Cohomologies for dummies

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The goal of this exposition is to explain some of the basics in cohomology tailored specifically towards understanding simplicial complexes and higher dimensional expanders. There is surely going to be massive over-simplification. Many of the terms used here have a broader meaning but throughout this exposition I would focus one just the goal of establishing just enough terminology to understand higher dimensional expanders.

1 Simplicial Complexes

In many areas of theoretical computer science, it is useful to model objects we study by associated graphs. Edges between vertices capture *relations* between various elements of the structure. A *simplicial complex*, to me, is a generalization of this where relations need not just be between two vertices but maybe even between larger number of them.

Definition 1.1 (Simplicial Complex)**.** *A* simplicial complex $\mathcal{X}$ *is a collection of finite subsets of a base set* X *that is* down-closed, *that is it has the property that if* $A \in \mathcal{X}$ *and* $B \subseteq A$ *then* $B \in \mathcal{X}$.

The elements of $\mathcal{X}$ *are called* cells *and the base set* X *is called the* vertices *of* $\mathcal{X}$.

We shall denote by $\mathcal{X}(i)$ *the set of elements of* $\mathcal{X}$ *of size* $(i+1)$ *(yes... I know... but if you think about it, it makes sense):*

$$\mathcal{X}(i) \quad := \quad \{A \in \mathcal{X} \ : \ |A| = i+1\}.$$

The dimension of $\mathcal{X}$, *denoted by* $\dim(\mathcal{X})$, *is the largest* i *such that* $\mathcal{X}(i) \neq \emptyset$. *In other words,* $\dim(\mathcal{X})$ *is the size of the largest cell in* $\mathcal{X}$ *minus one.* ◊

Recall graphs as a running example. We have vertices V and edges E between them. This is just a 1-dimensional simplicial complex $\mathcal{X}$ with $\mathcal{X}(0) = V$ and $\mathcal{X}(1) = E$. A 2-dimensional simplicial complex will have vertices, edges and triangles. And so on.

It is useful to have a picture like the following in mind when thinking about simplicial complexes.

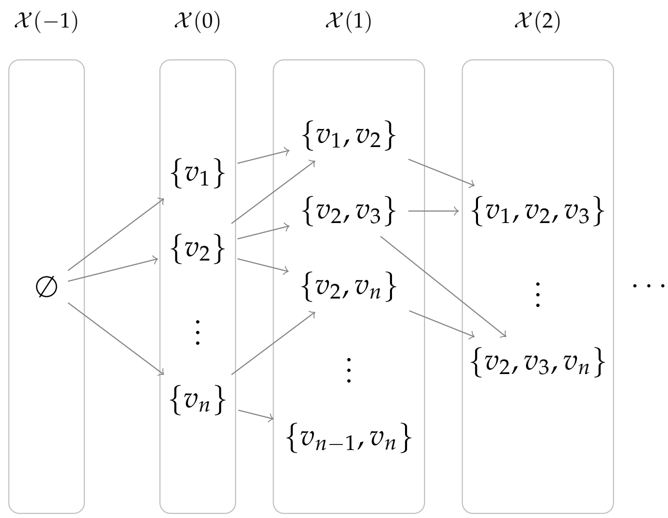

Figure 1: Layered representation of simplicial complexes

Throughout this discussion, we shall fix our field $\mathbb{F}$ to be $\mathbb{F}_2$ as it has the very useful property that $1 = -1$; this would greatly simplify the exposition here.

1.1 Co-chains := Functions on layers

The next object that we should know are called *co-chains*, which are just functions from a layer to $\mathbb{F}_2$.

Definition 1.2. *A* co-chain *on level i is a function $A : \mathcal{X}(i) \to \mathbb{F}_2$. We shall use C_i to denote the set of all co-chains on level i:*

$$C_i \quad := \quad \{A : \mathcal{X}(i) \to \mathbb{F}_2\}\,.$$

We will often think of a co-chain $A \in C_i$ as a subset of $\mathcal{X}(i)$ (that are "accepted" by A). $\diamondsuit$

Q: Why are they called 'co-chains'?

