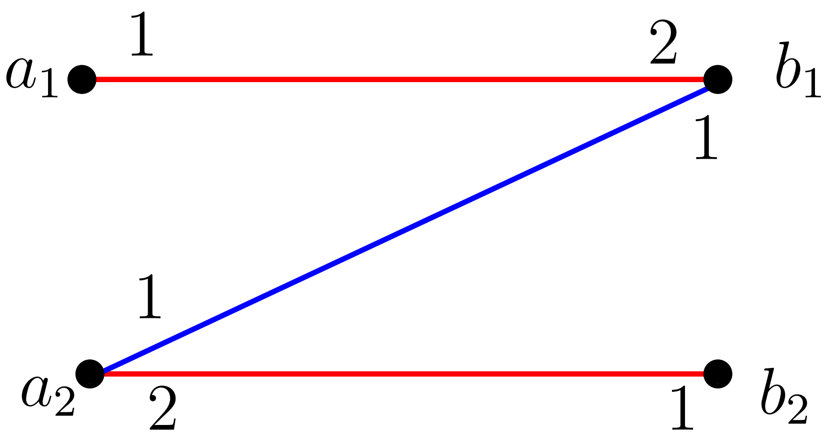

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- Characterization of a maximum size popular matching?

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- Delete from G all negative edges wrt M — call this subgraph G_M.

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  - That is, M will be a maximum size popular matching.

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    - Gale-Shapley algorithm on (L, R) yields such a matching.

Our first algorithm (Huang and K., 2011)

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- Run Gale-Shapley algorithm on (L_2, R_2): let M_2 be this matching.

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    - move unmatched women from right to left
    - start the next round.

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  - once a woman moves from right to left, she never moves back to the right side again.
- Running time: $O(m|\mathcal{B}|)$, where $m = |E|$.

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- A linear time algorithm for maximum size popular matching?

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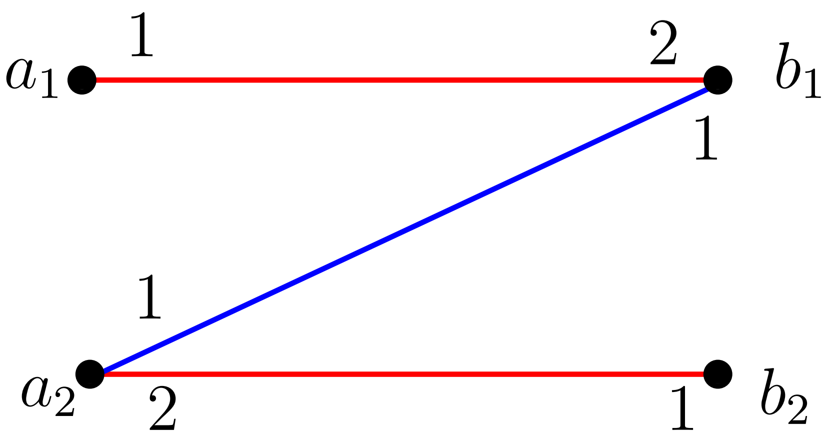

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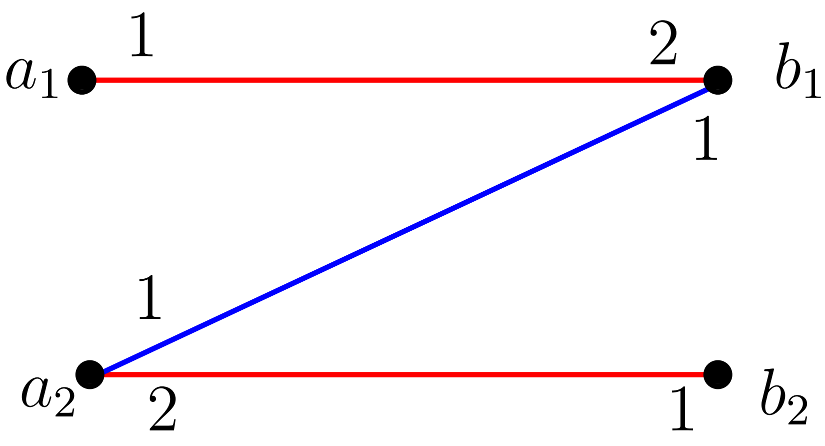

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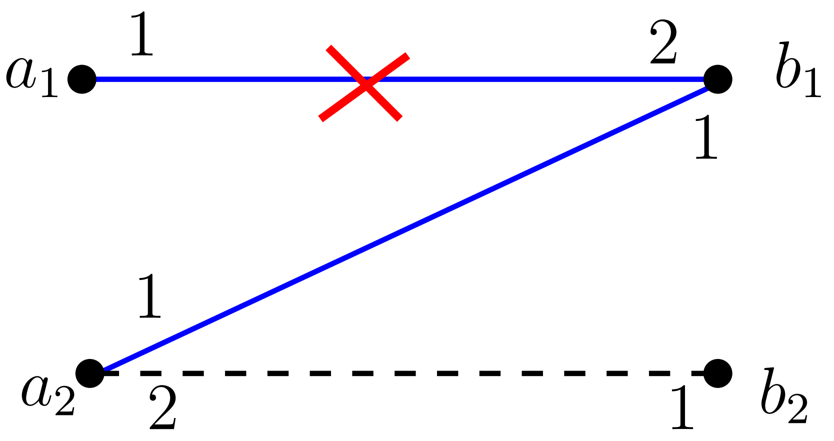
Modifying Gale-Shapley ...



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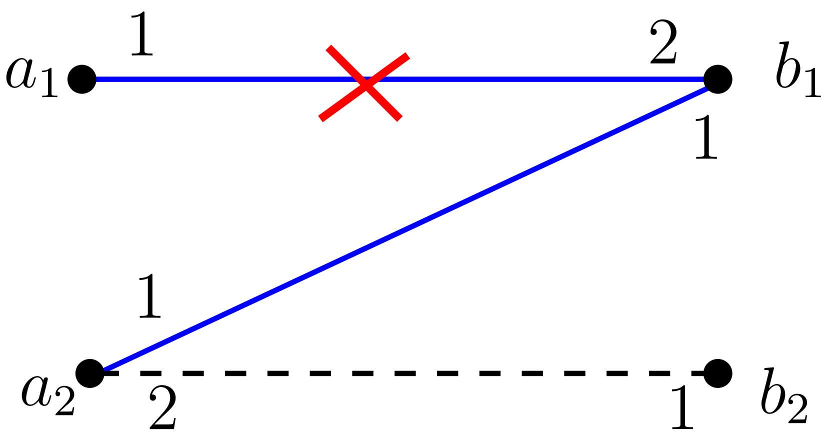

