Near-optimal Bootstrapping of Hitting Sets

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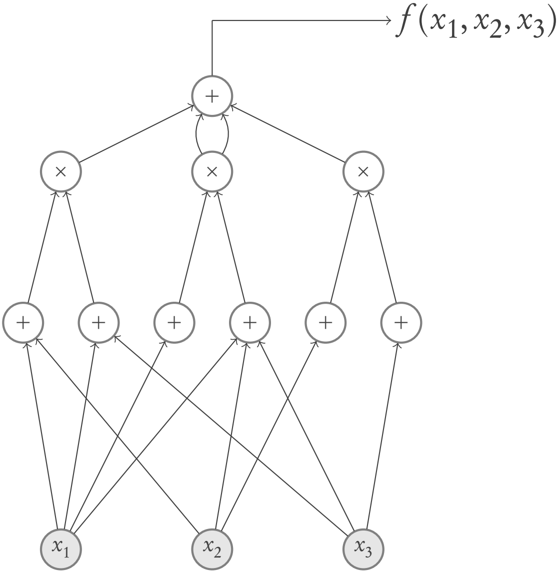
December 2018

Algebraic Circuits



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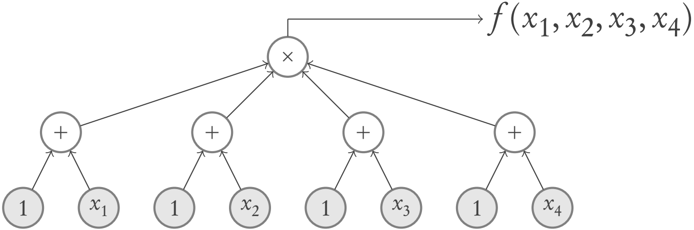

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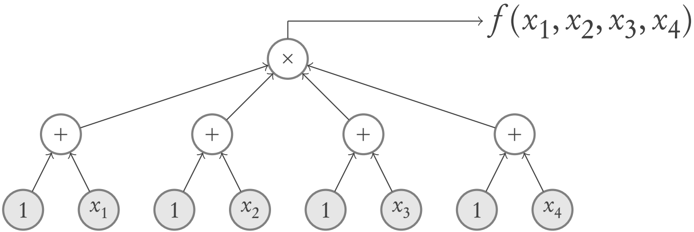

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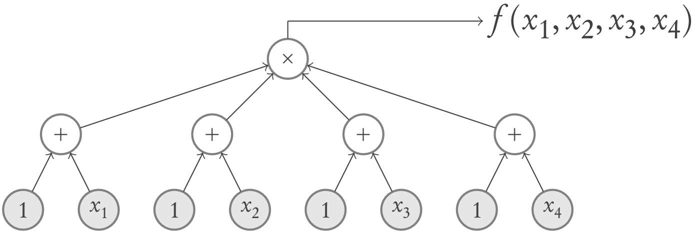

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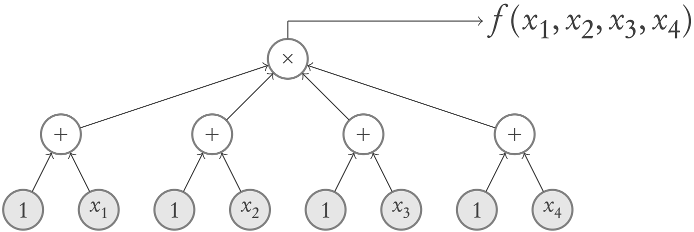

- A tree, made up of $+$ and $\times$ gates. Leaves containing variables or constants. Size = number of leaves
- $\mathrm{Size}(f(g_1, \ldots, g_n)) \leq \mathrm{Size}(f) \cdot \max_i (\mathrm{Size}(g_i))$

