Problem Set 1

- Due date: **5th March 2023** (released on 19th February 2023)
- The points for each problem is indicated on the side. The total for this set is **100** points but you can solve any **60** points worth of questions.
- Turn in your problem sets electronically (PDF; either LaTeXed or scanned etc.) via email.
- You are expected to solve the problems individually and not discuss with each other.
- Referring to sources other than the text book and class notes is STRONGLY DISCOURAGED. But if you do use an external source (eg.,other text books, lecture notes, or any material available online), ACKNOWLEDGE all your sources in your writeup. This will not affect your grades. However, not acknowledging will be treated as a serious case of academic dishonesty.
- Be clear in your writing.

1. **[Basic properties of algebraic sets]** (10)

   (a) Give an example of a countable collection of algebraic sets whose union is *not* an algebraic set.

   (b) Suppose $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ are algebraic sets, show that the *product*

   $$V \times W = \{(a_1, \ldots, a_n, b_1, \ldots, b_m) \ : \ (a_1, \ldots, a_n) \in V \ , \ (b_1, \ldots, b_m) \in W\}$$

   is also an algebraic set (in $\mathbb{A}^{m+n}$).

   (c) If you are working over a finite field $\mathbb{F}$, which subsets of $\mathbb{F}$ are algebraic sets?

2. **[Identifying algebraic sets]** (10)

   Which of the following are algebraic sets? Justify your answer with proofs.

   (a) $\left\{(\cos\theta, \sin\theta) \in \mathbb{A}^2(\mathbb{R}) \ : \ \theta \in \mathbb{R}\right\}$

   (b) $\left\{(\theta, \cos\theta, \sin\theta) \in \mathbb{A}^2(\mathbb{R}) \ : \ \theta \in \mathbb{R}\right\}$

   (c) $\left\{(z, w) \in \mathbb{A}^2(\mathbb{C}) \ : \ |z|^2 + |w|^2 = 1\right\}$ where $|a + \imath b|^2 = a^2 + b^2$.

3. **[Varieties and prime ideals]** (10)

   (a) Recall the proof that a an algebraic set V is a variety (irreducible algebraic set) if and only if $\mathbb{I}(V)$ is prime. Convince yourself that we did not need the underlying field to be algebraically closed.

   (b) Show that the polynomial $F(x, y) = y^2 + x^2(x-1)^2 \in \mathbb{R}[x, y]$ is an irreducible real polynomial.

   (c) Compute $\mathbb{V}(F) \subseteq \mathbb{A}^2(\mathbb{R})$ and show that it is reducible.

   (d) How do you reconcile the above answers?

4. **[A special algebraic set]** (5)

   (a) Show that the polynomial $f(x, y) = y^2 - x^3 + x$ is irreducible over $\mathbb{R}$.

   (b) Try and plot the set $\mathbb{V}(f) \subseteq \mathbb{A}^2(\mathbb{R})$.

(c) Is the algebraic set irreducible? Elaborate on the paradox.

5. **[Exhibiting primality of an ideal]** (10)

   Let $I = \langle x^2 - y^3, y^2 - z^3 \rangle \subseteq k[x,y,z]$. Define the map $\alpha : k[x,y,z] \to k[t]$ given by $x, y, z \mapsto t^9, t^6, t^4$ respectively.

   (a) Show that every element of $k[x,y,z]/I$ has a residue of the form $a(z) + x \cdot b(z) + y \cdot c(z) + xy \cdot d(z)$ where $a, b, c, d \in k[z]$.

   (b) Suppose $F = a(z) + x \cdot b(z) + y \cdot c(z) + xy \cdot d(z)$ as above, and $\alpha(F) = 0$, show that $F = 0$.

   (c) Use this to argue that $\ker(\alpha) = I$ and that I is prime and $\mathbb{V}(I)$ is therefore irreducible.

6. **[Zariski's lemma]** (20)

   Prove the following fact.

   > If k is a field and $D \subseteq k$ is a domain such that k is integral over D, show that D is infact a field.

   With the above, we can see a slightly different proof of Zariski's lemma. Suppose $k \subseteq L$ are fields with $L = k[\alpha_1, \ldots, \alpha_n]$ (i.e., L is ring-finite over k). We wish to show that L is actually an algebraic extension of k.