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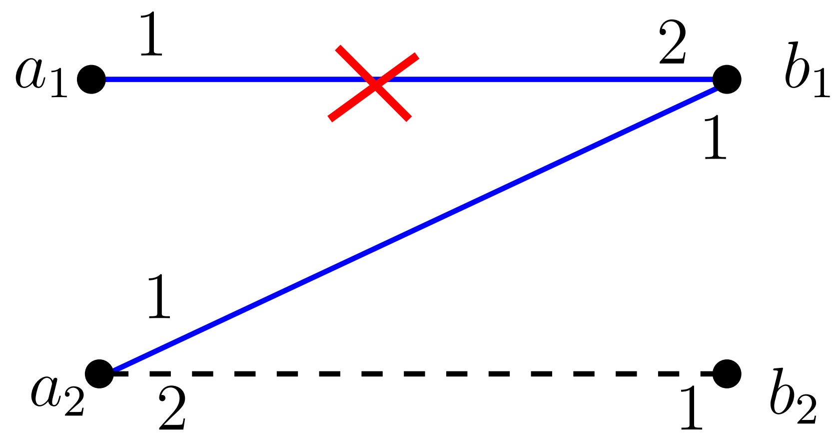

- Modify the Gale-Shapley algorithm so that a_1 gets a "second chance" to propose to b_1.
- when a_1 proposes for the *second* time to b_1, then b_1 should prefer a_1 to a_2.

Implementing this idea

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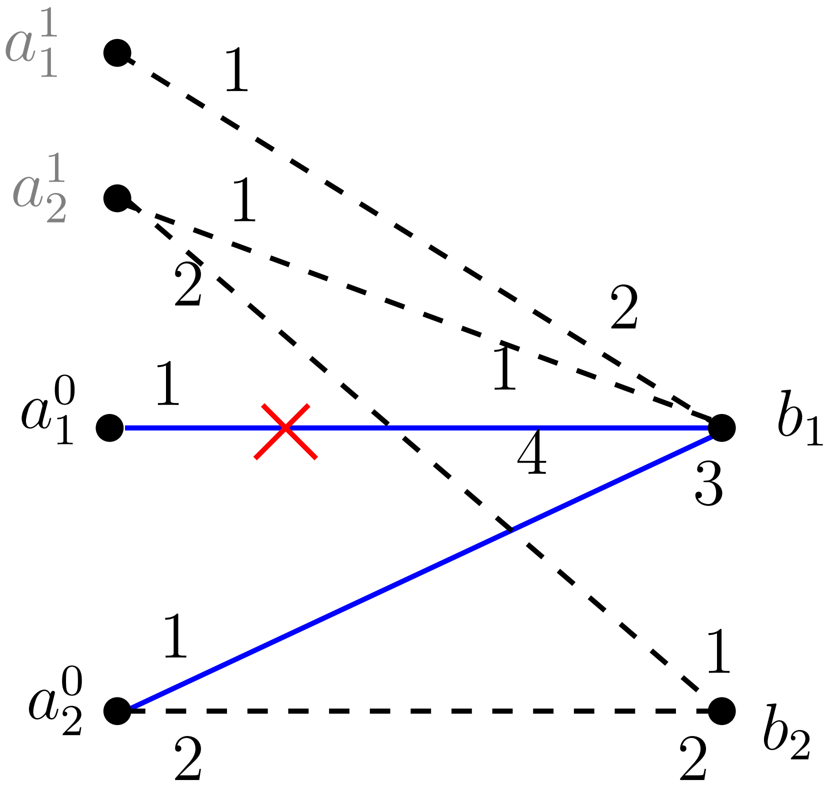
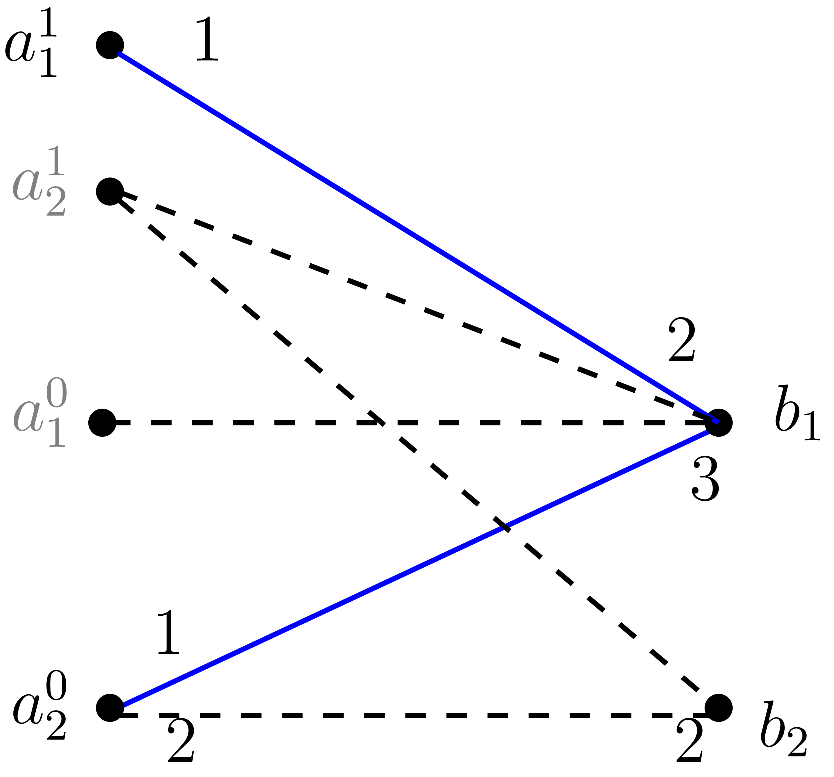


- a_1^0 is rejected by his only neighbor b_1.

In the new graph



- So a_1^1 becomes active and proposes to b_1.

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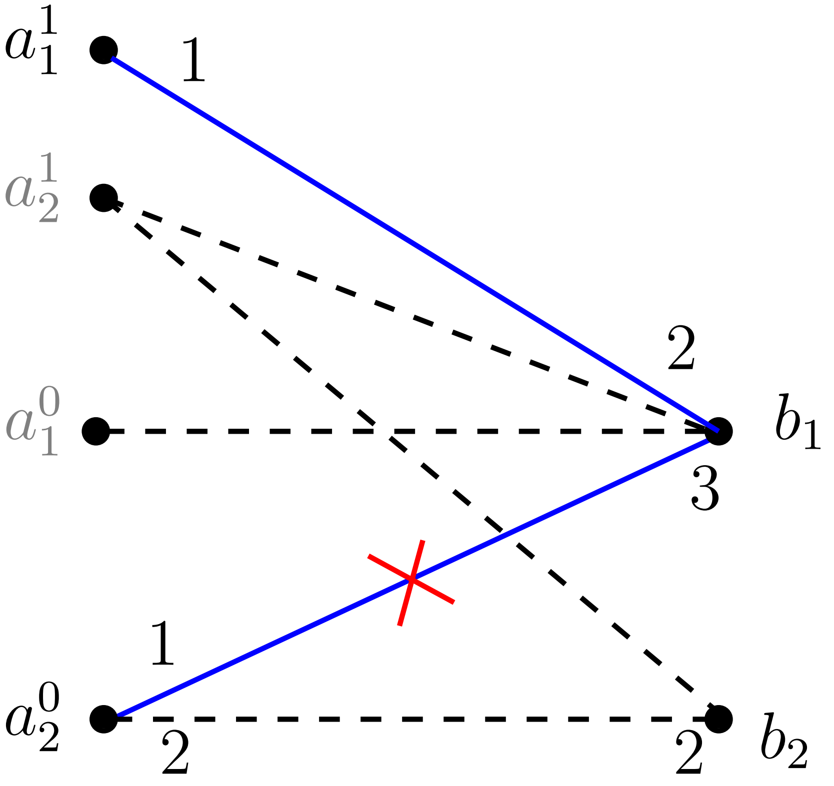

- b_1 accepts a_1^1 and rejects a_2^0.