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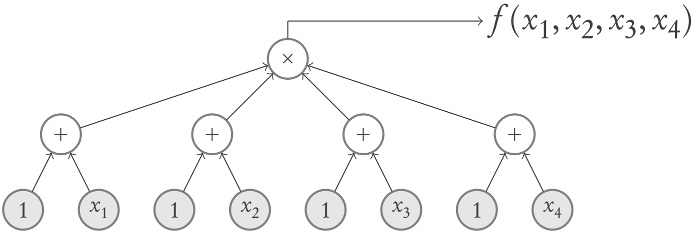
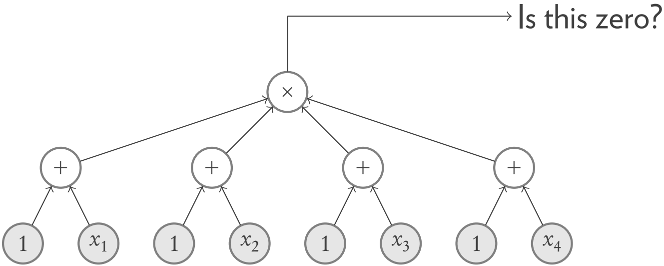


- A tree, made up of $+$ and $\times$ gates. Leaves containing variables or constants. Size = number of leaves
- $\text{Size}(f(g_1, \ldots, g_n)) \leq \text{Size}(f) \cdot \max_i(\text{Size}(g_i))$
- $\text{Formula}(n, d, s)$: n-variate, degree $\leq d$ polynomials computable by size s formulas. (note: $d \leq s$)

Polynomial Identity Testing



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Is this zero?

This box contains a polynomial from $\mathscr{C}(n, d, s)$

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Equivalent to constructing a hitting set H:

> *For every nonzero $P \in \mathscr{C}(n, d, s)$, there is some $\overline{a} \in H$ such that $P(\overline{a}) \neq 0$.*

Hitting Sets

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There are non-explicit hitting sets of $\mathrm{poly}(s)$ size for $\mathscr{C}(n, d, s)$.

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Lemma ([Ore*, DeMillo-Lipton, Schwartz-Zippel])

If $S \subseteq \mathbb{F}$ with $|S| \geq d + 1$, then S^n is a hitting set for $\mathscr{C}(n, d, s)$.

That is, we have an explicit, but trivial, hitting set of $(d+1)^n$ size.

*: See [Bishnoi-Clark-Potukuchi-Schmitt]

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Question: Are there small explicit hitting sets for $\mathscr{C}(n,d,s)$?

*: See [Bishnoi-Clark-Potukuchi-Schmitt]

Improving not-too-bad hitting sets

Theorem ([Agrawal-Ghosh-Saxena 2018])

Say n large enough.

Suppose, for each $s \geq n$, there is an explicit hitting set for $\mathrm{Circuits}(n, s, s)$ *of size at most*

$$(s+1)^{n^{0.49}}.$$

(Trivial hitting set size: $(s+1)^n$)

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Improving almost-trivial hitting sets

Theorem ([Kumar-S-Tengse])

Say n large enough.

Suppose, for each $s \geq n$, there is an explicit hitting set for $\mathrm{Circuits}(n, s, s)$ of size at most

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Then there is an explicit hitting set for $\mathscr{C}(s,s,s)$ of size at most

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(where $\mathscr{C}$ is any class well-behaved under sums, projections and compositions)

A *very* high-level overview

Non-trivial Hitting Sets

Explicit Lower Bounds

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Non-trivial Hitting Sets

[Heintz-Schnorr, Agrawal]

Explicit Lower Bounds

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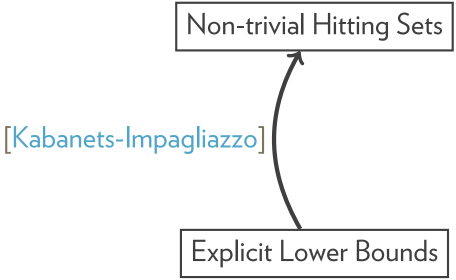

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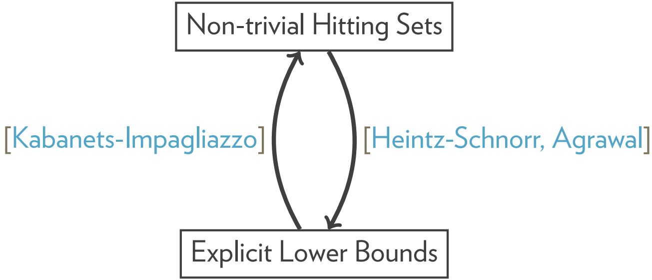

From a non-trivial hitting set, get a lower bound. Use that to get a *better* hitting set. And so on ...

