**CS369E: Expanders** April 11, 2005

# Lecture 2: Eigenvalues and Expanders

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## 2.1 Lecture Outline

1. A non-constructive proof that expanders exist.

   Our method of proof will be to pick a random graph and show that it is an expander with some non-zero probability. There do exist constructive proofs, but we won't see any today. Given this proof, in order to find an expander in practice, we might want to generate a graph at random and test to see if it is an expander – but testing is co-NP hard.

2. Explore the connection between expanders and the spectrum of the graph (that is, the set of eigenvalues of the graph).

   There is a connection between the expansion of a graph and the eigengap (or spectral gap) of the normalized adjacency matrix (that is, the gap between the first and second largest eigenvalues). Recall that the largest eigenvalue of the normalized adjacency matrix is 1; denote it by $\lambda_1$ and denote the second largest eigenvalue by $\lambda_2$. We will see that a large gap (that is, small $\lambda_2$) implies good expansion and vice versa.

   (a) Large spectral gap implies good expansion.

   (b) Expander Mixing Lemma
   Heuristically, this says that "an expander graph will behave like a random graph." Let $S$ and $T$ be disjoint subsets of a vertex set $V$. If $G$ is a random $d$-regular (multi)graph on $V$, then the expected number of $S$-$T$ edges (that is, edges with one endpoint in $S$ and one endpoint in $T$) is $d|S||T|/N$. The lemma says that in an expander on $V$, the number of $S$-$T$ edges will be $d|S||T|/N + \lambda_2(\text{error term})$.

3. Alon's proof ([Alo], 1986) that good expansion implies a large spectral gap.

   This can be viewed as a discrete analogue of a result of Cheeger. Though vertex expansion and spectral gap are very closely related, vertex expansion does not seem to be the combinatorial equivalent of spectral expansion. This is because, the connection between vertex expansion and spectral expansion does not seem to be tight. Bitu and Linial ([BL], 2004), via their (partial) converse to the expander mixing lemma, give what might be the combinatorial equivalent notion (at least in the case of constant degree graphs) of the second eigen value.

4. The relationship between $d$ and $\lambda_2$.

(Several of the proofs given in this Section are from Lectures 8 and 9 of Salil Vadhan's notes on Pseudo-randomness [Vad]).

## 2.2 Existence of Expanders

The best probabilistic result on the existence of expanders is:

**Theorem 2.1.** *Fix $d \geq 3$. A random $d$-regular graph is a $(\Omega(N), d-1.01)$-expander with high probability (as $N \to \infty$, the probability goes to 1).*

We will not prove this result. Instead we will show the existence of bipartite expanders. Let $\mathcal{G}_{d,N}$ denote the set of bipartite graphs with partite sets $L$ and $R$ of cardinality $N$ and left degree $d$.

**Theorem 2.2.** *For all $d$, there exists $\alpha(d) > 0$ such that for all $N$,*

$$Pr[G \textit{ is an } (\alpha N, d-2)\textit{-expander}] \geq 1/2,$$

*where $G$ is chosen uniformly at random from $\mathcal{G}_{d,N}$. (In fact, we can take $\alpha(d) = 1/(cd^4)$ for some constant $c$.)*

*Proof.* To choose $G$ in $\mathcal{G}_{d,N}$ uniformly at random, we choose $d$ (not necessarily distinct) neighbors for each vertex $L$ at random. For $k \leq \alpha N$, let

$$p_k = \Pr[\exists\ S \subseteq L \text{ such that } |S| = k,\ |\Gamma(S)| < (d-2)|S|].$$

Thus $p_k$ is the probability that $G$ is not a $(\alpha N, d-2)$-expander because the neighborhood of a set of size $k$ is not large enough. To prove the theorem, it suffices (by the union bound) to show that $\sum_k p_k \leq 1/2$.