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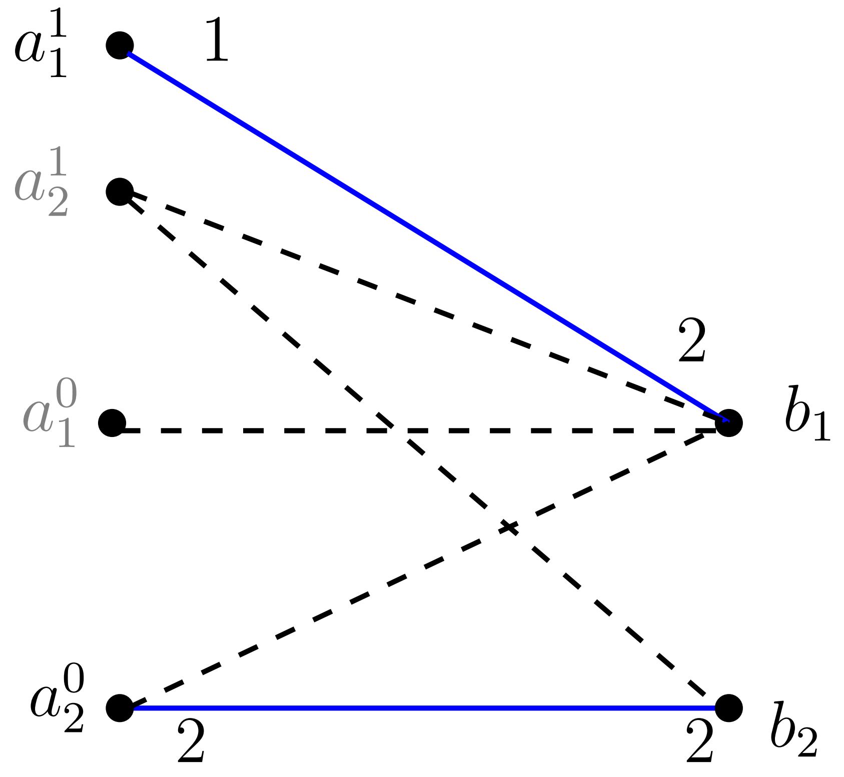
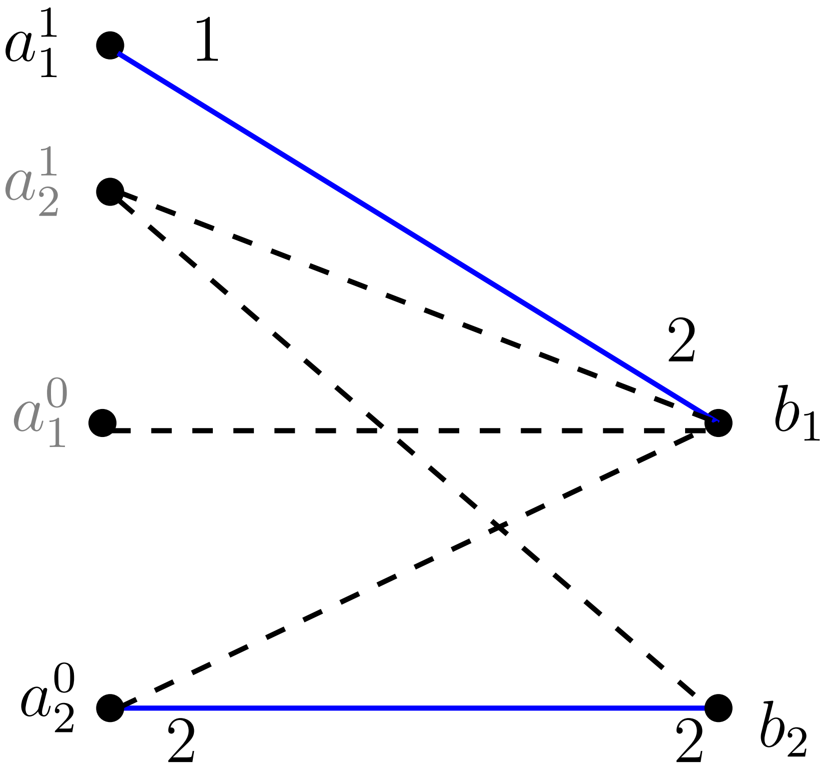


- So a_2^0 proposes to his next preferred neighbor b_2.

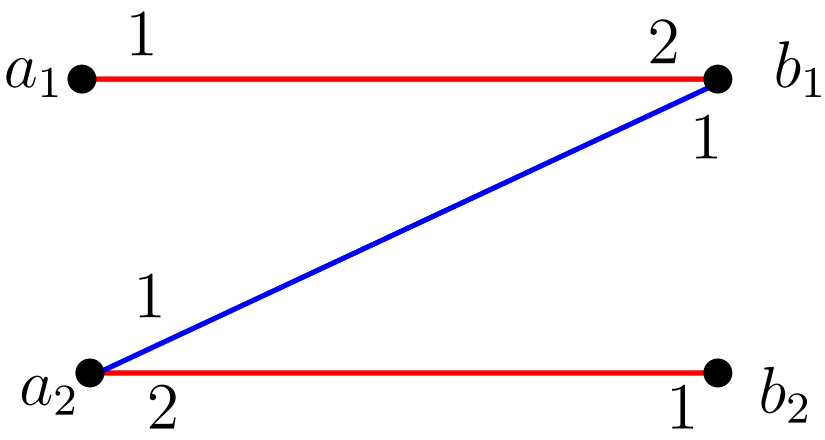
In the new graph



- The matching $\{(a_1^1, b_1),\ (a_2^0, b_2)\}$ is computed.

Back in the original graph

- Thus OPT $= \{(a_1, b_1), (a_2, b_2)\}$, the red matching, is found.



A linear time algorithm (K., 2012)

- Let $\tilde{G}_2$ be the graph on $\tilde{A}_2 \cup \mathcal{B}$ where $\tilde{A}_2$ consists of two copies a^0 and a^1 of each $a \in \mathcal{A}$.

