Today

- Space Complexity
- Configuration Graphs
- PSPACE
- TQBF & PSPACE-complete
- Savitch's Theorem

CSS.203.1
Computational Complexity
– Lecture #7 (8 Mar '21)

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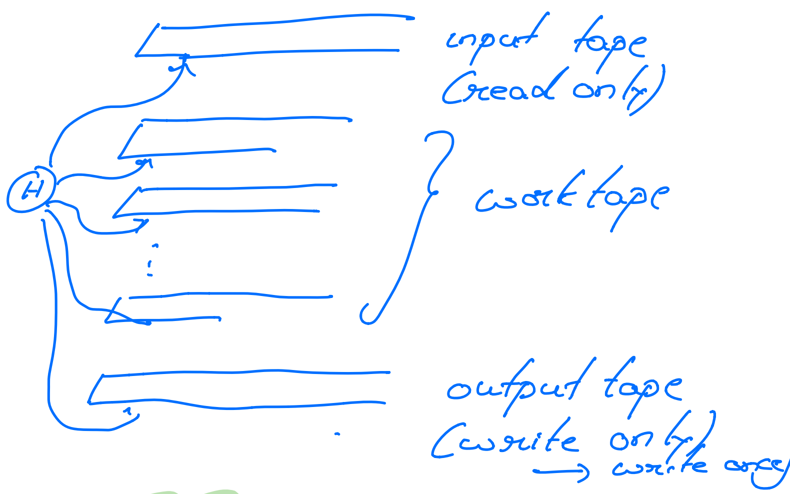
Space: the final frontier
– Capt James Kirk (Star Trek The Original Series)

Last time:



Assumption:
$S(n) \geq \log n$

SPACE$(S(n))$

NSPACE$(S(n))$

Write Once:
The head on o/p tape moves right the moment it writes a symbol.

Theorem: $\mathrm{DTIME}(S(n)) \subseteq \mathrm{SPACE}(S(n))$
$\subseteq \mathrm{NSPACE}(S(n))$
$\subseteq \mathrm{TIME}(2^{O(S(n))})$

Configuration of a TM M on i/p x

C
- State of TM
- Position on i/p tape

- Contents of all work tapes

Suppose TM M takes space at most S on input x.

the # configurations $\leq n \cdot |Q| \cdot 2^{O(S)}$

$= 2^{O(S)}$ (since $S \geq \log n$)

S(n) - space constructible

$S: \mathbb{N} \to \mathbb{N}$ is space constructible if $\exists$ a TM M that on i/p 1^n outputs $S(n)$ using space at most $S(n)$.

Turing machine M & input x

Configuration C.

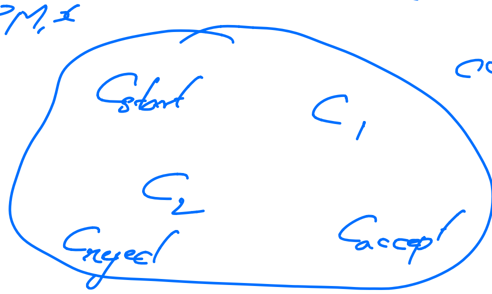
Configuration Graph $(G_{M,x})$ of TM M on i/p x

Directed Graph

Vertices = Configurations

Edges : (C, C') is an edge if transition fn moves config C to config C'.

$G_{M,x}$



$\# \text{Vertices} = 2^{O(S(n))}$.

Out-degree of a node ≤ 1 (if the TM is deterministic)

≤ 2 (if the TM is non-deterministic)

Question: Is x accepted by TM?

$\Updownarrow$

$\exists$ a path from start config to accept config.

$$SPACE(S(n)) \subseteq TIME(2^{O(S(n))})$$

Space-hierarchy Theorem: If f & g are space constructible $f(n) = o(g(n))$ then $SPACE(f(n)) \subsetneq SPACE(g(n))$

$PSPACE = \bigcup_{c: c \geq 0} SPACE(n^c)$

$NPSPACE = \bigcup_{c: c \geq 0} NSPACE(n^c)$

$L = SPACE(\log n)$

$NL = NSPACE(\log n)$.

Examples:

(1) $EVEN = \{x \in \{0,1\}^* / \#\{i,\ x_i = 1\} = \text{even}\}$.

$EVEN \in L$

(2) Compute DFS tree

- polynomial space. (deterministic)

(3) PATH = $\{(G,s,t) \mid G$ is a directed graph $\& \exists$ a path $s \rightsquigarrow t\}$.

PATH $\in$ PSPACE

What if we allow TM to be non-deterministic

$\left\{ \begin{array}{l} \text{Maintain current vertex} \\ \quad \& \text{ counter} \end{array} \right\}$ logspace

PATH $\in$ NL.

graph

$\to$ current address (initially s)

$\to$ counter (initialized to 0)

