Lecture 8 (Guest Lecturer: Yuval Filmus)

Lovász Θ Functions & its Applications (Notes by Prahladh)

Pronunciation Guide

Hungarian: s, sz, zs

Polish: s, sz

z, ż

Traffic Light Puzzle:

1 traffic light - controlled by n 3-way switches.
Light changes if all switches change.

Given a graph $G=(V,E)$
Goal. Find an upper bd on independent set of G.

Theorem (Lovász)

M - symmetric $V \times V$ matrix s.t.
$M(x,y)=1$ whenever (x,y) is not an edge

$$\Downarrow$$

$$\alpha(G) \leq \lambda_{max}(M)$$

Proof: F - independent set, $\mathbb{1}_F$ - char vector

$$\mathbb{1}_F^T M \mathbb{1}_F = \sum_{x,y \in F} M(x,y) = |F|^2$$

On the other hand

$$\mathbb{1}_F^T M \mathbb{1}_F \leq \lambda_{max} \mathbb{1}_F^T \mathbb{1}_F = \lambda_{max} |F|$$

Hence, $|F| \leq \lambda_{max}$ $\boxtimes$

Denote by $\vartheta(G)$ the best bound that this lemma can get

$$\vartheta(G) = \min \lambda_{max}(M) \quad \begin{cases} M(x,y) = 1 & (x,y) \notin E \\ M \preceq \lambda_{max} \end{cases}$$

Erdős-Ko-Rado Theorem:

Suppose $k \le n/2$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting then (1) $|\mathcal{F}| \le \binom{n-1}{k-1}$

(2) $k < n/2$ & $|\mathcal{F}| = \binom{n-1}{k-1}$, then $\mathcal{F}$ is a star.

Proof: Construct $G=(V,E)$ (using Hoffman bd)

$V = \binom{[n]}{k}$, $A, B \in V$, $A \sim B$ if $A \cap B = \emptyset$

Kneser graph. ($Kn(n,k)$)

(using M related adj matrix $Kn(n,k)$ yields (1))

Lovász Theorem (Strengthening)

Furthermore, if F is an independent set of size $\lambda_{max}(M)$, then $\mathbb{1}_F$ lies in eigenspace of λ_{max}

→ Yields Uniqueness

of gap between λ_{max} & λ_2 → stability results

→ Cross-Intersecting Families.

Theorem (Hoffman):

Let M be a $V \times V$ matrix s.t

① $M(x,y) = 0$ if $(x,y) \notin E$

② $M\mathbb{1} = \mathbb{1}$

Then, $\alpha(G) \leq \frac{-\lambda_{min}}{1-\lambda_{min}} |V|$

If F attains the bound then $\mathbb{1}_F - \frac{|F|}{|V|}\mathbb{1}$ lies in eigenspace of λ_{min}

Proof: Either directly or instead by redn to Lovász Bound

Let $M' = J - \frac{|V|}{1-\lambda_{min}} M$

M' satisfies the condn of Lovász bound.

J has e.val $\begin{cases} |V| & \text{w/ mult } 1 \\ 0 & \text{w/ mult } |V|-1 \end{cases}$

Hence, M' has e.val $\begin{cases} \left(1 - \frac{1}{1-\lambda_{min}}\right)|V| & \text{w/ mult } 1 \\ \frac{-|V|}{1-\lambda_{min}}\lambda & \end{cases}$

Hence, $\lambda(M') \leq \frac{-\lambda_{min}}{1-\lambda_{min}} |V|$

Schrijver Bound

$\theta_S(G) = \min \lambda$

$\begin{cases} M(x,y) \geq 1 & \text{if } (x,y) \notin E \\ M \preceq \lambda I & \end{cases}$

Similar strengthening of Hoffman Bd

Back to traffic light puzzle:

$G_n = (\mathbb{Z}_3^n, \{(x,y) \mid x_i \neq y_i, \forall i\})$

Soln to traffic light puzzle $\rightarrow$ 3-coloring of G.