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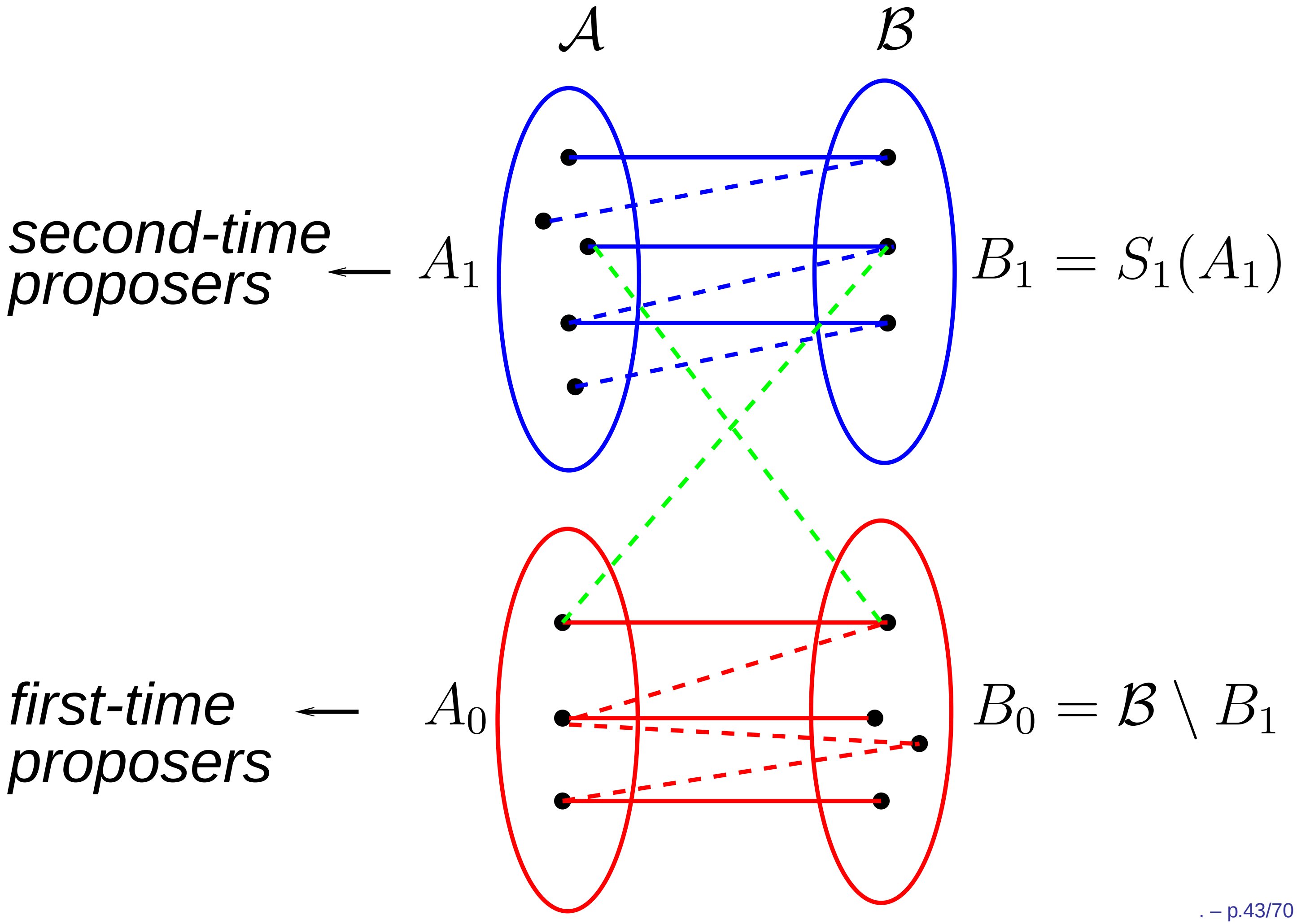
  - Running time is $O(m+n)$, which is $O(m)$.

Properties of the output matching S_1

- $S_1 \subseteq (A_0 \times B_0) \cup (A_1 \times B_1)$.



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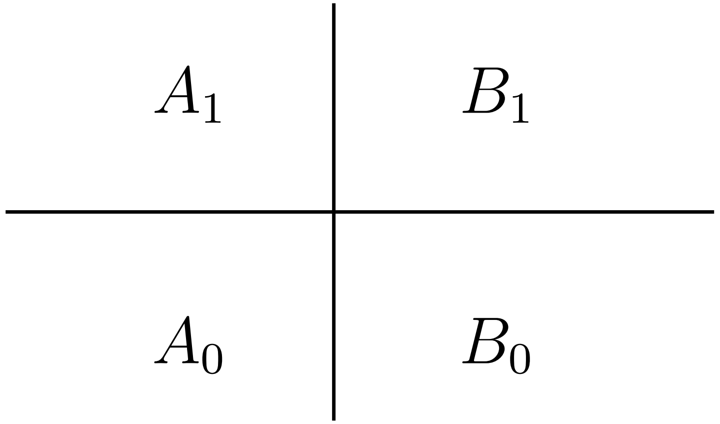

- Any blocking edge to S_1 has to be in $A_0 \times B_1$.

Partition of $\mathcal{A}$ and $\mathcal{B}$

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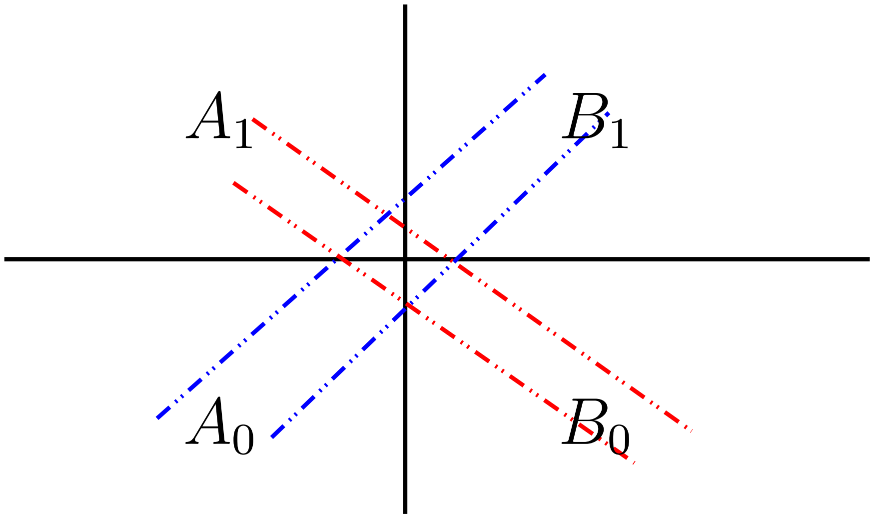

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- What about $|S_1|$ in terms of $|M_{max}|$?

