| CSS.205.1: Toolkit for TCS | | Jan-May 2025 |
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| | **HW1 : Spectral Methods** | |
| **Out:** 13 Feb, 2025 | | **Due:** 27 Feb, 2025 |

- *This homework has problems worth 30 points.*
- *Please take time to write clear and concise solutions. You are* STRONGLY *encouraged to submit LaTeXed solutions.*
- *Collaboration is OK, but please write your answers yourself, and include in your answers the names of* EVERYONE *you collaborated with and* ALL *references other than class notes you consulted. However,* ***not*** *acknowledging your collaborators and references will be treated as a serious case of* ***academic dishonesty****.*

1. **[bipartite]** (3 points)

   Let $A$ be the adjacency matrix of an $n$-vertex undirected *connected* graph $G$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the $n$ eigenvalues of $A$ arranged in non-increasing order. In lecture, we had shown that $\mu_1 > -\mu_n$.

   (a) (2 points) Prove that if $\mu_1 = -\mu_n$, then the graph is bipartite.

   (b) (1 point) Prove that if the graph is bipartite, then $\mu_1 = -\mu_n$.

2. **[Colouring using largest eigenvector]** (4 points)

   Let $A$ be the adjacency matrix of an undirected graph connected $G$, and let $\phi$ be its top eigenvector with eigenvalue $\kappa$. Note that $\kappa \geq 0$ and $\phi$ can be so chosen as to have non-negative entries. Let us assume further that we write $\phi$ so that its entries are arranged in descending order (so that $\phi_1 \leq \phi_2 \leq \cdots \leq \phi_n$). Note that this induces an ordering on the vertices of $G$.

   Consider now the following colouring procedure, which is based on the above order. We start with an empty list $L$ of colours. We then process the vertices $u_1, u_2, \ldots, u_n$ in order, and for any given $i$, construct the set $S$ of the colours assigned to neighbors $u_j$ of $u_i$ with $j < i$. If $S = L$, then we create a new color $c$, set $L = L \cup \{c\}$ and assign the colour $c$ to $u_i$. Otherwise, if $L \setminus S$ is non-empty, we choose a color from $L \setminus S$ (according to some pre-defined choice rule) and assign it to $u_i$. Note that this procedure produces a proper coloring of the graph.

   How large can $L$ can be at the end of the algorithm? Show that your bound is tight by giving an appropriate example.

3. **[Hall's drawing of graphs]** (3 points)

   Let $L_G = D_G - A_G$ be the Laplacian of an undirected graph $G = (V, E)$ with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and corresponding eigenvectots $\Psi_1 = \mathbb{1}/\sqrt{n}, \Psi_2, \ldots, \Psi_n$. For any positive integer $k < n$, Let $x_1, \ldots, x_k$ be orthonormal vectors that are all orthogonal to $\mathbb{1}$. Then prove that

   $$\sum_{i=1}^{k} \langle x_i, L_G x_i \rangle \geq \sum_{i=2}^{k+1} \lambda_i,$$

   and this inequality is tight only when $\langle x, \Psi_j \rangle = 0$ for all $j$ such that $\lambda_j > \lambda_{k+1}$.

4. **[Trevisan's robust characterization of bipartiteness]** (10 points)

   Let $G = (V, E)$ be an undirected unweighted connected graph. Let $W$ be the random-walk matrix and $L = I - W$ be the Laplacian. Recall that $W = D^{-1}A$ where $D = \text{Diag}(deg)$ is the diagonal matrix of degrees and $A$ is the adjacency matrix of $G$. Let $\pi$ be the stationary distribution and $\langle \cdot, \cdot \rangle_\pi$ the corresponding $\pi$-inner product. Recall that the quadratic form corresponding to the Laplacian satisfies the following.

   $$\langle f, Lf \rangle_\pi = \sum_{\{i,j\}} \pi(i) \cdot W(i,j) \cdot (f(i) - f(j))^2,$$

   where $\pi$ the stationary distribution of this random walk matrix The largest eigenvalue of the normalized Laplacian, denoted by $\gamma_n$, satisfies

   $$\gamma_n = \max_{f \neq 0} \frac{\langle f, Lf \rangle_\pi}{\langle f, f \rangle_\pi}.$$

(a) (1 point) **[bipartite $\Leftrightarrow \gamma_n = 2$]**
Prove that $\gamma_n \le 2$. Furthermore, prove that equality holds iff the graph $G$ is bipartite.

(b) (2 points) **[almost bipartite $\Rightarrow \gamma_n$ almost 2]**
Suppose the MAXCUT in $G$ has normalized cost at least $1-\varepsilon$. That is, there exists a cut $(S, V \setminus S)$ such $|E(S, V\setminus S)| \ge (1-\varepsilon)|E|$ where $E(S, V\setminus S) = \{\{u,v\} \in E : u \in S, v \notin S\}$. Prove that there is a non-zero vector $f : V \longrightarrow R$ such that

$$\langle f, Lf\rangle_\pi \ge 1 - \varepsilon,$$
$$\langle f, f\rangle_\pi = \pi(S) \le \frac{1}{2}.$$

Hence, conclude that $\gamma_n \ge 2(1-\varepsilon)$.