A: I don't know of a satisfactory answer for why they are like chains. But I do know what the 'co-' means; it is to point out that this is *contra-variant*. Let me explain.

Say you have a map $\Phi : G \to H$ between two graphs. Now say you want to understand $C_0(G)$, the functions on the vertices of G, and $C_0(H)$, the functions on vertices of H. Then, the map Φ induces a map $\tilde{\Phi} : C_0(H) \to C_0(G)$ as follows:

Given an $f \in C_0(H)$, map $g \in C_0(G)$ defined by $g(v) = f(\Phi(v))$.

The fact that a map from $G \to H$ induced a map between $C_0(H) \to C_0(G)$ is what is referred to as *contra-variant*. That's what the 'co-' is for.

Another way to interpret a co-chain at level-i as just labeling each element of slice i (in Figure 1) by an element of $\mathbb{F}$. The next thing we would want to do is if we can lift a function on vertices to another function on edges, or vice-versa.

1.2 Boundary and co-boundaries

Definition 1.3 (Boundary operator)**.** *The boundary operator, denoted by ∂, maps co-chains in C_i to co-chains in C_{i-1} in the following way: if $A : \mathcal{X}(i) \to \mathbb{F}$ then the function $\partial A : \mathcal{X}(i-1) \to \mathbb{F}$ is defined by*

$$(\partial A)(\sigma) := \sum_{\substack{\tau \in \mathcal{X}_i \\ \sigma \subset \tau}} A(\sigma) \quad \text{for every } \sigma \in \mathcal{X}(i-1). \qquad \Diamond$$

(Generally, there is a subscript on the ∂ to denote the layer but I am going to ignore it. There is a bigger chance of me putting the wrong index and confusing the reader than not putting it at all.)

For example, if we have a function A on the *edges* of a graph, then the function ∂A is defined on *vertices* by just adding the values on the edges incident on it.

Similarly, one can *lift* functions on vertices to functions on edges.

Definition 1.4 (Co-boundary operator)**.** *The* co-boundary operator*, denoted by δ, maps co-chains in C_i to co-chains in C_{i+1} in the following way: if $A : \mathcal{X}(i) \to \mathbb{F}$ then the function $\delta A : \mathcal{X}(i+1) \to \mathbb{F}$ is defined by*

$$(\delta A)(\sigma) := \sum_{\substack{\tau \in \mathcal{X}_i \\ \sigma \supset \tau}} A(\sigma) \quad \text{for every } \sigma \in \mathcal{X}(i+1). \qquad \Diamond$$

For example, if we have a function A on the *vertices* of a graph, then the function δA is defined on *edges* by just adding the values on the end-points.

I always keep getting confused between which way ∂ and δ operate. Irit Dinur suggested this:

> Keep Figure 1 in mind. The top of δ curves to the right, so it maps elements of C_i to C_{i+1}. And the top of ∂ curves to the left, so it maps C_i to C_{i-1}.

We have already seen that the co-boundary operator δ "lifts" functions from C_i to C_{i+1}. If we think of $i = 0$, we are lifting functions on vertices to functions on edges. Intuitively, any function on edges that is derived this way is in some sense "simple" as it really comes from a level below. In this language, the range of the co-boundary operator is synonymous to "simple". These are called *co-boundaries* and denoted by B_i.

$$B_i \quad := \quad \{\delta(f) \,:\, f \in C_{i-1}\}\,.$$

Almost there! Just one more definition and we can define cohomologies.

1.3 Co-cycles and cohomologies

Recall co-boundary operator δ maps elements of C_i to C_{i+1}. There may be functions $f \in C_i$ such that δf is the zero function on $\mathcal{X}(i+1)$. These are called *co-cycles*.

Definition 1.5 (Co-cycles)**.** *The co-cycles at level i, denoted by Z_i, is defined as the kernel of δ at level i. That is,*

$$Z_i \quad := \quad \{f \in C_i \;:\; \delta f = 0\} \qquad \Diamond$$

Now for the first theorem.