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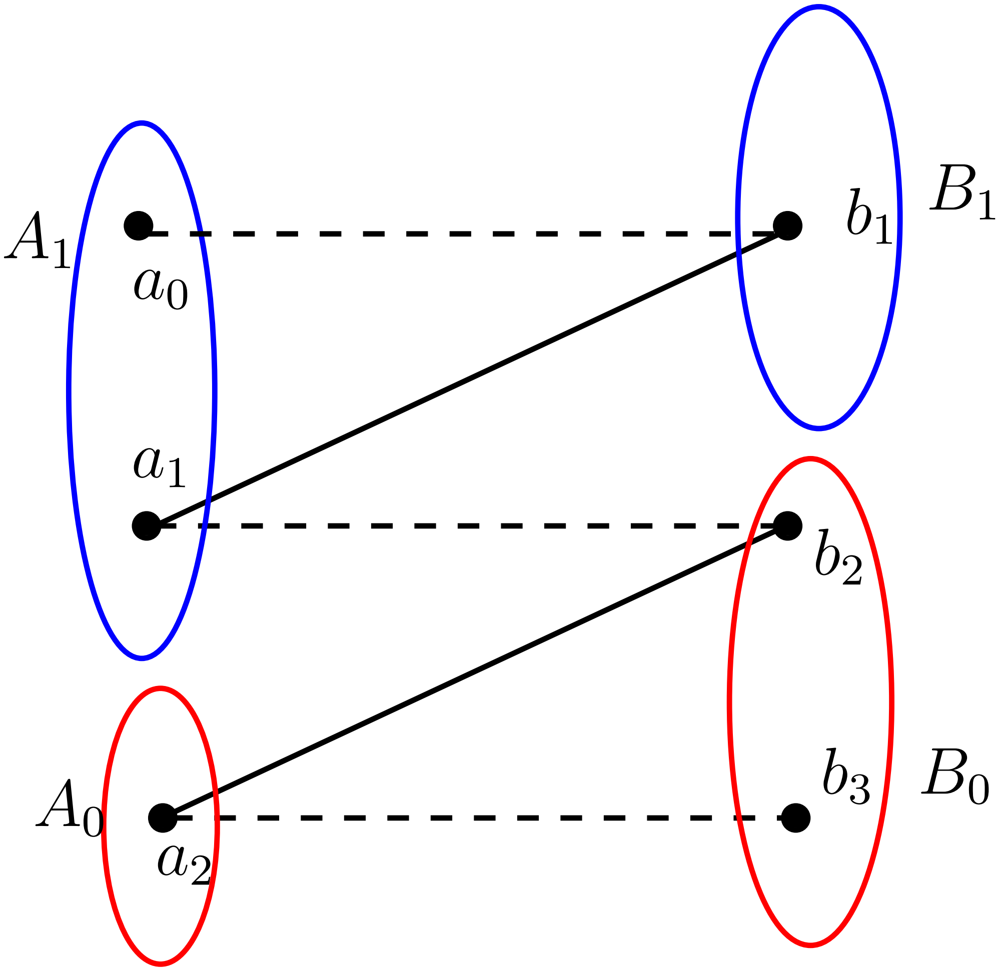
- Any augmenting path wrt S_1 in G has size ≥ 5:



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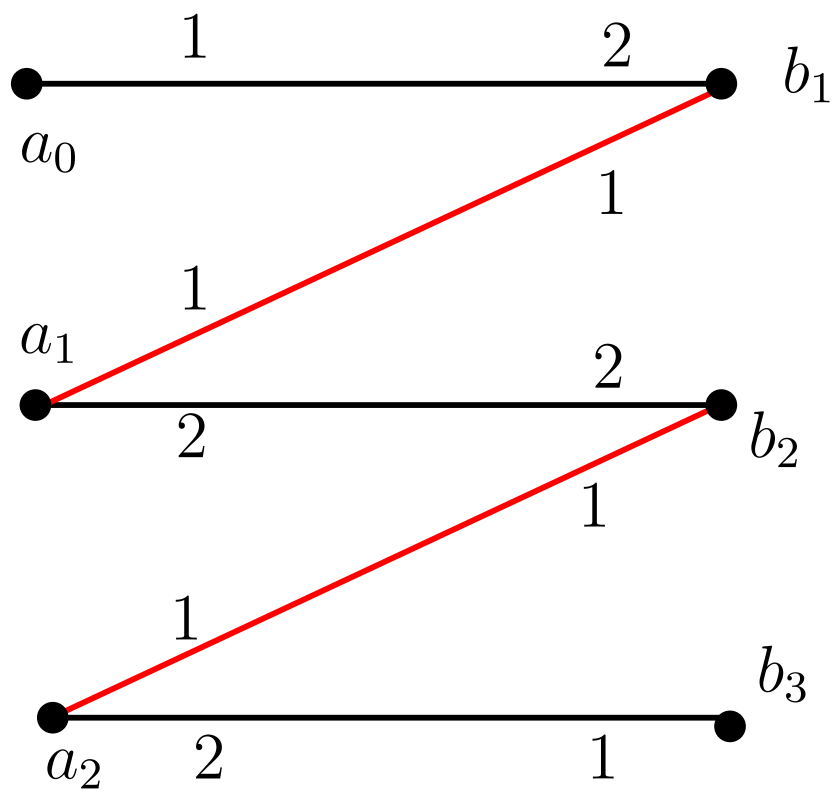
A tight example for the $2/3$ bound



- $|S_1| = 2$ while $|M_{max}| = 3$.

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$u(M) = \beta \Rightarrow$ for every matching M' we have:
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- Is there an $M^* \equiv$ a maximum cardinality matching s.t. for each max cardinality matching M: $M^* \succeq M$?

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    - For each $a \in \mathcal{A}$: at most one of $a^0, a^1, \ldots, a^{k-1}$ is active at any point.

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  - and at the bottom are level 0 neighbors.

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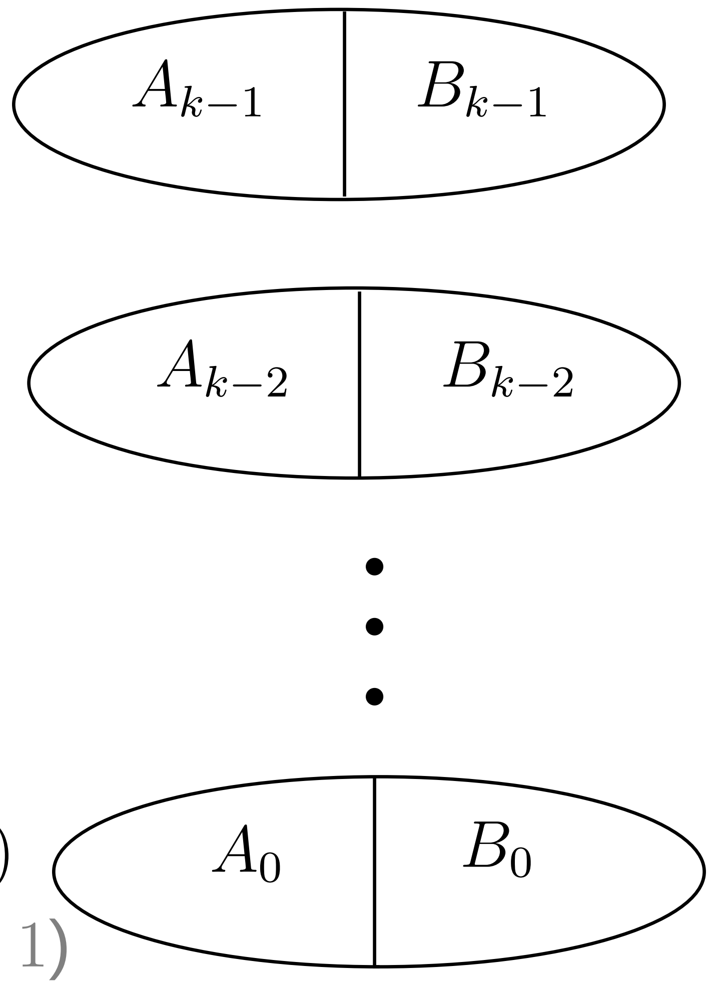
- Let S_{k-1} be the matching returned by this algorithm.

