Problem Set 2

- Due date: **10 November, 2024** (released on 22 October, 2024).
- The points for each problem is indicated on the side. The total for this set is **70** points. **Answer any 60 points worth.**
- Turn in your problem sets electronically (PDF; either LaTeXed or scanned etc.) via email.
- Collaboration with other students taking this course is encouraged, but collaboration with others is not allowed. Irrespective of this, all writeups must be done individually and must include names of all collaborators (if any).
- Referring to sources other than the text book and class notes is STRONGLY DISCOURAGED. But if you do use an external source (eg.,other text books, lecture notes, or any material available online), ACKNOWLEDGE all your sources (including collaborators) in your writeup. This will not affect your grades. However, not acknowledging will be treated as a serious case of academic dishonesty.

---

1. **[Rectangular Permanents]** (15)

   Let d, n be positive integers with $d \leq n$. Define the polynomial $\mathrm{RPerm}_{d,n}$ on dn variables and degree d as follows:

   $$\mathrm{RPerm}_{d,n}(X) = \sum_{\substack{\sigma:[d]\to[n] \\ f \text{ is injective}}} \prod_{i=1}^{d} x_{i,f(i)}$$

   (If $d = n$, then this is just the standard permanent). Show that

   $$\mathrm{RPerm}_{d,n}(X) = \sum_{\Gamma \in \mathcal{P}([d])} (-1)^{d-|\Gamma|} \cdot \prod_{S \in \Gamma} (|S|-1)! \cdot \sum_{j=1}^{n} \prod_{i \in S} x_{ij}$$

   where $\mathcal{P}([d])$ is the set of all partitions of the set $[d]$.

   What is the size of the above circuit? (You can look up 'Bell numbers' and get a more accurate estimate.)

   For example, for $d = 2$, we have two partitions in $\mathcal{P}(\{1,2\})$ namely $\{\{1\},\{2\}\}$ and $\{\{1,2\}\}$. The RHS thus reduces to

   $$\mathrm{RPerm}_{2,n}(X) = (x_{11} + \cdots x_{1n})(x_{21} + \cdots + x_{2n}) - (x_{11}x_{21} + \cdots + x_{1n}x_{2n}).$$

2. **[Depth reduction to non-homogeneous product-depth Δ]** (10)

   Assume that you are working over a field $\mathbb{F}$ of characteristic zero (or very large). Suppose $f(x_1, \ldots, x_n)$ is an n-variate degree d polynomial computed by a circuit of size $s = \mathrm{poly}(n, d)$. What is the smallest product-depth Δ circuit (possibly non-homogeneous) you can build for this polynomial?

   [Hint: In the last problem set, you use Agrawal-Vinay's depth reduction repeatedly to get a product-depth Δ formula of size $s^{O(d^{1/\Delta})}$. Try using the GKKS depth-reduction to $\Sigma\Pi\Sigma$ instead.]

3. **[Rank of the partial derivative matrix]** (10)

   What is the rank of the partial derivative matrix for the following polynomials? (You don't need to give the exact answer. I am more interested in just getting a ball-park of poly(n) or exp(n) etc. If you can get a precise answer, you are welcome to provide that.)

   (a) $f = \mathrm{ESYM}_d(x_1, \ldots, x_n)$, under the partition $Y = \{x_1, \ldots, x_{n/2}\}$ and $Z = X \setminus Y$. (Take $d = 0.1n$.)

   (b) $h_d(x_1, \ldots, x_n)$ is the *complete homogeneous polynomial* of degree d (i.e., h_d is the sum of all monomials of degree d, each with coefficient 1), under the partition $Y = \{x_1, \ldots, x_{n/2}\}$ and $Z = \{x_{n/2+1}, \ldots, x_n\}$. (Take $d = 0.1n$.)

   (c) $f = \mathrm{Det}_n$ under the partition where Y consists of variables in the first $n/2$ rows, and Z consists of variables in the last $n/2$ rows.

   (d) $f = (x_1 + \cdots + x_n)^d$, under the partition $Y = \{x_1, \ldots, x_{n/2}\}$ and $Z = \{x_{n/2+1}, \ldots, x_n\}$.

4. **[q-product representations]** (2 + 2 + 2 + 3 + 1)

   The goal of the question is to prove a variant of the log-product representation for homogeneous constant-depth formulas.

   Let C be a size s product-depth Δ homogeneous formula computing an n-variate degree d polynomial $f(x_1, \ldots, x_n)$, with $\Delta < \ln d$. We will assume without loss of generality that the children of all multiplication gates are arranged in increasing order (from left to right), and that there are no gates computing scalars.

   A leaf ℓ of the formula C is said to be a "right-leaning" leaf if the unique path from the root to ℓ takes the right-most child of each multiplication gate encountered on the path.