Theorem 1.6 ("$\delta\delta = 0$")**.** *For any function $f \in C_i$, we have $\delta(\delta(f)) \in C_{i+2}$ is the zero function.*

Proof. Important exercise! Don't go beyond this point without working this out! □

Note that B_i and Z_i are spaces of functions over $\mathbb{F}_2$ and are in fact *vector spaces*. The above theorem shows that B_i is a subspace of Z_i. The whole point of cohomologies is to study if Z_i is bigger than B_i or not.

Definition 1.7 (Cohomologies)**.** *The i-th* cohomology*, denoted by H_i, is defined as the group quotient*

$$H_i \quad := \quad \frac{Z_i}{B_i} \qquad \Diamond$$

These are quite a few definitions and they are best understood by taking examples.

2 Examples

Let us just work with graphs for now. A graph $G = (V, E)$ is just a 1-dimensional simplicial complex $\mathcal{X}$ with $\mathcal{X}(0) = V$ and $\mathcal{X}(1) = E$ (and $\mathcal{X}(-1) = \{\emptyset\}$).

Question. *What is B_0, the set of functions on vertices that are co-boundaries?*

Recall that $B_0 = \{\delta(f) \;:\; f \in C_{-1}\}$. But what is C_{-1}? These are functions on $\mathcal{X}(-1) = \{\emptyset\}$ and there are just two of them – the function f_0 that maps $\emptyset$ to zero, and the function f_1 that maps $\emptyset$ to one.

What is δf_0? By defintion, δf_0 on $\{v\}$ is equal to $\sum_{\tau \subset \{v\}} f_0(\tau) = f_0(\emptyset) = 0$. Thus, δf_0 is just the all zero function on the vertices, and similarly δf_1 is the all ones function on vertices.

In other words, the co-boundaries at level-0 are the constant functions.

Question 2.1. *What is B_1, the set of functions on edges that are co-boundaries?*

Recall that $B_0 = \{\delta(f) : f \in C_0\}$ and functions from V to $\mathbb{F}_2$ can just be thought of as choosing a subset of vertices via $S = \{v : f(v) = 1\}$. What is δf? Fix an edge (u, v). Then by definition, $(\delta f)(\{u, v\}) = f(u) + f(v)$. Since we are working over $\mathbb{F}_2$, this would be one if and only if exactly one of $\{u, v\} \in S$. Hence, the function (δf) "accepts" only those edges (u, v) such that exactly one of its end-points is in S, which is just the *cut-edges* induced by S.

In other words, a subset $A \subseteq E$ of edges is a co-boundary if and only if $A = E(S, \overline{S})$ for some $S \subseteq V$.

Question 2.2. *What is Z_0, the set of functions f on vertices such that $\delta f = 0$? And what is H_0, the 0-th cohomology?*

Recall that $(\delta f)(\{u, v\}) = f(u) + f(v)$. Hence, if $\delta f = 0$, we must have $f(u) = f(v)$ for every edge $(u, v) \in E$. Certainly the constant functions satisfy this property. Is that all? Or are there non-constant functions that also satisfy $f(u) + f(v) = 0$ for every $(u, v) \in E$?

This answer depends on whether or not the graph G is connected. If the graph has two disconnected components G_1 and G_2, we could take a function f that is 1 on all the vertices in G_1 and 0 on all vertices of G_2. That would still satisfy the $f(u) + f(v) = 0$ for every $(u, v) \in E$. It is not hard to see that these are precisely all the functions that satisfy this property.

Therefore, Z_0 is just the set of functions that are constant on each connected component of G. We therefore get the following observation.

Lemma 2.3. *For any graph G, the dimension of the 0-th cohomology H_0 is exactly the number of connected components of G.* □

Now suppose we have a 2-dimensional simplicial complex $\mathcal{X} = (V, E, T)$ (where T is a set of triangles), we can also try and understand Z_1.

