End-semester exam: conceptual part

- The points for each problem is indicated on the side. The total for this set is **50 points**.
- You are allowed 90 minutes for this exam.
- Collaboration with other students is **not allowed**!
- You can refer to the summary scribe notes and any handwritten notes that you may have. All other sources are disallowed.
- Be clear in your writing. Try to fit your answer within the space provided. If you *really* feel like you need more space (although you shouldn't have to), you can use blank sheets.
- Even if you do not manage to solve some questions, explain your attempts and partial thoughts.
- If you don't write the word 'pineapple' on the last page, it will cost you 5 points.

Your name: ______________________________

1. **(5 points)**

   Which of the following statements imply $\mathsf{P} \neq \mathsf{NP}$? And are any of them already known to be true?

   Give brief but sufficient justifications for your answers.

   (a) There is a constant $c > 1$ such that $\mathsf{SAT} \notin \mathsf{DTIME}(n^c)$.

   (b) For all constants $c > 1$, $\mathsf{SAT} \notin \mathsf{DTIME}(n^c)$.

   (c) There is a constant $c > 1$ such that $\mathsf{NP} \not\subseteq \mathsf{DTIME}(n^c)$.

   (d) For all constants $c > 1$, $\mathsf{NP} \not\subseteq \mathsf{DTIME}(n^c)$.

2. **(5 points)**

In class, when we did the proof of Razborov-Smolensky, we saw two different notions of approximating polynomials:

> Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function.
> A polynomial $P(x_1, \ldots, x_n)$ *weakly-approximates* f with error ε if
>
> $$\Pr_{x \in \{0,1\}^n}[P(x) \neq f(x)] \leq \varepsilon.$$
>
> A polynomial $Q_{r_1,\ldots,r_t}(x_1, \ldots, x_n)$ *strongly-approximates* f with error ε if
>
> $$\text{for every } x \in \{0,1\}^n, \quad \Pr_{r \in \mathbb{F}^t}[Q_r(x) \neq f(x)] \leq \varepsilon.$$

We showed that OR and AND have very simple weakly-approximating polynomials. We also showed that any $f \in \mathsf{AC}^0$ can actually be strongly-approximated using $O((\log s)^d)$ degree polynomials.

Recall that we showed that Parity cannot even be weakly-approximated unless the degree is $\Omega(\sqrt{n})$. Hence, it should have been sufficient to show that AC^0 can be weakly-approximated by $O((\log s)^d)$ degree polynomials.

Why did we work with strong-approximation? Couldn't we have just used the weakly-approximating polynomials for OR and AND to construct weakly-approximating polynomials for AC^0?

3. **(10 points)**

Consider the following function.

```
def f(n):
    if n = 0 then return 24
    if n = 1 then return 42
    curr_sum = 0
    for i = 1, ..., n − 1 do
        a = f(i)
        curr_sum = (curr_sum + a) mod 1000
    return curr_sum
```

What is the time and space complexity of the above code (think of the input n being provided in unary as 1^n)? (Just provide ball-park answers such as $O(\log n), O(n), 2^{O(n)}$ etc. but with sufficient justification)

4. **(10 points)**

   (a) For any space-constructible function $s : \mathbb{N} \to \mathbb{N}$, and any constant $\varepsilon > 0$, prove that

   $$\mathsf{DSPACE}(s(n)) \not\subseteq \mathsf{DTIME}(s(n)^{2-\varepsilon}).$$

   [Hint: This is the first time I'm asking this question in an exam / problem set.]

   (b) Suppose $\mathsf{P} = \mathsf{LOGSPACE}$, does this imply that $\mathsf{EXP} = \mathsf{PSPACE}$? Give brief but sufficient justification.

5. **(10 points)**

   Suppose you are told that Circuit-SAT can be solved in $T_1(n)$ time. What is the time complexity $T_2(n)$ to solve Σ_2 -SAT? (As usual, n is the length of the input.)

   Which of the following values of $T_1(n)$ results in $T_2(n) = 2^{o(n)}$?

   - $T_1(n) = \mathrm{poly}(n)$
   - $T_1(n) = n^{O\left((\log n)^{10}\right)}$
   - $T_1(n) = 2^{n^{1/100}}$

6. True or false (with brief but sufficient justification):

   (a) **(5 points)** $\mathsf{NP}^{\mathsf{NP} \cap \mathsf{coNP}} = \mathsf{NP}$.

   (b) **(5 points)** $\mathsf{LOGSPACE}^{\mathsf{LOGSPACE}} = \mathsf{LOGSPACE}$.
   (For space-bounded machines, assume that the oracle tape is a write-once tape, and the space on the oracle tape does not count towards workspace tape (just like the output tape).)