If $S \subseteq L$ has cardinality $k$, then the total number of neighbors of vertices in $S$, counted with multiplicity, is $dk$. So if $|\Gamma(S)| < (d-2)k$, then there must be $2k$ repeats among the neighbors of vertices in $S$. We can compute this probability:

$$Pr[\text{at least } 2k \text{ repeats among the } kd \text{ neighbors of vertices in } S] \leq \binom{kd}{2k}\left(\frac{kd}{N}\right)^{2k}.$$

Here, the binomial coefficient represents the number of ways to choose $2k$ neighbors to be repeats, and the fraction $kd/N$ represents an upper bound on the probability that any given choice of a neighbor is a repeat. That this is an upper bound follows from the union bound. Since there are $\binom{N}{k}$ possibilities for $S$, we have

$$\begin{aligned} p_k &\leq \binom{N}{k}\binom{kd}{2k}\left(\frac{kd}{N}\right)^{2k} \\ &\leq \left(\frac{Ne}{k}\right)^{k}\left(\frac{kde}{2k}\right)^{2k}\left(\frac{kd}{N}\right)^{2k} \\ &= \left(\frac{cd^4k}{N}\right)^{k}, \end{aligned}$$

where $c = e^3/4$. When $\alpha = 1/(cd^4)$ and $k \leq \alpha N$, we see that $p_k \leq 4^{-k}$. Then

$$Pr[G \text{ is not an } (\alpha N, d-2)\text{-expander}] \leq \sum_{k=1}^{\alpha N} p_k \leq \sum_{k=1}^{\alpha N} 4^{-k} < 1/2.$$

This completes the proof. $\square$

## 2.3 Exploring the spectral connection

Let $G$ be a $d$-regular multigraph with normed adjacency matrix $A$. The largest eigenvalue of $A$ is $\lambda_1 = 1$ with eigenvector $u = (1/N, \dots, 1/N)$. Then the second largest eigenvalue is given by

$$\lambda_2 = \max_{\|x\|=1, x \perp u} \|Ax\|.$$

If $\pi$ is a probability distribution on the vertices of $G$ (represented as a vector), we can write $\pi = u + \pi^{\perp}$, where $\pi^{\perp} \perp u$. View $A$ as the transition matrix for a Markov chain and use the initial distribution $\pi$. Then

$$A\pi - u = A(u + \pi^{\perp}) - u = Au - u + A\pi^{\perp} = A\pi^{\perp}.$$

Thus

$$\|A\pi - u\|^2 = \|A\pi^{\perp}\|^2 \leq \lambda_2^2 \|\pi^{\perp}\|^2 = \lambda_2^2 \|\pi - u\|^2.$$

**Definition 2.3.** *$G$ has* spectral expansion $\lambda$ *if $\lambda_2(G) \leq \lambda$.*

So if $G$ has spectral expansion $\lambda$, at each step of the Markov chain, distance to uniformity shrinks by at least $\lambda$. Note that the term spectral expansion suggests that large $\lambda$ is good for expansion, but the opposite is true.

**Definition 2.4.** *Given a probability distribution $\pi$, the* collision probability *of $\pi$ is* $\mathrm{Coll}(\pi) = \|\pi\|^2 = \sum_x \pi_x^2$.

**Lemma 2.5.** $\mathrm{Coll}(\pi) = \|\pi - u\|^2 + 1/N$.

*Proof.* Write $\pi = u + \pi^{\perp}$. Then

$$\|\pi\|^2 = \|u\|^2 + \|\pi^{\perp}\|^2 = 1/N + \|\pi - u\|^2.$$

□

Note that $A\pi$ is also a probability distribution and using the lemma, we can compute the associated collision probability:

$$\mathrm{Coll}(A\pi) - 1/N = \|A\pi - u\|^2 \leq \lambda^2 \|\pi - u\|^2 = \lambda^2(\mathrm{Coll}(\pi) - 1/N).$$

Given a probability distribution $\pi$, let the *support* of $\pi$ be $\mathrm{support}(\pi) = \{x : \pi_x \neq 0\}$.

