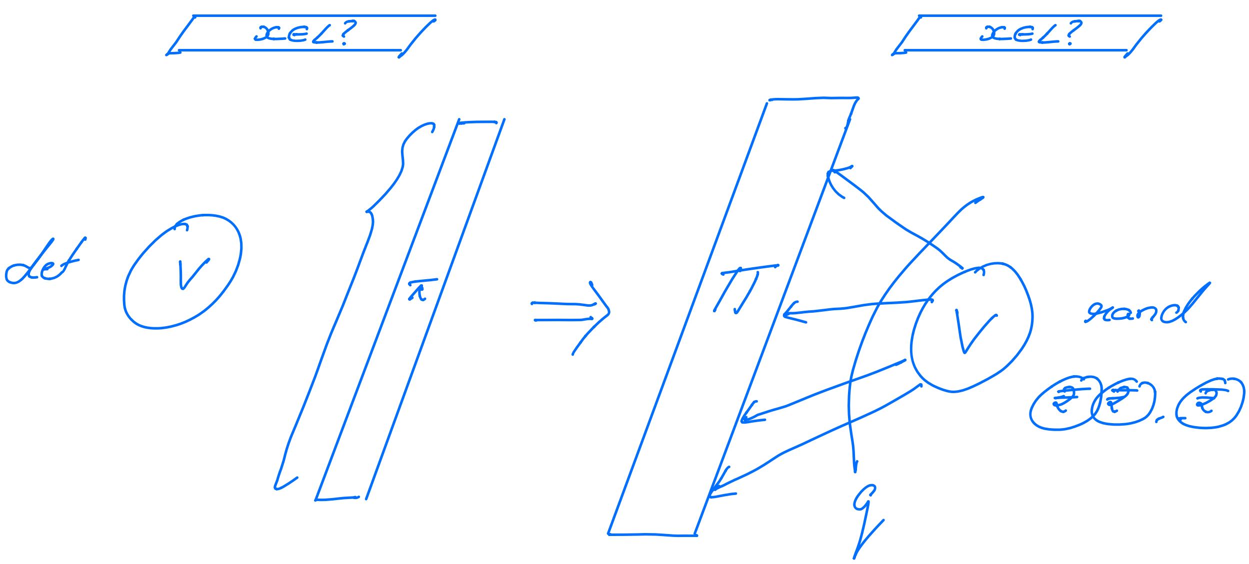
Today

- PCPs and hardness of approximation.

CSS.203.1
Computational Complexity
- Lecture #29 (31 May 21)
Instructor: Prahladh Harsha

Recap (from last time)

L - language.



$L \in PCP_{1,1/2}[r, q, m, t, a]$

$\begin{cases} r = \text{\# random coins} \\ q = \text{\# queries} \end{cases}$

$\begin{cases} m = \text{proof length} \\ t = \text{running time of verifier} \\ a = \text{size of predicate} \end{cases}$

Drop

$\begin{Bmatrix} m \le q \cdot 2^r \\ t \le \text{poly}(n) \\ a = \text{poly}(q) \end{Bmatrix} \Leftrightarrow$ (the m, t, a above)

PCP Theorem I:

There exists a Q s.t.
$\forall L \in NP$, $\exists$ a constant c
$L \in PCP_{1,1/2}[c \log n, Q]$

Today

Approximation Algorithms:

NP-hard combinatorial optimization problem

- Vertex Cover
- Max Clique
- TSP
- Max SAT

Can we design approximation alg for these problems instead.

Maximization Problem: Φ

$\alpha \in (0,1)$

A - α-approx for Φ

$$x \longrightarrow \boxed{A} \longrightarrow A(x)$$

$$\alpha \cdot OPT_{\Phi}(x) \leq A(x) \leq OPT_{\Phi}(x)$$

(For minimization problems $OPT_{\Phi}(x) \leq A(x) \leq \frac{1}{\alpha} \cdot OPT_{\Phi}(x)$)

Vertex Cover: 1/2-approximation

Maximal matching MM

$VC(G) \leq 2\,|MM(G)| \leq 2 \cdot VC(G)$

MAX3SAT: 7/8-approximation

Input: $\varphi = C_1 \wedge C_2 \ldots \wedge C_m$

C_i - clause of 3 literals

Goal: Find the max # of clauses that can be simultaneously satisfied by an assign.