Question 2.4. *What is Z_1, the set of functions f on edges such that $\delta f = 0$?*

If $\delta f = 0$, then for any triangle $\{u, v, w\} \in T$ we must have $f(\{u, v\}) + f(\{v, w\}) + f(\{u, w\}) = 0$. Thus, if we were to interpret f as a subset $A = \{(u, v) : f(\{u, v\}) = 1\}$ of edges, we must have the property that each triangle in T must either include exactly two edges from A or no edges from A.

Note that if $A = E(S, \overline{S})$ for some $S \subseteq V$, then indeed each triangle will include either two or no edges from A. But are there other subsets of edges that have this property? Once again, this depends on the underlying structure of the simplicial complex. For now, we shall give an example of a simplicial complex where there are indeed subsets that are in Z_1 but are not cuts.

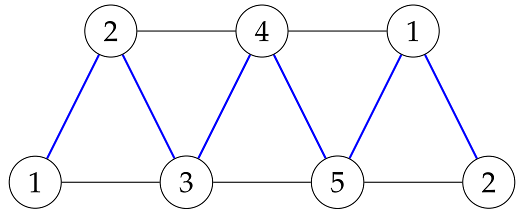

Figure 2: Möbius triangulation – An example where $Z_1 \neq B_1$

Consider the above example where $\mathcal{X}(2) = \{\{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{4,5,1\}, \{5,1,2\}\}$, and $\mathcal{X}(1)$ and $\mathcal{X}(0)$ are just subsets of these triangles. If

$$A \quad = \quad \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}\},$$

observe that every triangle in $\mathcal{X}(2)$ includes exactly two edges in A and hence $\delta A = 0$. But A is *not* a cut.

2.1 Cohomologies as property testing of "simple" functions

Although we defined cohomologies in an abstract setting, it is important to understand the underlying philosophy that it captures. We are interested in a certain class of *simple* objects, which in this case was co-boundaries. What we have are tests that all co-boundaries satisfy. A necessary condition for a function to be a co-boundary is that they must become zero when hit with co-boundary operator again since we know that $\delta\delta = 0$ (by Theorem 1.6). The question is whether this is a sufficient condition as well. The cohomology being trivial is just saying that this is indeed a sufficient condition.

3 Connectivity and co-boundary expansion

Since connectivity of a graph is captured by the 0-th cohomology being trivial, we shall generalize this notion to complexes and define *homological connectivity* in the following natural way.

Definition 3.1 (Homological Connectivity)**.** *A simplicial complex $\mathcal{X}$ is said to be* homologically connected *if $Z_i = B_i$ for every $i \geq 0$ (or in other words, H_i is trivial for all $i \geq 0$).* ♢

In other words, a function $f \in C_i$ is a co-boundary *if and only if* it becomes zero when hit with the co-boundary operator again. Thus $\delta f = 0$ is a necessary and sufficient test for $f = \delta g$ for some $g \in C_{i-1}$.

Write the proof that the *complete simplicial complex* is homologically connected.

We now want to generalize the notion of *expansion* to higher dimensional simplicial complexes. Let us first revisit the notion of expansion in graphs and try to state things in terms of cohomologies.

3.1 Revisiting Cheeger

Expanders graphs are captured by what is known as the Cheeger constant.

Definition 3.2 (Cheeger Constant and vertex expansion)**.** *For a graph G, define the parameter $h(G)$ to be*

$$h(G) \quad := \quad \min_{\emptyset \neq S \subsetneq V} \frac{\left\|E(S,\overline{S})\right\|}{\left\|\min(|S|,|\overline{S}|)\right\|}$$

where $\left\|E(S,\overline{S})\right\| = \left|E(S,\overline{S})\right| / |E|$ *and* $\left\|\min(|S|,|\overline{S}|)\right\| = \min(|S|,|\overline{S}|)/|V|$.