**Lemma 2.6.** *Let $\pi$ be a probability distribution. Then* $\mathrm{Coll}(\pi) \geq 1/|\mathrm{support}(\pi)|$.

*Proof.* Let $m = |\mathrm{support}(\pi)|$. We claim that if $x_1 + \cdots + x_m = x$, then $x_1^2 + \cdots + x_m^2$ is minimized (with value $x/m$) when $x_1 = \cdots = x_m = x/m$. This easily follows from the fact that $x^2 + y^2 \geq ((x+y)/2)^2 + ((x+y)/2)^2$. Thus $\mathrm{Coll}(\pi) \geq 1/m$ and we are done. □

**Theorem 2.7.** *If $G$ has spectral expansion $\lambda$, then for all $\alpha > 1$, $G$ has vertex expansion $\left(\alpha N, \frac{1}{(1-\alpha)\lambda^2 + \alpha}\right)$.*

*Proof.* Let $|S| \leq \alpha N$. Choose $\pi$ a probability distribution that is uniform on $S$ and 0 on the complement of $S$. Then

$$Coll(\pi) = 1/|S| \qquad and \qquad Coll(A\pi) \geq 1/|support(A\pi)| = 1/|\Gamma(S)|.$$

Then

$$1/|\Gamma(S)| - 1/N \leq \lambda^2(1/|S| - 1/N).$$

But $N \geq |S|/\alpha$, so solving the above inequality gives

$$|\Gamma(S)| \geq \frac{|S|}{(1-\alpha)\lambda^2 + \alpha}.$$

Thus $G$ is an $(\alpha N, 1/((1-\alpha)\lambda^2 + \alpha))$-expander. □

Now we turn to a theorem on the spectral expansion of random graphs.

**Theorem 2.8 (Alon's Conjecture, Friedman ([Fri], 2003)).** *For any $d$ and any constant $\varepsilon > 0$, a random $d$-regular graph has spectral expansion at most $2\sqrt{d-1}/d + \varepsilon$ with probability $1 - 1/N^{\Omega(d)}$.*

This theorem says that with high probability, the spectral expansion of a random $d$-regular graph is approximately bounded by $2/\sqrt{d}$. The previous theorem implies that such a graph has expansion at least $d/4$. In fact, there do exist graphs with $\lambda_2 \leq 2/\sqrt{d}$ and expansion greater than $d/2$.

There is a theorem of Alon and Boppana that gives a lower bound for spectral expansion, showing that Alon's Conjecture is essentially sharp.

**Theorem 2.9 (Alon-Boppana (stated in [Alo])).** *Any infinite family of $d$-regular graphs has spectral expansion (as $N \to \infty$) at least $2\sqrt{d-1}/d - o(1)$.*

### 2.3.1 Expander Mixing Lemma

Heuristically, the following lemma, due to Alon and Chung [AC], says that an expander graph behaves like a random graph.

**Theorem 2.10 (Expander Mixing Lemma, [AC] 1988).** *For any subsets $S$ and $T$ of $V(G)$, let $e(S,T)$ denote the set of $S-T$ edges in $G$ (edges with one endpoint in $S$ and one endpoint in $T$). Let $G$ be $d$-regular with $\lambda_2 = \lambda$. Then*

$$|\#e(S,T) - d|S||T|/N| \leq \lambda d\sqrt{|S||T|}.$$

*Proof.* Let $\chi_S$ and $\chi_T$ be the characteristic vectors of $S$ and $T$ respectively. First note that

$$\#e(S,T) = \sum_{u\in S, v\in T} (dA)_{uv} = \sum_{u,v} \chi_S(u)(dA)_{uv}\chi_T(v) = \chi_S^t(dA)\chi_T.$$

Write $\chi_S$ in terms of something parallel to $u$ and $\chi_S^{\perp}$. Then the coefficient of $u$ is the projection

$$\frac{\chi_S \cdot u}{\|u\|^2} = \frac{(1/N)\sum_i \chi_S(i)}{(1/N)} = |S|.$$

So

$$\chi_S = |S|u + \chi_S^\perp \qquad \text{and} \qquad \chi_T = |T|u + \chi_T^\perp.$$

(The intuition should be that the term $|S|u$ "spreads the weight evenly" and $\chi_S^\perp$ is an error term.)