TSP: 2/3-approximation (Christofides alg, 1976)

(metric instances)

Qn: Given a max/min problem, what is the best factor upto which we can can approximate?

Limit to approximation ??

—

Understanding hardness of approximation problem

$\downarrow$

Construct the corresponding decision problems

Focus attention on MAX3SAT

α-approximation for MAX3SAT

Definition : $\alpha \in (0,1)$. gap_α-MAX3SAT (promise problem)

YES $= \{(\varphi, k) \mid \exists$ an assign that satisfies $\geq k$ clauses$\}$

NO $= \{(\varphi, k) \mid$ Every assign sats $< \alpha k$ number of clauses$\}$

Prop: α-approximating MAX3SAT is polytime

$\Updownarrow$

gap_α-MAX3SAT is polytime

Pf: ($\Rightarrow$) Suppose A is an α-approx alg for MAX3SAT

B: On input (φ, k)

1. Run A on φ & let $k' = A(\varphi)$
2. Accept if $k' \geq \alpha k$.

$(\varphi, k) \in \text{YES} \Rightarrow \text{OPT}(\varphi) \geq k$

$\Rightarrow A(\varphi) \geq \alpha \cdot \text{OPT}(\varphi) \geq \alpha k$

$\Rightarrow B$ o/p YES ✓

$(\varphi, k) \in NO \Rightarrow \quad OPT(\varphi) < \alpha k$

$\qquad \Rightarrow \quad A(\varphi) < OPT(\varphi) < \alpha k$

$\qquad \Rightarrow \quad B$ o/p NO ✓

$(\Leftarrow)$ Suppose B is an alg for gap$_\alpha$-MAX3SAT

A: On input φ

1. $m \leftarrow$ # clauses in φ
2. Run $B(\varphi, i)$ for $i \leftarrow 1$ to m
3. Let $k^* \leftarrow$ largest k s.t. $B(\varphi, k) = YES$
4. Output αk^*.

$$B(\varphi, k^*+1) = NO \quad \& \quad B(\varphi, k^*) = YES$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$(\varphi, k^*+1) \notin YES \qquad (\varphi, k^*) \notin NO$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$OPT(\varphi) \le k^* \qquad OPT(\varphi) \ge \alpha k^*$$

$$\alpha \cdot OPT(\varphi) \le \alpha k^* \le OPT(\varphi)$$

Hence, A is an α-approx $\boxtimes$

PCP Theorem II: $\exists$ an $\alpha \in (0,1)$ such that SAT is ptime reducible to gap_α-MAX3SAT. (ie, $\exists$ ptime redn R s.t

$\psi \in SAT \Rightarrow R(\psi) = (\varphi, k) \in YES$

$\psi \notin SAT \Rightarrow R(\psi) = (\varphi, k) \in NO$.

ie, $\exists \alpha \in (0,1)$, gap_α-MAX3SAT is NP-hard

Cor: $\exists \alpha \in (0,1)$, α-approximating MAX3SAT is NP-hard.

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gap_α-MAX3SAT*

YES $= \{\varphi \mid \varphi \in SAT\}$

NO $= \{\varphi \mid \forall$ assignments at most αm clauses are satisfied $\}$

PCP Theorem III: $\exists \alpha \in (0,1)$ gap_α-MAX3SAT* is NP-hard.

Obs: PCP Thm III $\Rightarrow$ PCP Thm II.

Lemma: PCP Theorem I $\iff$ PCP Theorem III

Proof:

($\Leftarrow$) PCP Thm III $\Rightarrow$ PCP Thm I

Assume $SAT \leq_p \text{gap}_\alpha\text{-MAX3SAT}^*$ for some $\alpha \in (0,1)$.

Let $L \in NP$

$L \leq_p \text{gap}_\alpha\text{-MAX3SAT}^*$.

$x \xmapsto{R} \varphi$

$x \in L \Rightarrow \varphi \in SAT$

$x \notin L \Rightarrow$ Every assign sat at most αm clauses ($m = \#\text{clauses}(\varphi)$)

Restricted Verifier. V

On input x

1. Run redn R to obtain $\varphi = R(x)$
2. Pick a random clause C of φ
3. Query the 3 vars of φ
   > accept if ~~all~~ they satisfy the

Expect as proof π an assign to φ

clause.