The partition of $\mathcal{A}$ and $\mathcal{B}$

- $A_i = \{a \in \mathcal{A}$ such that a is in level i at the end$\}$.

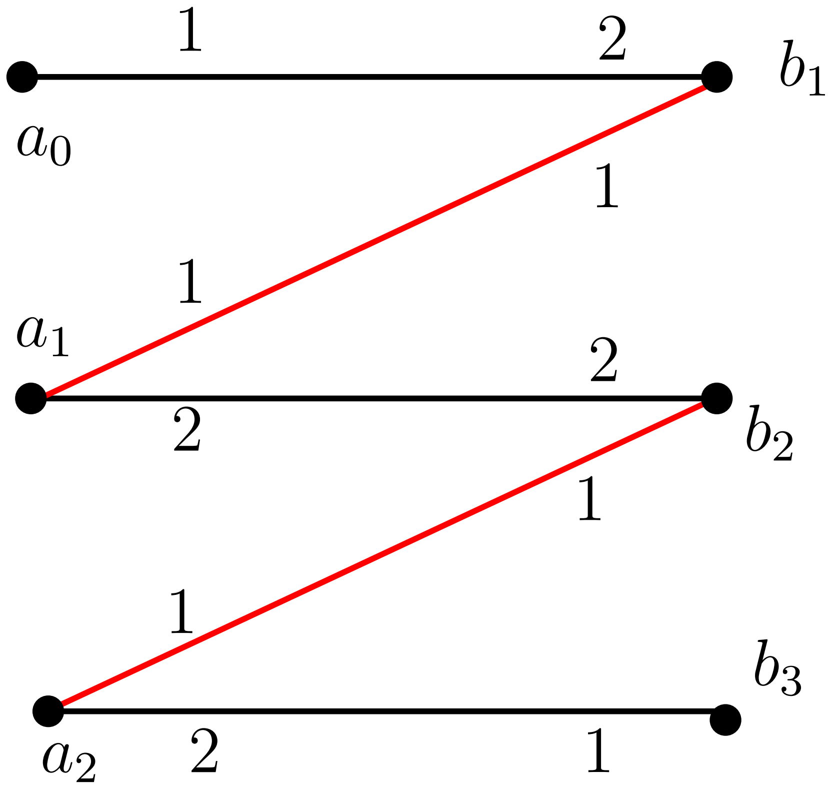


- $B_i = S_{k-1}(A_i)$

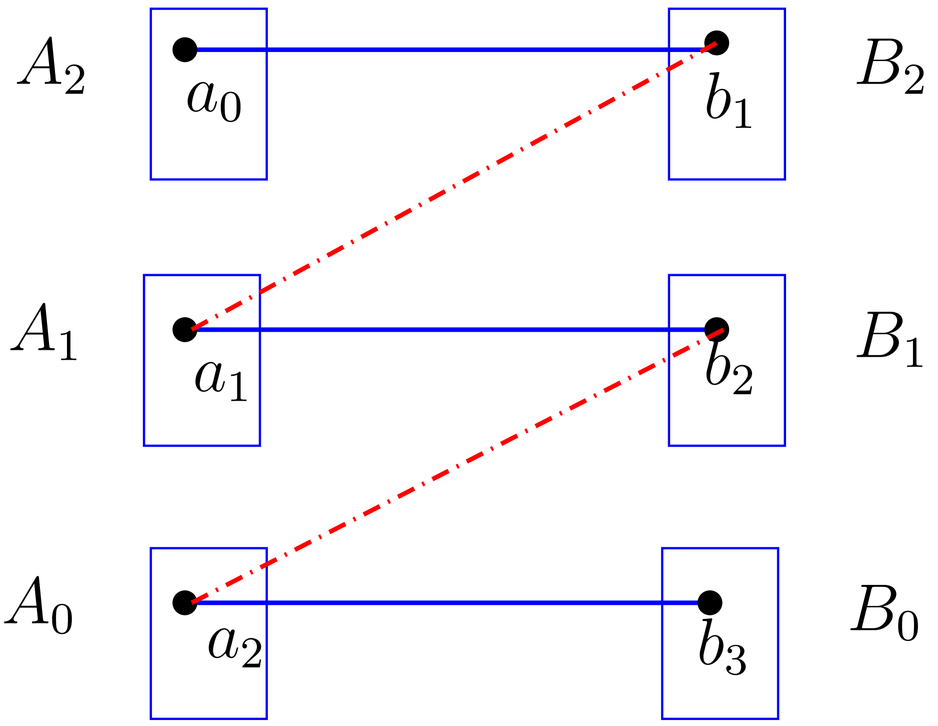
(for $1 \leq i \leq k-1$)

The 3-level algorithm

- Say we run the 3-level algorithm on our tight example for the 2-level algorithm ...