Now:

$$\begin{aligned}
\#e(S,T) &= (|S|u + \chi_S^\perp)^t (dA)(|T|u + \chi_T^\perp) \\
&= d|S||T|(u \cdot u) + d|S|u^t A \chi_T^\perp + d|T|(\chi_S^\perp)^t A u + d(\chi_S^\perp)^t A \chi_T^\perp
\end{aligned}$$

Since $\chi_T^\perp \cdot u = 0$, we see that $u^t A \chi_T^\perp = 0$, and similarly $\chi_S^\perp A u = 0$. Then

$$\begin{aligned}
\#e(S,T) &= d|S||T|/N + d(\chi_S^\perp)^t A \chi_T^\perp \\
&\le d|S||T|/N + \|\chi_S^\perp\| \|A\chi_T^\perp\| \\
&\le d|S||T|/N + d\lambda \|\chi_S\| \|\chi_T\| \\
&= d|S||T|/N + d\lambda\sqrt{|S||T|}.
\end{aligned}$$

From the first line, it is evident that $\#e(S,T) \ge d|S||T|/N$. Thus

$$|\#e(S,T) - d|S||T|/N| \le d\lambda\sqrt{|S||T|}.$$

□

There is a partial converse to this theorem.

**Theorem 2.11 (Bilu-Linial, ([BL], 2004)).** *Let $G$ be a $d$-regular graph and fix $\theta$. If for all $S, T \subset V$, the inequality*

$$|\#e(S,T) - d|S||T|/N| \le \theta d\sqrt{|S||T|}$$

*holds, then $G$ has spectral expansion $\lambda = O(\theta(1 + \log(d/\theta)))$.*

In particular, this means that for a $d$-regular graph, $\lambda$ is essentially (up to $\log d$ factor), the best constant that can occur in the expander mixing lemma.

## 2.4 Vertex Expansion Implies Spectral Expansion

The following theorem (due to Alon) is a discrete version of Cheeger's result. We first prove for the special case when the normalized adjacency matrix $A$ has only non-negative eigenvalues.

**Theorem 2.12 (Alon, ([Alo], 1986)).** *Let $G$ be a $d$-regular $(N/2, 1+\alpha)$-expander and let $\lambda_2(G)$ be the second largest eigenvalue of the normalized adjacency matrix $A(G)$ of $G$ in absolute value. If the matrix $A = A(G)$ has all non-negative eigen-values, then $G$ is a $\lambda$-spectral expander for $\lambda = 1 - \alpha^2/(d(8+4\alpha^2))$.*

*Proof.* Let $x$ be an eigenvector with eigenvalue $\lambda_2(A)$. Since $x \perp u$, the vector $x$ has both positive and negative entries. Let $V_+ = \{i : x_i > 0\}$ and $V_- = \{i : x_i \leq 0\}$. Without loss of generality $|V_+| \leq N/2$. Let $\overline{x}$ be the vector that agrees with $x$ on $V_+$ and is 0 elsewhere.

Note that $\langle \overline{x}, \overline{x} \rangle = \langle x, \overline{x} \rangle$, so it can be shown that

$$\lambda_2(A) = \frac{\lambda_s \langle x, \overline{x} \rangle}{\langle x, \overline{x} \rangle} = \frac{\lambda_2 \langle x, \overline{x} \rangle}{\langle \overline{x}, \overline{x} \rangle} = \frac{\langle Ax, \overline{x} \rangle}{\langle \overline{x}, \overline{x} \rangle}.$$