Each color class an independent set

F - one color class.

Want to apply Hoffman's bound w/ matrix $B = A/2^n$ where

$A_n := \mathrm{Adj}(G_n)$

$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad A_n = A_1^{\otimes n}$

$A_1 \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ 2 & -\omega & -\omega^2 \\ 2 & -\omega^2 & -\omega \end{pmatrix}$ e-val. $\{2, -1, -1\}$

Eigenvalues of $A_n = 2^n, -2^{n-1}, 2^{n-2}, -2^{n-3}, \ldots$

$\lambda_{min}(A) = -2^{n-1}$

B satisfies Hoffman condns

$B\mathbb{1} = \mathbb{1}$

& $\lambda_{min}(B) = -1/2$

$\therefore$ Hence, $\Theta_H(G) = \dfrac{-1/2}{1-(-1/2)} \cdot 3^n = \dfrac{3^n}{3} = 3^{n-1}$

All color classes contain at most 3^{n-1} points

Each color class is exactly 3^{n-1} points

So by Hoffman Bound

$\mathbb{1}_F - \frac{1}{3}\mathbb{1}$ lies in the eigenspace of $\lambda_{min}(A)$ (i.e., -2^{n-1})

Eigenspace of $-2^{n-1} = \text{span}\{\varphi_i : \mathbb{Z}_3 \to \mathbb{R} \mid \sum_{x_i} \varphi_i(x_i) = 0\}$

$$\mathbb{1}_F(x_1 \dots x_n) = \frac{1}{3} + \sum_{i=1}^{n} \varphi_i(x_i) \quad =: g(x_1 \dots x_n)$$

Suppose φ_1, φ_2 - non-constant

$$\left.\begin{array}{l} \varphi_1(a) < \varphi_1(b) \\ \varphi_2(c) < \varphi_2(d) \end{array}\right\} \Rightarrow \text{Then } g(a, c, x_3 \dots x_n) < g(b, c, x_3 \dots x_n) < g(b, d, x_3 \dots x_n)$$

Impossible since g attains at most 2 possible values.

Hence, at most one of φ_i is non-constant

$$\mathbb{1}_F = \frac{1}{3} + \varphi_i(x_i) = \psi(x_i)$$

So $F = \{x \mid x_i = j\}$

Since all 3 color classes are disjoint, the corresponding i's are same.

Hence, traffic light depends only on one switch.

Applications of Hoffman Bound in Extremal Combinatorics

A subset $F \subseteq S_n$ is intersecting if $\forall \pi_1, \pi_2 \in F$ $\exists i, \ \pi_1(i) = \pi_2(j)$

Upper Bound $F = \{\pi \mid \pi(1) = 1\}$
$(n-1)!$

Lower Bound: Any F contains at most 1 of

$$\left.\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{matrix}\right\} \quad \text{Hence } |F| \leq \frac{n!}{n} = (n-1)!$$

Thm [Ellis-Friedgut-Pilpel]
For all t, large enough n (depending on t) the maximal size of a t-intersecting family of permutations is $(n-t)!$
Moreover, this is achieved only on "t-double-coset."

: Let $q = p^m$, k
Consider $\binom{[n]}{k}_q = \{$subspaces of $\mathbb{F}_q^n$ of dim $k\}$

A family of subspaces is t-intersecting if intersection of any 2 subspaces has dim $\geq t$.

Theorem [Frankl-Wilson]

$$\text{max-size} = \max\left\{\binom{n-t}{k-t}_q, \binom{2k-t}{k}_q\right\}$$

Domain: all perfect matching in K_{2n}
(or almost p.m in K_{2n+1})

Thm: [Lindzey 2019] Same as EFP.

Next lecture:

Domain: all subgraphs of K_n

Δ-intersecting family of graphs is a collection of graphs, the intersection of any two of which contains a Δle.