(c) (7 points) **[$\gamma_n$ almost 2 $\Rightarrow$ almost bipartite]**
In this part, we will prove the following theorem.

**Theorem.** *Let $\gamma_n \ge 2(1-\varepsilon)$ or equivalently there exists a non-zero vector $f\colon V \longrightarrow \mathbb{R}$ such that $\langle f, (I+W)f\rangle_\pi \le 2\varepsilon \cdot \langle f, f\rangle_\pi$. Then there exists non-zero vector $y \in \{-1,0,1\}^V$ such that*

$$\frac{\sum_{\{i,j\}\in E} |y_i + y_j|}{\sum_{i\in V} d_i |y_i|} \le \sqrt{8\epsilon}.$$

To this end, we define the following randomized process that constructs a random non-zero vector $Y \in \{-1,0,1\}^V$ given a non-zero vector $f\colon V \longrightarrow \mathbb{R}$ satisfying $\langle f, (I+W)f\rangle_\pi \le 2\varepsilon \cdot \langle f, f\rangle_\pi$. Since this latter condition is scale-invariant, we may assume wlog. that $\max_i |f(i)| = 1$ and let $i_* \in V$ such that $|f(i_*)| = 1$.

- Pick a value $t$ uniformly in $[0,1]$.
- Define $Y \in \{-1,0,1\}^V$ as follows:

$$Y_i = \begin{cases} -1 & \text{if } f(i) < -\sqrt{t}, \\ 1 & \text{if } f(i) > \sqrt{t}, \\ 0 & \text{otherwise, i.e., } |f(i)| \le \sqrt{t}. \end{cases}$$

i. (1 point) Prove that $\mathbf{P}[\exists i \in V, Y_i \ne 0] = 1$.

ii. (2 points ) Prove that for any $i, j \in V$, $\mathbf{E}[|Y_i|] = f(i)^2$ and $\mathbf{E}\left[|Y_i + Y_j|\right] \le |f(i) + f(j)| \cdot (|f(i)| + |f(j)|)$.

iii. (3 points) Prove that $\mathbf{E}\left[\sum_{\{i,j\}\in E} |Y_i + Y_j|\right] \le \sqrt{8\epsilon} \cdot \mathbf{E}\left[\sum_i d_i |Y_i|\right]$.
**Hint:** Cauchy-Schwarz Inequality.

iv. (1 point) Hence, conclude that there exists a non-zero vector $y \in \{-1,0,1\}^V$ such that $\sum_{\{i,j\}\in E} |y_i + y_j| \le \sqrt{8\epsilon}. \cdot \sum_{i\in V} d_i |y_i|$.

**Discussion.** *It is known that $G$ is connected iff $\gamma_2 \ne 0$. Or equivalently, $\phi(G) \ne 0$ iff $\gamma_2 \ne 0$. Cheeger's inequalities give a "quantitative strengthening" of this statement by showing that*

$$\sqrt{2\gamma_2} \ge \phi(G) \ge \gamma_2/2.$$

*The problem is similar in spirit but works with $\gamma_n$ and "bipartiteness" instead of $\gamma_2$ and "connectedness".*

*Define the* bipartiteness ratio number *of a graph $G$ to be*

$$\beta(G) := \min_{y\in\{-1,0,1\}^V} \frac{\sum_{\{u,v\}\in E} |y_u + y_v|}{2d \sum_{u\in V} |y_u|},$$

*which is equivalent to*

$$\beta(G) = \min_{S\subseteq V, (L,R) \text{ partition of } S} \frac{2\partial(L,L) + 2\partial(R,R) + \partial(S, V\setminus S)}{d|S|},$$

*Observe that $\beta(G) = 0$ iff $G$ is bipartite. Problem 4a shows that $\beta(G) = 0$ iff $\gamma_n = 2$. Problems 4b–4c are a quantitative strengthening of this claim as they demonstrate that*

$$\sqrt{2(2-\gamma_n)} \geq \beta(G) \geq \frac{1}{2} \cdot (2-\gamma_n).$$

*This result is due to Luca Trevisan.*

5. **[Chernoff bound for Expander Random-Walks]** (10 points)

In lecture, we proved the following hitting-set lemma for random-walks on spectral expanders.

**Lemma** (hitting-set lemma for expanders)**.** *Let $W$ be a reversible random walk on a set $V$ of $n$ vertices and $1 = \lambda_1 \geq \lambda_2 \geq \cdot \geq \lambda_n \geq -1$ be its eigenvalues and $\pi$ the stationary distribution. Let $\lambda = \max\{\lambda_2, |\lambda_n|\}$. Let $B \subseteq V$ such that $\mu := \pi(B)$. Let $X_1, \ldots, X_t$ be a random-walk of length $t$ according to $W$ where the first vertex $X_1$ is chosen according to the stationary distribution. Then*

$$\mathbf{P}\left[\bigwedge_{i\in[t]} (X_i \in B)\right] \leq \mu \cdot (\mu + \lambda(1-\mu))^{t-1}.$$

The proof discussed in class actually proves the following more general lemma.