Also,

$$\begin{aligned}
\lambda_2(A) \langle \overline{x}, \overline{x} \rangle &= \langle Ax, \overline{x} \rangle \\
&= \sum_{i,j} A_{ij} x_j \overline{x}_i \\
&= \|x\|^2 - \frac{1}{d} \left( d\|x\|^2 - \sum_{i \in V_+, \{i,j\} \in E} \overline{x}_i x_j \right) \\
&= \|x\|^2 - \frac{1}{d} \left( d\|x\|^2 - 2 \sum_{i,j \in V_+, \{i,j\} \in E} \overline{x}_i \overline{x}_j - \sum_{i \in V_+, j \in V_-, \{i,j\} \in E} \overline{x}_i \overline{x}_j \right) \\
&\leq \|x\|^2 - \frac{1}{d} \left( d\|x\|^2 - 2 \sum_{\{i,j\} \in E} \overline{x}_i \overline{x}_j \right) \\
&= \|x\|^2 - \frac{1}{d} \sum_{\{i,j\} \in E} (\overline{x}_i - \overline{x}_j)^2 \\
\lambda_2 &\leq 1 - \frac{\sum_{\{i,j\} \in E} (\overline{x}_i - \overline{x}_j)^2}{d \sum_{i \in V} \overline{x}_i^2}. \qquad (1)
\end{aligned}$$

Build a new (directed) graph $H$ as follows. Let

$$V(H) = \{s\} \cup \{v_i : i \in V_+\} \cup \{w_j : j \in V\} \cup \{t\}.$$

For all $i \in V_+$, put the arcs $(s, v_i)$ in $H$ with capacity $1 + \alpha$. For each $i \in V_+$ and $j \in V$ where $j$ is a neighbor of $i$ in $G$, put the arcs $(v_i, w_j)$ in $H$ with capacity 1. Finally, for each $j \in V$, put the arcs $(w_j, t)$ in $H$ with capacity 1.

We claim that the minimum cut in this graph is $(1+\alpha)|V_+|$. A cut of this size is given by the set of arcs $\{(s, v_i) : i \in V_+\}$. Given any other cut $C$, let $W = \{i \in V_+ : (s, v_i) \notin C\}$. For each $j \in N(W)$, there must be an arc in $C$ adjacent to $w_j$. But $|N(W)| \geq (1+\alpha)|W|$, so the capacity of $C$ must be at least $(1+\alpha)|V_+ - W| + |N(W)| \geq (1+\alpha)|V_+|$ (since $|W| \leq N/2$). So the minimum cut has capacity $(1+\alpha)|V_+|$.

By the min-cut max-flow theorem, there exists a flow on $H$ of size $(1+\alpha)|V_+|$. In particular, note that the flow through each vertex $v_i$ must be $1 + \alpha$. Reading off the flow along arcs $(v_i, w_j)$, it follows that there is a function $F : V \times V \to \mathbb{R}$ satisfying the following conditions (here $\tilde{E}$ denotes the set of ordered pairs $(i,j)$ where $\{i,j\} \in E$, so that each edge in $E$ is counted twice in $\tilde{E}$):

1. $0 \leq F(i,j) \leq 1$ for all $i, j \in V$.

2. $F(i,j) = 0$ if $i \notin V_+$ or $(i,j) \notin \tilde{E}$.

3. $\sum_{j\,:\,(i,j)\in\tilde{E}} F(i,j) = 1+\alpha$ for each $i \in V_+$.

4. $\sum_{i\,:\,(i,j)\in\tilde{E}} F(i,j) \leq 1$ for each $j \in V$.

