Problem Set 1

- Due date: **22 September, 2024** (released on 10 September, 2024).
- The points for each problem is indicated on the side.
- At the moment, the total for this set is **80** points.
- The problem set has a fair number of questions so please do not wait until close to the deadline to start on them. Try and do one question every couple of days.
- Turn in your problem sets electronically (PDF; either LaTeXed or scanned etc.) on Piazza via private post.
- Collaboration with other students taking this course is encouraged, but collaboration with others is not allowed. Irrespective of this, all writeups must be done individually and must include names of all collaborators (if any).
- Referring to sources other than the text book and class notes is STRONGLY DISCOURAGED. But if you do use an external source (eg.,other text books, lecture notes, or any material available online), ACKNOWLEDGE all your sources (including collaborators) in your writeup. This will not affect your grades. However, not acknowledging will be treated as a serious case of academic dishonesty.
- Be clear in your writing.

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1. **[Formulas and constant-width ABPs]** (10)

   Assume that $f(x_1, \ldots, x_n)$ is computable by an *algebraic branching program (ABP)* of size s. Additionally, if you knew that the ABP computing f is a *layered* graph (that is, the nodes are partitioned into disjoint layers and edges go only from layer i to layer $i+1$) with at most 10 vertices in each layer. Show that f is computable by an *algebraic formula* of size $\mathrm{poly}(s)$.

2. **[Homogenising ABPs]** (10)

   We shall say that an ABP is *homogeneous* if

   - the nodes can be partitions into disjoint layers, with edges only going from layer i to layer $i+1$,
   - all edge weights are of the homogeneous linear forms (i.e. linear functions without a constant term) such as $\alpha_1 x_1 + \cdots + \alpha_n x_n$ where $\alpha_i \in \mathbb{F}$ for all i.

   Show that any ABP computing a degree-d homogeneous polynomial $f(x_1, \ldots, x_n)$ can be converted to a *homogeneous ABP* of size $\mathrm{poly}(s, d)$ computing the same polynomial.

3. **[Complete homogeneous symmetric polynomials]** (10)

   Consider the *complete homogeneous symmetric polynomial* of degree d, denoted by $h_d(x_1, \ldots, x_n)$, that is a sum of all monomials of degree d (including non-multilinear monomials) with coefficient 1 each. For instance,

   $$h_3(x_1, x_2, \ldots, x_n) = \sum_i x_i^3 + \sum_{\substack{1 \le i,j \le n \\ i \ne j}} x_i^2 x_j + \sum_{1 \le i < j < k \le n} x_i x_j x_k.$$

   For any d, show that the polynomial $h_d(x_1, \ldots, x_n)$ is computable by a $\mathrm{poly}(n, d)$-size algebraic circuit over any large enough field.

4. **[Computing partial derivatives of circuits]** (3 + 3 + 3 + 3 + 8)

   Say we are given an algebraic circuit $C(x_1, \ldots, x_n)$ of size s that computes a degree-d polynomial $f(x_1, \ldots, x_n)$. Assume that the underlying field is $\mathbb{Q}$.

   (a) Show that the polynomial $\frac{\partial f}{\partial x_1}$ can be computed by an algebraic circuit without much blow-up in size.

   (b) What can you say if you are instead given a formula for f? Can you construct a formula computing $\frac{\partial f}{\partial x_1}$?

   (c) What if we have a $\Sigma\Pi\Sigma$ circuit computing f? Can you construct a $\Sigma\Pi\Sigma$ circuit computing $\frac{\partial f}{\partial x_1}$ of not-much-larger size?

   (d) What about computing $\frac{\partial^i f}{\partial x_1^i}$, for $i = \frac{d}{2}$ say?

   (e) Suppose we want to compute the higher-order derivative

   $$g = \frac{\partial^{n/2} f}{\partial x_1\, \partial x_2 \, \cdots \, \partial x_{n/2}}.$$

   Given f as a small algebraic circuit, do you think it is possible to compute g by a small algebraic circuit? Try and justify your answer.

5. **[Depth reduction to product-depth Δ]** (10)

   Define the product-depth of any circuit to be the maximum number of multiplication gates encountered on any root-to-leaf path.

   Show that for any polynomial n-variate degree d polynomial $f(x_1, \ldots, x_n)$ that can be computed by a size s algebraic circuit Φ, there is a circuit Φ' of size $s^{O(d^{1/\Delta})}$ and product-depth Δ computing f .

6. **[Newton Identities via generating functions]** (3 + 3 + 7 + 7)

   We shall use E_d to and P_d to denote the elementary symmetric polynomials and the power symmetric polynomials respectively (on $x_1, \ldots, x_n$):

   $$E_d = \sum_{S \subseteq [n], |S| = d} \prod_{i \in S} x_i \quad \text{and} \quad P_d = \sum_{i \in [n]} x_i^d.$$

   (a) Consider the *generating function* $Q(t) = (1 + x_1 t) \cdots (1 + x_n t) = \sum_{i=0}^n E_i t^i$. Show that

   $$t \cdot \frac{\partial Q}{\partial t} = Q(t) \cdot \sum_{i=1}^n \frac{x_i t}{(1 + x_i t)} = Q(t) \cdot \sum_{i=1}^n \left(x_i t - x_i^2 t^2 + x_i^3 t^3 - x_i^4 t^4 + \cdots \right).$$

   (b) By comparing coefficients above, conclude that for all $0 \leq k \leq n$ we have the following identity:

   $$kE_k = \sum_{i=1}^k (-1)^{i-1} E_{k-i} \cdot P_i.$$

   (c) Unpacking the above, for any $0 \leq k \leq n$, show that there is a polynomial $G_k(y_1, \ldots, y_k)$ such that $G(P_1, \ldots, P_k) = E_k$ and that G_k is computable by an homogeneous algebraic circuit of size $\operatorname{poly}(k)$.

   (d) Complete the sketch of the argument outlined in class to give a polynomial sized circuit for Det_n by looking at $\operatorname{Tr}(X), \operatorname{Tr}(X^2), \ldots, \operatorname{Tr}(X^n)$.