   (a) Suppose ℓ is a "right-leaning" leaf, and say the path from root to ℓ encounters multiplication gates $g_1, \ldots, g_t$ (with g_1 closest to the root), and let g_i have r_i other children (besides the path leading to g_{i+1}). We can write $f = A_\ell \cdot [\ell] + C_{\ell=0}$ for some polynomial A_ℓ (like we have done in class several times). Show that the polynomial A_ℓ is a product of at least $(r_1 + \cdots + r_t) - t$ non-trivial polynomials.

   (b) Show that

   $$f = \sum_{\text{right-leaning } \ell} A_\ell \cdot [\ell].$$

   [Hint: What parse-trees are accounted for by $A_\ell \cdot [\ell]$?]

   (c) Show that $r_1 \cdots r_t \geq d$.

   [Hint: If $d_i = \deg(g_i)$, what can you say about $\frac{d_i}{d_{i+1}}$?]

   (d) Show that $td^{1/t} - t \geq \Delta d^{1/\Delta} - \Delta$ for all $t < \Delta$ as long as $\Delta < \ln d$.

   [Hint: Look at $g'(t)$ for the function $g(t) = td^{1/t} - t$.]

   (e) Conclude that we can write f as

   $$f = \sum_{i=1}^{s} g_{i1} \cdots g_{iq}$$

   with $q \geq \Delta d^{1/\Delta} - \Delta$

5. **[Evaluation dimension]** (5 + 5)

Suppose $X = Y \sqcup Z$ is a partition. Define the evaluation dimension of $f \in \mathbb{F}[X]$, denoted by $\mathrm{evalDim}_{Y,Z} f$, as follows:

$$\mathrm{evalDim}_{Y,Z}(f) = \dim \operatorname{span} \left\{ f(\mathbf{a}, Z) \;:\; \mathbf{a} \in \mathbb{F}^{|Y|} \right\}.$$

In words, consider the space of all "partial evaluations" of f, and look at the dimension of the span.

(a) What is the evaluation dimension of the following polynomials under an equipartition? (Again, a ball-park answer would be sufficient)
- $f = \mathrm{ESYM}_d(x_1, \ldots, x_n)$. (Take $d = 0.1n$.)
- $h_d(x_1, \ldots, x_n)$ is the *complete homogeneous polynomial* of degree d (i.e., h_d is the sum of all monomials of degree d, each with coefficient 1). (Take $d = 0.1n$.)
- $f = (x_1 + \cdots + x_n)^d$.

(b) Show that for any polynomial $f(X)$, we have that $\mathrm{evalDim}_{Y,Z}(f) = \operatorname{rank} M_{Y,Z}(f)$ where $M_{Y,Z}(f)$ is the communication matrix discussed in class.
(You'll get full credit even if you are able to show that the two are the same up to $\mathrm{poly}(n,d)$ factors.)

[Hint: Show that each of the partial evaluations $f(\mathbf{a}, Z)$ can be written as a linear combination of the rows of $M_{Y,Z}(f)$, also that each row of $M_{Y,Z}(f)$ can be written as a linear combination of the partial evaluations.]

6. **[The LST lower bound for arbitrary constant depth]** (3 + 6 + 6)

Let $\alpha = \frac{1}{\sqrt{2}}$.

(a) If a, b are positive integers, show that

$$|a\alpha - b| \geq \frac{1}{10(a+b)}.$$

(You can probably replace 10 by a smaller constant but any constant would do.)

[Hint: Consider $(a\alpha - b) \cdot (a\alpha + b)$ and use the fact that α is irrational and that $a, b > 0$.]

(b) Let $w_1, \ldots, w_d \in \{\lfloor \alpha k \rfloor, -k\}$, and let $X = X_1 \sqcup \cdots \sqcup X_d$ with $|X_i| = 2^{|w_i|}$. Let $Y = \bigcup_{i : w_i < 0} X_i$ and $Z = \bigcup_{i : w_i > 0} X_i$. Suppose $C = Q_1 \cdots Q_r$ is a set-multilinear product with $\deg(Q_i) \leq \tau$ for all i. Show that

$$\mathrm{relRank}_{Y,Z}(C) \leq \exp_2\left(-\frac{kd}{10\tau^2} + d \right)$$

(where $\exp_2(x) := 2^x$).

(c) Prove the following claim by induction on Δ:

Assume $k \gg d$. If C is a size s set-multilinear formula of product-depth Δ over the above partition, then

$$\mathrm{relRank}_{Y,Z}(C) \leq s \cdot \exp_2\left(-\frac{k d^{1/(2^\Delta - 1)}}{20} \right).$$

[Hint: Take $\tau = d^{(2^{\Delta-1}-1)/(2^\Delta - 1)}$]