We need to calculate two bounds involving $F$ in order to bound $\lambda_2(G)$. Keeping in mind that $2(a^2+b^2) \geq (a+b)^2$ for all real $a$ and $b$, we find:

$$\begin{aligned}
\sum_{(i,j)\in\tilde{E}} F^2(i,j)(\overline{x}_i+\overline{x}_j)^2 &\leq 2\sum_{(i,j)\in\tilde{E}} F^2(i,j)(\overline{x}_i^2+\overline{x}_j^2) \\
&= 2\sum_{i\in V}\overline{x}_i^2\left(\sum_{(i,j)\in\tilde{E}} F^2(i,j) + \sum_{(i,j)\in\tilde{E}} F^2(j,i)\right) \\
&\leq (4+2\alpha^2)\sum_{i\in V}\overline{x}_i^2. \\
\sum_{(i,j)\in\tilde{E}} F(i,j)(\overline{x}_i^2-\overline{x}_j^2) &= \sum_{i\in V}\overline{x}_i^2\left(\sum_{(i,j)\in\tilde{E}} F(i,j) - \sum_{(i,j)\in\tilde{E}} F(j,i)\right) \\
&\geq \alpha\sum_{i\in V}\overline{x}_i^2
\end{aligned}$$

Note that in the third line, we used the fact that if $x_1+\cdots+x_n = 1+\alpha$ and $0 \leq x_i \leq 1$ for all $i$, then $x_1^2+\cdots+x_n^2 \leq 1+\alpha^2$. Multiplying equation (1) by

$$1 = \frac{\sum_{(i,j)\in\tilde{E}} F^2(i,j)(\overline{x}_i+\overline{x}_j)^2}{\sum_{(i,j)\in\tilde{E}} F^2(i,j)(\overline{x}_i+\overline{x}_j)^2}$$

and using Cauchy-Schwarz, we get

$$\begin{aligned}
\lambda_2(G^2) &\leq 1 - \frac{\sum_{\{i,j\}\in E}(\overline{x}_i-\overline{x}_j)^2}{d\sum_{i\in V}\overline{x}_i^2} \\
&= 1 - \frac{\sum_{\{i,j\}\in E}(\overline{x}_i-\overline{x}_j)^2\cdot\sum_{(i,j)\in\tilde{E}} F^2(i,j)(\overline{x}_i+\overline{x}_j)^2}{d\sum_{i\in V}\overline{x}_i^2\cdot\sum_{(i,j)\in\tilde{E}} F^2(i,j)(\overline{x}_i+\overline{x}_j)^2} \\
&\leq 1 - \frac{\left(\sum_{(i,j)\in\tilde{E}} F(i,j)(\overline{x}_i^2-\overline{x}_j^2)\right)^2}{2d(4+2\alpha^2)\left(\sum_{i\in V}\overline{x}_i^2\right)^2} \\
&\leq 1 - \frac{\alpha^2}{d(8+4\alpha^2)}.
\end{aligned}$$

This completes the proof. □

We now move to the general case, when the eigen-values of $A(G)$ need not all be non-negative.

**Corollary 2.13.** *If $G$ is a $d$-regular $(N/2, 1+\alpha)$-expander, then $G$ is also a $\lambda$-spectral expander for $\lambda = \sqrt{1 - \alpha^2/(d^2(8+4\alpha^2))}$.*

*Proof.* Consider the graph $G^2$. If the normalized adjacency matrix of $G$ is $A$, then the normalized adjacency matrix of $G^2$ is $A^2$. Also $G^2$ is $d^2$-regular and has all non-negative eigenvalues. Finally, $G^2$ is a $(N/2, 1+\alpha)$-expander, as follows: If $S$ is a subset of the vertices of size at most $N/2$, then $|N(S)| \geq (1+\alpha)|S|$. Choose a subset $S'$ of $N(S)$ with $|S| \leq |S'| \leq N/2$. Then $|N(N(S))| \geq |N(S')| \geq (1+\alpha)|S'| \geq (1+\alpha)|S|$. But $N(N(S))$ is the neighborhood of $S$ in $G^2$, and $S$ was an arbitrary subset of vertices of size at most $N/2$, so $G^2$ is a $(N/2, 1+\alpha)$-expander.

By Theorem 2.12, $\lambda_2(G^2) \leq 1 - \alpha^2/(d^2(8+4\alpha^2))$. Since the eigenvalues of $G^2$ are the squares of the eigenvalues of $G$, taking the square root of the right-hand side proves the corollary. □

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