In the 3-level algorithm



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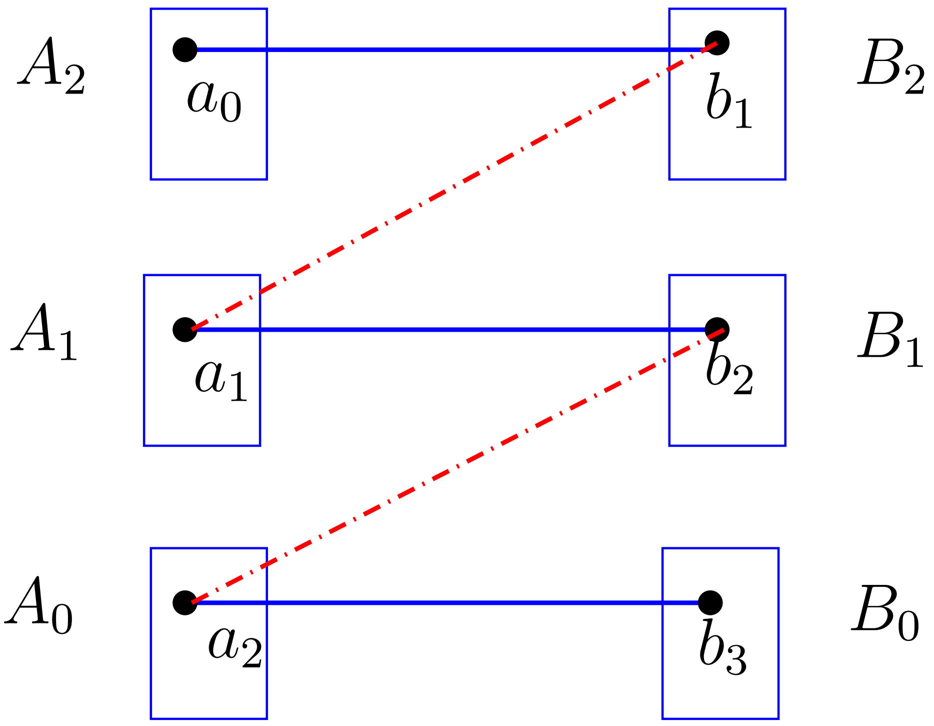
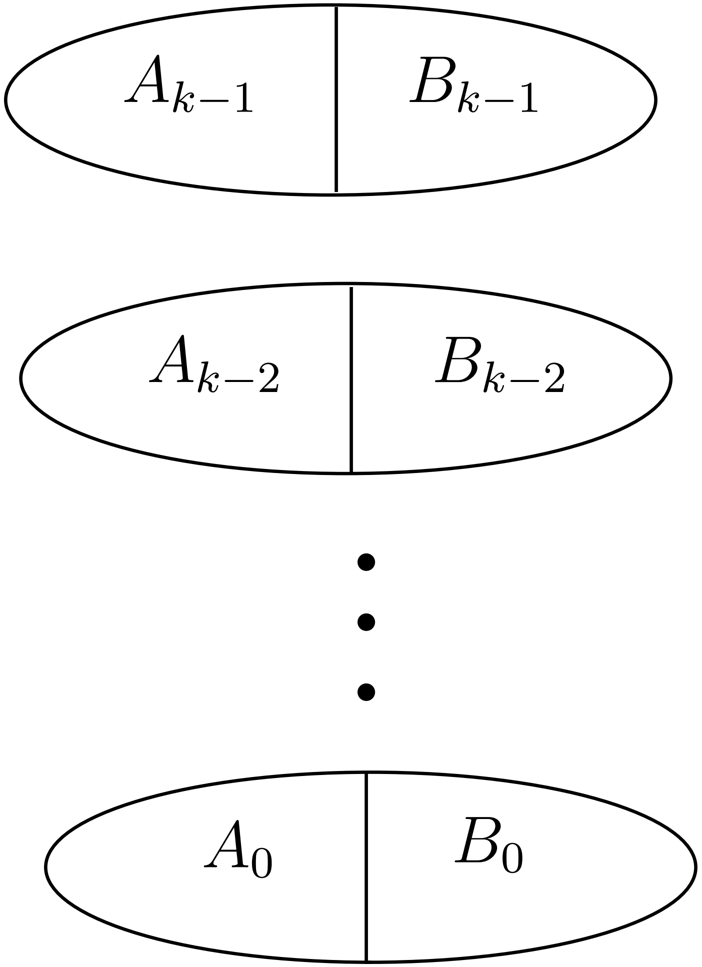


- The matching $S_2 = \{(a_0, b_1), (a_1, b_2), (a_2, b_3)\}$ is output by the 3-level algorithm.

Properties of the matching S_{k-1}

- $S_{k-1} \subseteq (A_{k-1} \times B_{k-1}) \cup (A_{k-2} \times B_{k-2}) \cup \cdots \cup (A_0 \times B_0)$.



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    - no alternating path has k blocking edges.

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- S_{k-1} can be computed in $O(mk)$ time.

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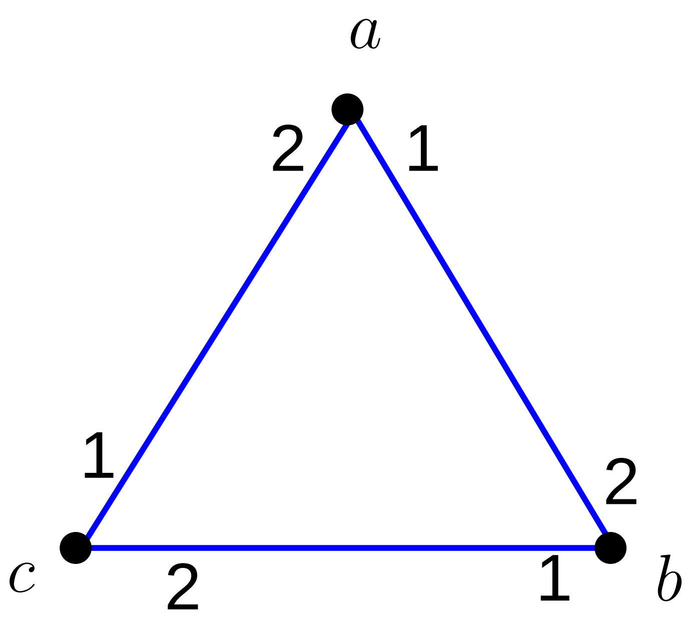

In general graphs

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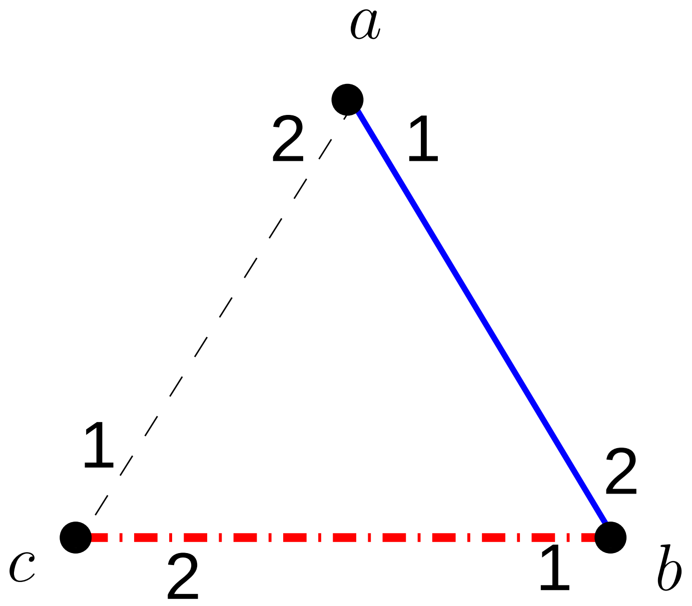