*A graph G is said to be an ϵ-*vertex-expander *if $h(G) \geq \epsilon$.* ◊

Let us try to write this in terms of notation from cohomologies so that we can generalize it to higher dimensional simplicial complexes. The Cheeger constant is a minimum over sets of vertices that are neither empty nor full. If we were to think of such a subset S as a function f_S on vertices, then we are essentially excluding the constant functions. Note that the constant functions were precisely B_0, the co-boundaries at level 0. Hence, we can think of the minimum as being over all functions $f_S \in C_0 \setminus B_0$.

Let us now focus on the numerator. For a set $S \subset V$, can we express $\left|E(S,\overline{S})\right|$ in the language of cohomologies? Indeed we can. This is precisely the hamming weight of δS, that is the number of edge $(u,v) \in E$ such that $(\delta S)(\{u,v\}) = 1$. Therefore, $\left\|E(S,\overline{S})\right\|$ is the normalized weight of δS defined as

$$\|\delta S\| \quad := \quad \frac{|\{\sigma \in \mathcal{X}(1) \ : \ (\delta S)(\sigma) = 1\}|}{|\mathcal{X}(1)|} \quad = \quad \left\|E(S,\overline{S})\right\|$$

Now for the denominator. How can $\min(|S|,|\overline{S}|)$ be expressed in the language of cohomologies? This is precisely the distance of the function f_S from B_0, the constant functions. Normalizing it again, we have

$$\begin{aligned}
\mathrm{dist}(f_S, B_1) \quad &:= \quad \min_{g \in B_1} |\{\sigma \in \mathcal{X}(0) \ : \ f_S(\sigma) = g(\sigma)\}| \\
&= \quad \min(|S|,|\overline{S}|) \\
\|\mathrm{dist}(f_S, B_0)\| \quad &:= \quad \frac{\mathrm{dist}(f_S, B_0)}{|\mathcal{X}(0)|}
\end{aligned}$$

Therefore, the expression for $h(G)$ can be rewritten as

$$h(G) \quad = \quad \min_{f_S \in C_0 \setminus B_0} \frac{\|\delta f_S\|}{\|\text{dist}(f_S, B_0)\|}.$$

In other words, G is an ϵ-vertex-expander if the normalized weight of δf_S is *proportionally large* compared to the distance of f_S from B_0. This expression is now in a form that can certainly be lifted to higher dimensional simplicial complexes as well.

Definition 3.3 (Co-boundary expanders)**.** *Let $\mathcal{X}$ be a simplicial complex. We shall say that $\mathcal{X}$ is an ϵ-coboundary expander if for all $i \geq 0$, we have*

$$\mathcal{E}_i \quad := \quad \min_{f \in C_i \setminus B_i} \frac{\|\delta f\|}{\|\text{dist}(f, B_i)\|} \quad \geq \quad \epsilon. \qquad \Diamond$$

In the case of connectivity, we said that a function f is a co-boundary if and only if $\delta f = 0$. Co-boundary expansion is a robust version of this statement where we are saying that if f is *far* from being a co-boundary, then δf is far from zero.

For example, consider a 2-dimensional simplicial complex that is an ϵ-coboundary expander. Then this implies that if we have a subset A of edges such that we need to change α-fraction of the total number of edges to transform A to a cut, then there must be an $\epsilon\alpha$-fraction of all triangles in $\mathcal{X}(2)$ that involve and odd number of edges from A.

What are some explicit simplicial complexes that are co-boundary expanders? Certainly the first complex to try is Δ_d, the *complete* d-dimensional simplicial complex that involves *all* possible subsets of size at most $d+1$ over a vertex set. Indeed, it is known that Δ_d is a 1-coboundary expander. The proof is not hard, apparently but I don't yet know how to prove it.

Theorem 3.4 (Gromov)**.** *For any $d \geq 0$, Δ_d is a* 1*-coboundary expander.*

What about other examples? Are there *low-degree* co-boundary expanders? We do not have explicit families of bounded degree co-boundary expanders yet! But we do have some candidates of bounded degree simplicial complexes that are conjectured to be co-boundary expanders. These are the Ramanujan Complexes (which will be dealt with in a different exposition).