

Computational Complexity: Lecture 9. (Part 2)

Recap: ▷ Oracle TMs.

TMs with "oracle access" to a certain language.

P^{NP} could solve "is the smallest VC of size exactly k ?"

NP^{NP} could solve "min formula"

▷ Quantified boolean formulas

$$\exists x \forall y \exists z$$

$$\exists x. \forall y \exists z \varphi(x, y, z)$$

▷ TQBF = $\{ \exists x_1 \exists x_2 \dots \exists x_n \varphi(x_1 \dots x_n) : \text{that are true} \}$

TQBF is PSPACE complete (under polytime reductions)
(under logspace redns).

Polynomial Hierarchy: (PH)

$$\Sigma_1^P = NP$$

$$\Pi_1^P = coNP$$

Defn: Σ_2^P is the collection of languages L s.t
a poly time TM M and a poly q . s.t

$$x \in L \iff \exists y \in \{0,1\}^{q(|x|)} \forall z \in \{0,1\}^{q(|x|)} M(x, y, z) = 1.$$

(Generalise to higher i)

starts with \exists

$$\Sigma_i^P$$

→ # quantifiers

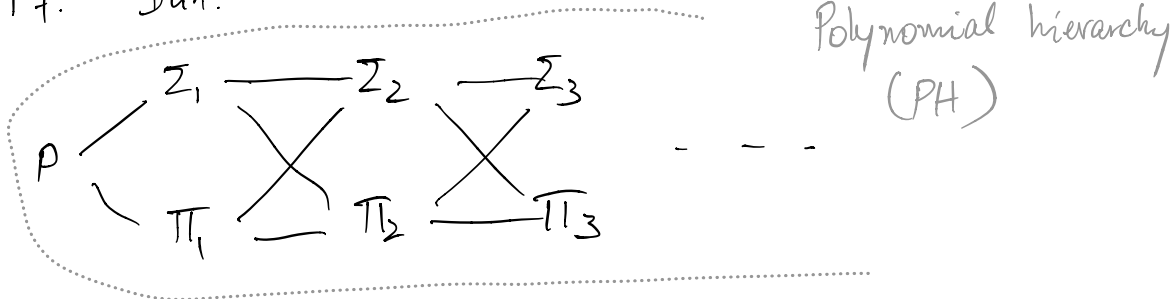
$$\Pi_2^P = co. \Sigma_2^P$$

$$\text{ie } L \in \Pi_2^P \iff \bar{L} \in \Sigma_2^P.$$

$$\Sigma_0 = \Pi_0 = P$$

Obs: $\Sigma_i^P, \Pi_i^P \subseteq \Sigma_{i+1}^P, \Pi_{i+1}^P$

Pf: Duh!



$$\text{Formally } PH = \bigcup_{i \geq 0} \Sigma_i^P = \bigcup_{i \geq 0} \Pi_i^P$$

Recall: TQBF is in PSPACE

∴ PH = PSPACE, right? **WRONG!**

$$P = \bigcup_{c \geq 0} \text{TIME}(n^c) \quad \text{TIME}(n^n) \subseteq P.$$

Current belief: All these classes are distinct.

"Polynomial hierarchy collapses" $\Leftrightarrow PH = \Sigma_i / \Pi_i$

What if $\Sigma_i = \Pi_i$?

Thm: For any $i \geq 1$, if $\Sigma_i = \Pi_i$, then $PH = \Sigma_i$.

Pf (for $i=1$) Assume $\Sigma_1^P = \Pi_1^P$ i.e. NP = coNP.

Say $L \in \Sigma_2^P$. \Rightarrow There is a poly time M
and g

$$\text{s.t. } x \in L \Leftrightarrow \exists y \in \{0,1\}^{q(|x|)} \forall z \in \{0,1\}^{q(|x|)} . M(x,y,z)=1$$

$$L' = \left\{ (x,y) : \forall z \in \{0,1\}^{q(|x|)} M(x,y,z)=1 \right\} .$$

$\in \text{coNP} = \text{NP}$ (by assumption).

\Rightarrow There is M' poly time, and a polynomial q' s.t

$$(x,y) \in L' \Leftrightarrow \exists w \in \{0,1\}^{q'(|x|+|y|)} M'(x,y,w)=1$$

$$\therefore x \in L \Leftrightarrow \exists y : (x,y) \in L'$$

$$= \exists y \exists w . M'(x,y,w)=1$$

$\in \text{NP}.$

$$\therefore \Sigma_2 = \text{NP}.$$

□

Next class:

More on PH

(Doesn't Σ_2^P look a lot like NP^{NP} ?)