Computational Complexity: Lecture 9. (Part 2)

Recap: ▷ Oracle TMs.

TMs with "oracle access" to a certain language.

P^{NP} could solve "is the smallest VC of size exactly k?"

NP^{NP} could solve "min formula"

▷ Quantified boolean formulas

$\exists x \; \forall y \quad x \neq y$

$$\exists x . \; \forall y \; \exists z \quad \varphi(x, y, z)$$

▷ $\text{TQBF} = \left\{ Q x_1 Q x_2 \ldots Q x_n \; \varphi(x_1, \ldots, x_n) : \text{that are true} \right\}$

TQBF is PSPACE complete (under polytime reductions)

(under logspace redns).

Polynomial Hierarchy: (PH)

$\Sigma_1^P = NP \qquad \Pi_1^P = coNP$

Defn: Σ_2^P is the collection of languages L s.t

a poly time TM M and a poly q. s.t

$$x \in L \iff \exists y \in \{0,1\}^{q(|x|)} \; \forall z \in \{0,1\}^{q(|x|)} \; M(x, y, z) = 1.$$

(Generalise to higher i)

starts with $\exists$ Σ_i^P # quantifiers

$\Pi_2^P = co . \Sigma_2^P$

ie $L \in \Pi_2^P \iff \bar{L} \in \Sigma_2^P$.

$\Sigma_0 = \Pi_0 = P$

Obs: $\Sigma_i^P, \Pi_i^P \subseteq \Sigma_{i+1}^P, \Pi_{i+1}^P$.

Pf: Duh!

$$\begin{tikzcd}
& \Sigma_1 \arrow[r, dash] \arrow[ddr, dash] & \Sigma_2 \arrow[r, dash] \arrow[ddr, dash] & \Sigma_3 & \\
P \arrow[ur, dash] \arrow[dr, dash] & & & & \cdots \\
& \Pi_1 \arrow[r, dash] \arrow[uur, dash] & \Pi_2 \arrow[r, dash] \arrow[uur, dash] & \Pi_3 &
\end{tikzcd}$$

Polynomial hierarchy (PH)

Formally $PH = \bigcup_{i \geq 0} \Sigma_i^P = \bigcup_{i \geq 0} \Pi_i^P$

Recall: TQBF is in PSPACE

$\therefore$ PH = PSPACE, right? WRONG!

$P = \bigcup_{c \geq 0} TIME(n^c)$ $\qquad$ $TIME(n^n) \subseteq P$.

Current belief: All these classes are distinct.

"Polynomial hierarchy collapses" $\Leftrightarrow PH = \Sigma_i / \Pi_i$

What if $\Sigma_i = \Pi_i$?

Thm: For any $i \geq 1$, if $\Sigma_i = \Pi_i$, then $PH = \Sigma_i$.

Pf (for $i = 1$) Assume $\Sigma_1^P = \Pi_1^P$ ie $NP = coNP$.

Say $L \in \Sigma_2^P$. $\Rightarrow$ There is a poly time M and q

s.t. $x \in L \iff \exists y \in \{0,1\}^{q(|x|)} \; \forall z \in \{0,1\}^{q(|x|)} . \; M(x,y,z)=1$

$$L' = \left\{ (x,y) : \; \forall z \in \{0,1\}^{q(|x|)} \quad N(x,y,z)=1 \right\}.$$

$\in$ co NP $=$ NP (by assumption).

$\Rightarrow$ There is M' poly time, and a polynomial q'
s.t.
$(x,y) \in L' \iff \exists w \in \{0,1\}^{q'(|x|+|y|)} . \quad M'(x,y,w)=1$

$\therefore \; x \in L \iff \exists y : \; (x,y) \in L'$

$= \exists y \; \exists w . \quad M'(x,y,w)=1$

$\in$ NP.

$\therefore \; \Sigma_2 = NP.$ $\square$

Next class:

More on PH

(Doesn't Σ_2^P look a lot like NP^{NP} ?)