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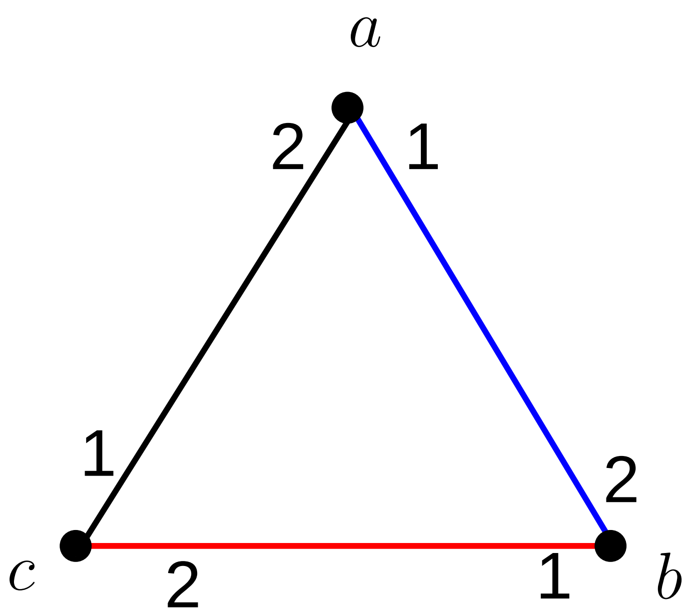

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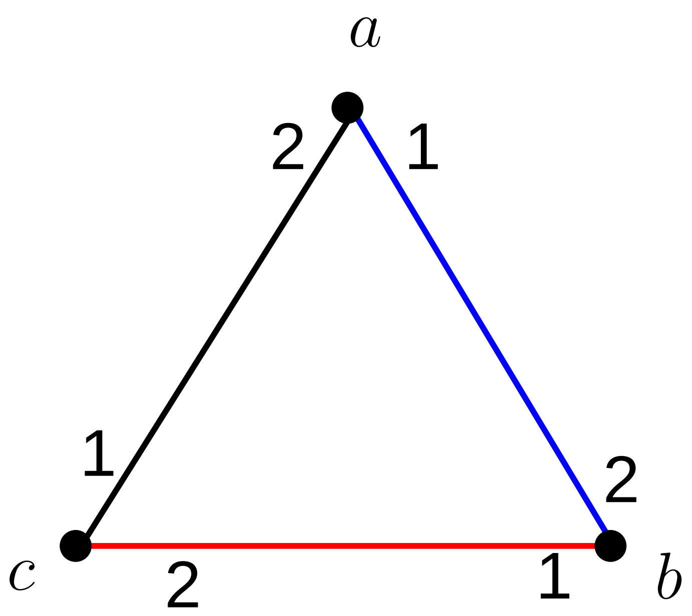

- We have $M_1 \prec M_2 \prec M_3 \prec M_1$ here, where $M_1 = \{(a,b)\}$, $M_2 = \{(b,c)\}$, and $M_3 = \{(a,c)\}$.

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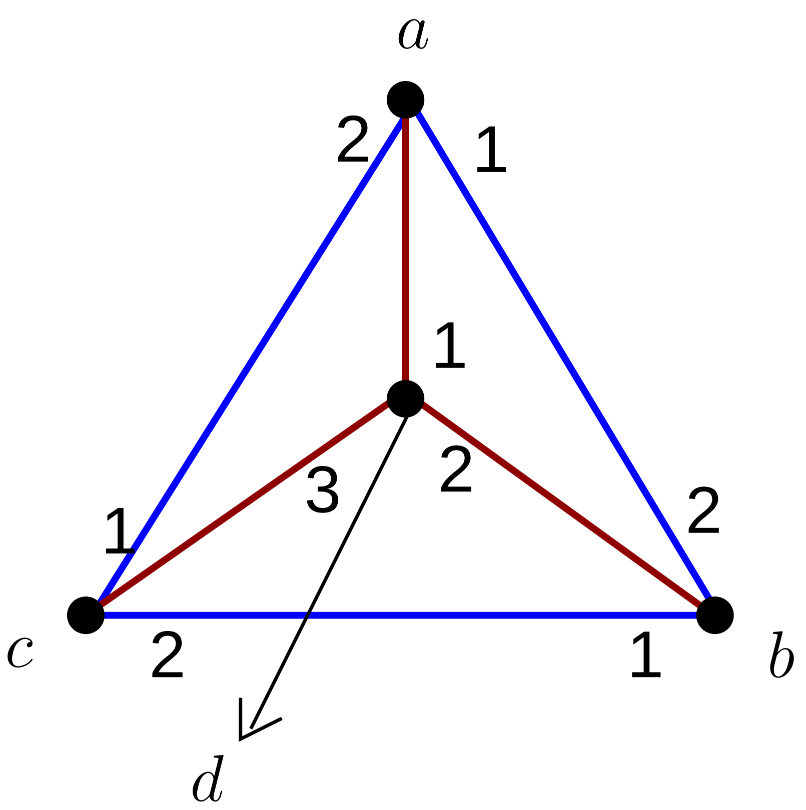

- d is the least preferred neighbor for a, b, c.

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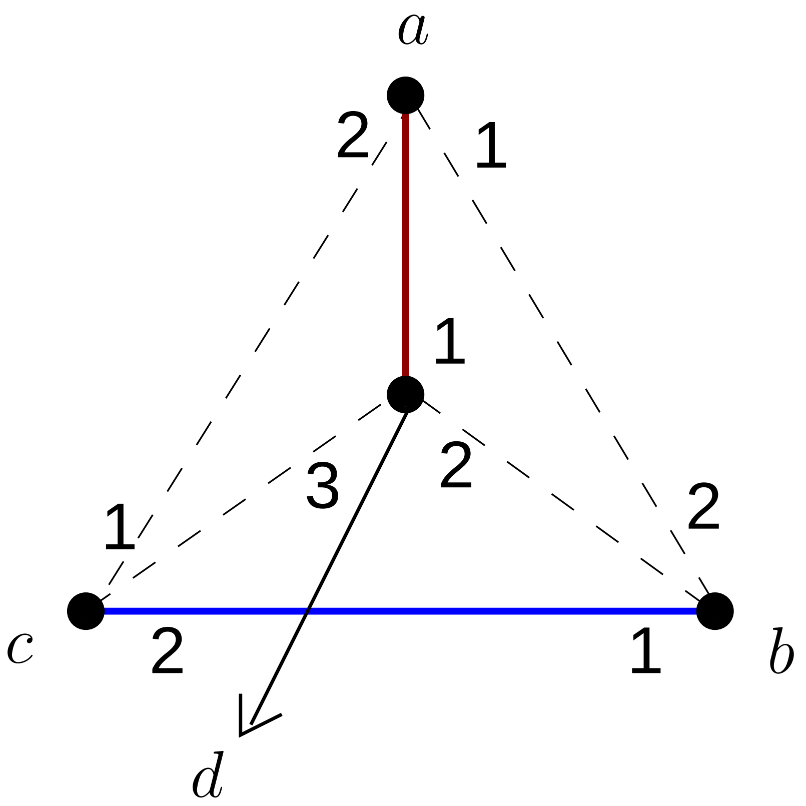

- $\{(a, d), (b, c)\}$ is popular.

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  - Such a matching can be computed in linear time.
- Computing a *least* unpopularity factor matching is NP-hard.
- *Open problem*: complexity of determining if G admits a popular matching or not.

Thank you!