**Lemma** (generalized hitting-set lemma for expanders)**.** *Let $W^(1), \ldots, W^{(t-1)}$ be $(t-1)$ reversible random walk matrices all on the same set $V$ of $n$ and sharing the same stationary distribution $\pi$. Let $1 = \lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \cdot \geq \lambda_n^{(i)} \geq -1$ be the eigenvalues of the $i$-th random walk $W^{(i)}$. Let $\lambda = \max_i\{\lambda_2^{(i)}, |\lambda_n^{(i)}|\}$. Let $B \subseteq V$ such that $\mu := \pi(B)$. Let $X_1, \ldots, X_t$ be a random sequence of $t$ vertices where the first vertex $X_1$ is chosen according to the (common) stationary distribution $\pi$ and $X_i$ is a random neighbour of $X_{i-1}$ according to the random walk $W^{(i)}$. Then*

$$\mathbf{P}\left[\bigwedge_{i\in[t]} (X_i \in B)\right] \leq \mu \cdot (\mu + \lambda(1-\mu))^{t-1}. \tag{1}$$

In this problem, we will extend this to obtain the following Chernoff-like bound on expander random-walks.

**Theorem.** *Let $W$ be a reversible random walk on a set $V$ of $n$ vertices and $1 = \lambda_1 \geq \lambda_2 \geq \cdot \geq \lambda_n \geq -1$ be its eigenvalues and $\pi$ the stationary distribution. Let $\lambda = \max\{\lambda_2, |\lambda_n|\}$. Let $B \subseteq V$ such that $\mu := \pi(B)$. Let $X_1, \ldots, X_t$ be a random-walk of length $t$ according to $W$ where the first vertex $X_1$ is chosen according to the stationary distribution. Then for any $\delta \in (0, 1)$, we have*

$$\mathbf{P}\left[\#\{i\colon X_i \in B\} \geq (\mu + \lambda(1-\mu) + \delta)t\right] \leq \exp\left(-\lambda(\delta^2 t)\right).$$

Let $S \subseteq [t]$ be a random subset chosen as follows: for each $i \in [t]$, independently add $i$ to $S$ with probability $q$. This satisfies that for any fixed set $s \subseteq [t]$, we have $\mathbf{P}[S = s] = q^k \cdot (1-q)^{t-k}$ where $k = |s|$.

(a) (2 points) Use (1) to conclude that for each integer $0 \leq k \leq t$

$$\mathbf{P}_{X_1,\cdots,X_t,S}\left[\bigwedge_{i\in S}(X_i \in B) \;\middle|\; |S| = k\right] \leq \mu \cdot (\mu + \lambda(1-\mu))^{k-1} \leq (\mu + \lambda(1-\mu))^k.$$

Note that the above probability is over the random choice of the walk $X_1, \ldots, X_t$ as well as the set $S$ conditioned on the fact that $|S| = k$.

(b) (4 points) Show that

$$\mathbf{P}_{X_1,\cdots,X_t,S}\left[\bigwedge_{i\in S}(X_i \in B)\right] \leq \left(q \cdot (\mu + \lambda(1-\mu)) + 1 - q\right)^t.$$

(c) (3 points) Let $X_B$ be the random subset of $[t]$ defined as follows:

$$X_B := \{i \in [t] \colon X_i \in B\}.$$

Show that

$$\mathbf{P}\left[|X_B| \geq (\mu+\varepsilon)t\right] \leq \left(\frac{q \cdot (\mu + \lambda(1-\mu)) + 1 - q}{(1-q)^{1-\mu-\varepsilon}}\right)^t .$$

(d) (1 point) Let $\varepsilon > \lambda(1-\mu)$. Use calculus to show that the right hand side of the above expression is minimized when

$$q = \frac{\varepsilon - \lambda(1-\mu)}{(1-\mu-\lambda(1-\mu)) \cdot (\mu+\varepsilon)},$$

to obtain

$$\begin{aligned}\mathbf{P}\left[|X_B| \geq (\mu+\varepsilon)t\right] &\leq \left[\left(\frac{mu + \lambda(1-\mu)}{\mu+\varepsilon}\right)^{\mu+\varepsilon} \cdot \left(\frac{1-\mu-\lambda(1-\mu)}{1-\mu-\varepsilon}\right)^{1-\mu-\varepsilon}\right]^t \\ &= \exp\left(-D_{KL}(\mu+\varepsilon \| \mu+\lambda(1-\mu)) \cdot t\right).\end{aligned}$$

**Discussion.** *This proof of the Chernoff bound on expander random-walks is due to Impagliazzo and Kabanets. A proof can of the standard Chernoff bound can also be obtained along similar lines. For expander random-walks, a stronger Chernoff bound is known due to Gillman.*