Algebra and Computation

Problem Set 1

Due date: February 10th, 2019

Instructions

1. The problem set has **10 questions** with a total score of **70 points**.
2. The due date is **Sunday, February 10$^{\text{th}}$, 2019**. Questions marked with a ($\star$) are optional (but recommended).
3. You are welcome to discuss with other classmates **as long as these discussion are reasonable**; these are not meant for one to solve the problem for the other. You are eventually expected to find and write your own solutions and code.

   If you do discuss, you are expected to explicitly mention who you discussed with and which parts of your solution came from these discussions.
4. Solutions are expected as a LaTeX documents and sage worksheets (`.ipynb` files).

Questions (for sage)

Question 1. [5 points] Construct a subgroup of `SymmetricGroup(20)` that is generated by the all 3-cycles. Compute its order. What is your guess on what this group is?
($\star$): Can you prove your guess?

Question 2. [10 points] Construct the group for the $2 \times 2 \times 2$ Rubik's cube. Compute its order. Can you explain that number?

Question 3. [10 points] Construct the group for the Top-spin puzzle for all n in the range $\{20, 21, \ldots, 30\}$. Compute the orders for each. Can you make a guess about what this group is for general n?

You aren't expected to prove your guess.

Btw, in python, `range(1,5) = [1,2,3,4]`, and `tuple(range(1,5)) = (1,2,3,4)`.

Question 4. [5 points] Sage allows you to pick a random element from a group via the function `G.random_element()`. Pick a random element `rho` from `G` and do the following for a few times:

> Pick a random element τ from G. Replace ρ with $\rho\tau\rho^{-1}\tau^{-1}$. Keep printing ρ. Do you see anything unusual?

Now let `H = G.sylow_subgroup(2)` (we will learn about Sylow groups later in the course). Now pick a random ρ from H and repeat the previous experiment:

> Pick a random element τ from H. Replace ρ with $\rho\tau\rho^{-1}\tau^{-1}$. Keep printing ρ. What do you see?

Questions

Question 5. [5 points] Which of the following maps are legitimate group actions of a group G on itself?

1. $g(h) = gh$	5. $g(h) = ghg$
2. $g(h) = g^{-1}h$	6. $g(h) = g^{-1}hg$
3. $g(h) = hg$	7. $g(h) = ghg^{-1}$
4. $g(h) = hg^{-1}$	8. $g(h) = hgh^{-1}$

Question 6. [5 points] Let G, H be two graphs. If $\mathrm{Iso}(G, H)$ is the set of all isomorphisms between G and H, show that it is either empty or a coset of $\mathrm{Aut}(H)$.

Question 7. [5 points] Suppose a group G acts on Ω and let $\alpha, \beta \in \Omega$ with $\beta \in \alpha^G$. If $G_\alpha = \mathrm{stab}_G(\alpha)$ and $G_\beta = \mathrm{stab}_G(\beta)$, show that there is some $g \in G$ such that $G_\alpha = g \cdot G_\beta \cdot g^{-1}$.

Question 8. [5 points] Let $\Phi : G \to H$ be a homomorphism between two groups that is surjective (onto). Show that the map is 1-1 if and only if $\ker(\Phi) = \{\mathrm{id}_G\}$.

[5 points] Furthermore, show that $\frac{G}{\ker(\Phi)} \cong H$. That is, there is a 1-1 and onto homomorphism from $\frac{G}{\ker(\Phi)}$ to H.

Question 9. [10 points] Let $H \le G$ and say $\{a_1H, \ldots, a_rH\}$ are the set of cosets of H in G. We would like to define a group structure on the cosets of H via

$$(a_1H) \cdot (a_2H) := (a_1a_2H).$$

Show that this is a well-defined operation *if and only if* $aha^{-1} \in H$ for every $a \in G$ and $h \in H$.

Question 10. [5 points] Prove or disprove:

> If $A \trianglelefteq B$ and $B \trianglelefteq C$, then $A \trianglelefteq C$.