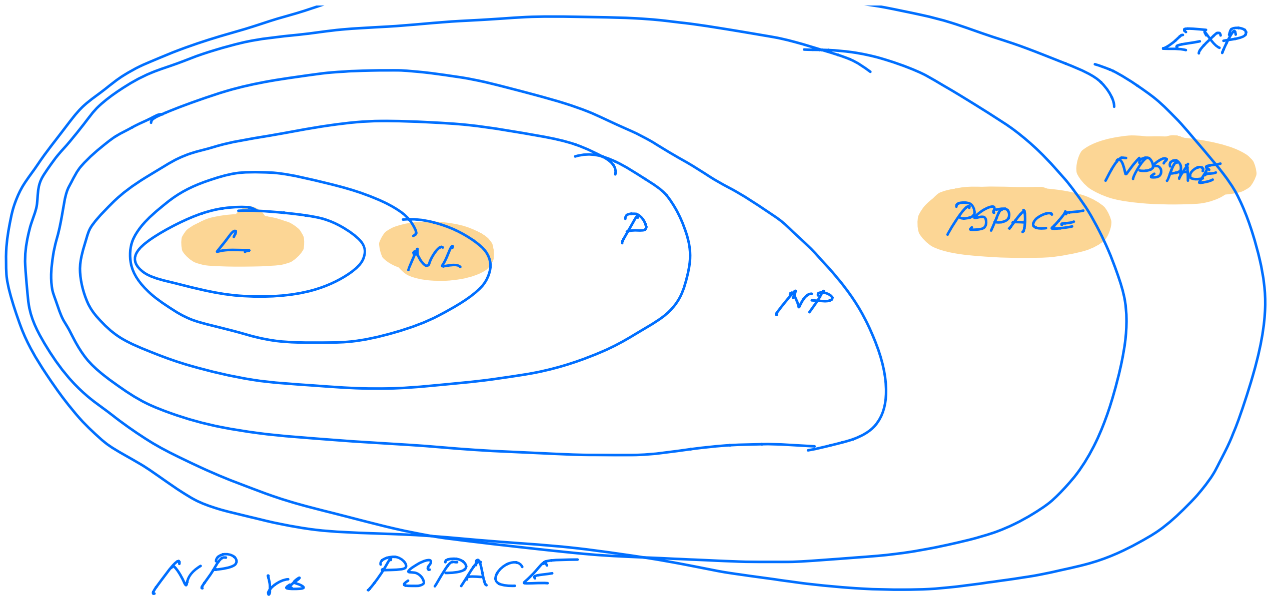
(4) Computing the adjacency list rep^n given the adjacency matrix rep of graph.

input $\to$ n^2. bit long (regular)

$\to$ $O(dn)$ - bit string

d - degree

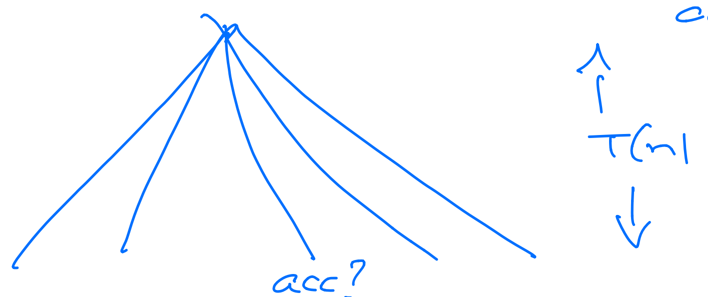
- logspace. (deterministic).



NP vs PSPACE

$TIME(T(n)) \subseteq SPACE(T(n))$

$NTIME(T(n)) \subseteq SPACE(T(n))$ } We can go over all $T(n)$- non deterministic choices in space $T(n)$.



PSPACE:

Nice complete problem for PSPACE.

$$L \triangleq \{(x, \alpha, 1^s) \mid M_\alpha \text{ accepts } x \text{ in space } s\}$$

L is PSPACE-complete.

PSPACE-hard

L is PSPACE-hard if $\forall L' \in PSPACE$
$L' \leq_p L$

Ex: Let A be any lang s.t
$\phi \subsetneq A \subsetneq \{0,1\}^*$

For any $L \in PSPACE$,
L is reducible to A under polyspace reductions.

$\times \times \times$

Quantified Boolean Formula:

$\varphi(x_1 .. \quad x_n)$

$\varphi(x,y) \triangleq \underbrace{(x \vee \bar{y}) \wedge (\bar{x} \vee y)}_{\text{formulae.}}$

Quantified formula.

$\exists x \; \exists y \; (x \vee \bar{y}) \wedge (\bar{x} \vee y).$

$\exists x \; \forall y \; (x \vee \bar{y}) \wedge (\bar{x} \vee y)$

$\forall x \; \exists y \; (x \vee \bar{y}) \wedge (\bar{x} \vee y)$

$Q_1 x_1 \, Q_2 x_2 \quad Q_n x_n \; \underbrace{\varphi(x_1 .. \quad x_n)}_{\text{arbitrary formula in n vars}}$

$Q_i \in \{\exists, \forall\}.$

$TQBF = \{ \psi = Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \; \varphi(x_1 \ldots x_n) \mid \psi \text{ is true} \}.$

$Q_1 = \ldots = Q_n = \exists$, SAT – NP

$Q_1 = \ldots \; Q = \forall$, TAUT – coNP

Lemma: TQBF $\in$ PSPACE.

Pf: n = #vars ; m – size of the formula φ.

$S_{n,m}$ = space reqd to solve TQBF instances on n vars of size m.

$S_{0,m} = O(m)$

$S_{n,m} \leq S_{n-1,m} + O(m)$

$S_{n,m} \leq (n+1) \cdot O(m) = O(n \cdot m).$

$\boxtimes$

Next

Theorem: TQBF is PSPACE-hard

Pf: $L \in PSPACE$ $\quad L \leq_p TQBF$

$x \mapsto \psi_x$

$G_{M,x}$