$x \in L \Rightarrow \varphi \in SAT \Rightarrow \Pr_{R}[V^{\pi}(x;R) = acc] = 1$

$x \notin L \Rightarrow$ Every assign sat at most αm clauses ($m = \#$clauses(φ))

$\Rightarrow \forall \pi, \Pr_{R}[V^{\pi}(x;R) = acc] \le \alpha$

$L \in PCP_{1,\alpha}[\log m, 3]$
$= PCP_{1,\alpha}[c \log n, 3]$ (since $m = n^c$)
$\le PCP_{1,\alpha^k}[kc\log n, 3k]$

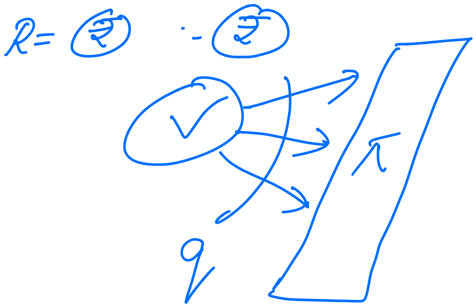
($\Rightarrow$) PCP Theorem I $\Rightarrow$ PCP Theorem II

Assume SAT has $(c\log n, Q)$-restricted verifier for some constant c & Q.

We need to find a <u>ptime</u> redn from SAT to gap_{α}-MAX3SAT* for some $\alpha \in (0,1)$.

$SAT \longrightarrow gap_{\alpha}\text{-}MAX3SAT^*$

First $SAT \longrightarrow gap_{1/2}\text{-}MAXQSAT^*$



MAX Q "SAT" instance.

Vars: Proof bits of π

Clauses:

One for each R

D_R - predicate on Q locations

$$\Phi = \bigwedge_R \underbrace{D_R}_{Q\text{-arity}}$$

Running time of redn $= t \cdot 2^r = \text{poly}(n)$

$$\text{SAT} \longrightarrow \text{gap}_{1/2}\text{-MAXQSAT}^*$$

Observation: $\forall Q$, $\exists$ $\ell(Q)$ & $k(Q)$ s.t any Boolean fn D on Q vars can be encoded by a 3CNF formula φ_D on $\underbrace{Q + \ell(Q)}$ vars & $k(Q)$ clauses

Q - original var

$\ell(Q)$ - additional vars. s.t

$D(a) = 1 \Rightarrow \exists$ a setting b to additional var $\varphi_D(a,b) = 1$

$D(a) \neq 1 \Rightarrow \forall$ settings b of additional vars $\varphi_D(a,b) \neq 1$.

$$\text{SAT} \longrightarrow \text{gap}_\alpha\text{-MAXQSAT} \longrightarrow \text{gap}_\alpha\text{-MAX3SAT}$$

$$x \longmapsto \underbrace{\bigwedge_R D_R}_{=\Phi} \longrightarrow \underbrace{\bigwedge_R \varphi_{D_R}}_{=\Psi}$$

YES: Ψ is a satisfiable instance

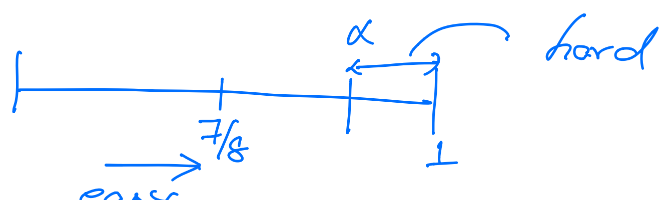
NO: Max fraction #clauses satisfied, $\leq \frac{1}{2}\cdot 1 + \frac{1}{2}\left(1-\frac{1}{k}\right)$

$$= 1 - \frac{1}{k} + \frac{1}{2k}$$

$$= \underbrace{1 - \frac{1}{2k}}_{=\alpha} < 1$$

Redn SAT $\longrightarrow$ $\text{gap}_{1-1/2k}\text{-MAX3SAT}^*$.

Conclusion: $\exists\ \alpha \in (0,1)$ s.t. α-approximating MAX3SAT is NP-hard.



[Hastad]: $\forall \varepsilon > 0$: $\left(\frac{7}{8}+\varepsilon\right)$-approx MAX3SAT is NP-hard.

Next lecture: [Feige-Goldwasser-Lovasz-Safra-Szegedy]

PCP Theorem I $\Rightarrow$ NP-hardness of approximating CLIQUE $\boxtimes$