   Let us assume the contrary, and assume without loss of generality that α_1 is *not* algebraic over k. Then, $L = k(\alpha_1)[\alpha_1, \ldots, \alpha_n]$. By induction, we have that $\alpha_2, \ldots, \alpha_n$ are all algebraic over $k(\alpha_1)$.

   (a) Show that there exists $f_2, \ldots, f_n \in k[\alpha_1]$ such that $f_i \alpha_i$ is integral over $k[\alpha_1]$.

   (b) Show that $D = k\left[\alpha_1, \frac{1}{f_2}, \ldots, \frac{1}{f_n}\right]$ is a domain and that $\alpha_2, \ldots, \alpha_n$ are integral over D.

   (c) Use the above fact to conclude that D must be a field and hence $D = k(\alpha_1)$.

   Thus, this leads us to conclude that $D = k(\alpha_1)$ is ring-finite over k as $D = k\left[\alpha_1, \frac{1}{f_2}, \ldots, \frac{1}{f_n}\right]$ which we have already seen is absurd as $k(\alpha_1) \equiv k(x)$. That concludes the proof of Zariski's lemma.

7. **[Integral basis for an algebraic extension]** (5)

   Let D be a domain and k its fraction field. Suppose L is an algebraic extension of k. Show that there exists $v_1, \ldots, v_r \in L$ such that each $\{v_1, \ldots, v_r\}$ is a basis for L as a k-vector space such that each v_i is integral over D.

8. **[Inverse image under polynomial maps]** (10)

   Suppose $F : k^n \to k^m$ is a polynomial map, and say $W \subseteq \mathbb{A}^m(k)$ is an algebraic set with $W = \mathbb{V}(I)$. Show that the set $F^{-1}(W) = \{(a_1, \ldots, a_n) \in k^n \ : \ F(a) \in W\}$ is an algebraic set by proving formally that

   $$F^{-1}(W) = \mathbb{V}\left(\left\langle G^F = G(F_1, \ldots, F_m) \ : \ G \in I \right\rangle\right).$$

   Note: This is an important exercise. This shows that if $F : k^n \to k^m$ is a polynomial map, then inverse images of algebraic sets are algebraic sets (and thus inverse images of complements of algebraic sets are complements of algebraic sets). In other words, polynomial maps are continuous maps for the Zariski topology.

If $V \subseteq A^n(k)$ is an algebraic set, is it always the case that image of V under F, given by $\{F(a_1, \ldots, a_n) \ : \ (a_1, \ldots, a_n) \in V\}$, is an algebraic set? Justify your answer.

9. **[Pole sets]** (10)

   (a) Let $V = \mathbb{V}(x^2 + y^2 - 1) \in A^2(k)$ and consider $f = \frac{y}{1-x} \in k(V)$. What is the pole set of f? What is the pole set of f^2?

   (b) Let $V = \mathbb{V}(xw - yz) \in \mathbb{A}^4(k)$. Consider $f = \frac{x}{y} \in k(V)$. Formally show that f is defined whenever $y \neq 0$ or $w \neq 0$.

   Also show that, although f is defined everywhere on $V \setminus \mathbb{V}(yw)$, it is impossible to express $f = \frac{a}{b}$ for polynomials a, b in such a way that that b is defined everywhere on $V \setminus \mathbb{V}(yw)$. (That is, if you wish to represent the function $f \in k(V)$ over the entire domain of definition, you would need to use more than one rational representation of f.)

10. **[Local to global properties]** (10)

   Let V be an algebraic set in $\mathbb{A}^n(k)$ where k is algebraically closed. For every point $\mathbf{a} = (a_1, \ldots, a_n) \in V$, we have an associated maximal ideal $\mathfrak{m}_{\mathbf{a}} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.

   (a) What do you think $\cap_{\mathbf{a} \in V} \mathfrak{m}_{\mathbf{a}}$ should be? Prove your answer formally.

   [Hint: Think about what an $f \in \cap_{\mathbf{a} \in V} \mathfrak{m}_{\mathbf{a}}$ means as a function on V.]

   (b) Suppose V is an irreducible algebraic set (i.e., an algebraic variety). Suppose $\alpha \in k(V)$ such that α is a unit in $\mathcal{O}_{\mathbf{a}}(V)$ for all $\mathbf{a} \in V$. Show that α is infact a unit in $\Gamma(V)$.