Preliminaries:

Hardness vs Randomness

for algebraic models

Lower bounds from hitting sets

H is a hitting set for $\mathscr{C}(n,d,s)$ if

for all $0 \neq P \in \mathscr{C}(n,d,s)$, there is some $\overline{a} \in H$ such that $P(\overline{a}) \neq 0$.

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Observation

If P is a nonzero polynomial that *vanishes* on H, then P cannot be a member of $\mathscr{C}(n,d,s)$.

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Theorem ([Heintz-Schnorr, Agrawal])

For any $k \leq n$ such that $k\,|H|^{1/k} \leq d$, we can find a nonzero k-variate polynomial Q of individual degree less than $|H|^{1/k}$ such that Q requires size more than s.

Hitting sets from lower bounds

Theorem ([Kabanets-Impagliazzo] (Informal))

If Q is hard-enough, then for any small algebraic circuit computing P, we have

$$P(x_1, \ldots, x_m) \neq 0 \quad \Longleftrightarrow \quad P(Q(\overline{y}_1), \ldots, Q(\overline{y}_m)) \neq 0$$

even if $\overline{y}_1, \ldots, \overline{y}_m$ are almost disjoint.

Hitting sets from lower bounds

Aside: Combinatorial Designs

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Definition (Combinatorial designs)

$\{S_1, \ldots, S_m\} \subseteq [\ell]$ is an (ℓ, k, r)-design if $|S_i| = k$ and $\left|S_i \cap S_j\right| < r$.

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For all $\ell \geq k^2$ and $r \leq k$, we have explicit (ℓ, k, r)-designs with $m = \left(\frac{\ell}{k}\right)^r$.*

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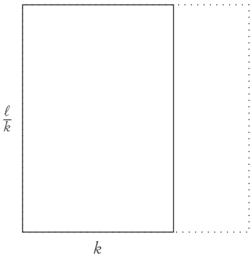

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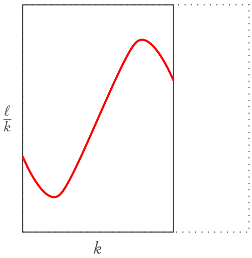

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For $p(z) \in \mathbb{F}[z]$ with $\deg(p) < r$,

$S_p = \{(i, p(i)) \,:\, i \in [k]\}$.

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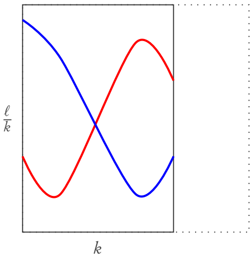

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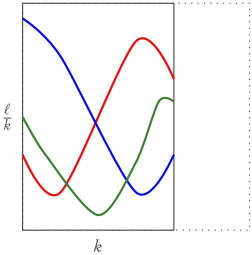

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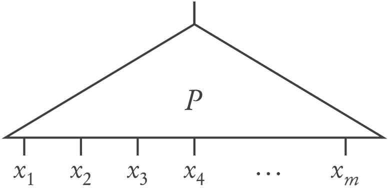

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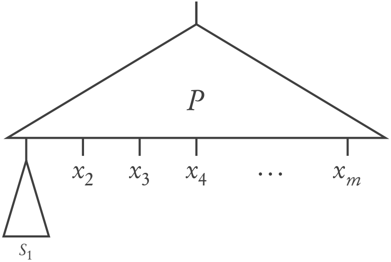

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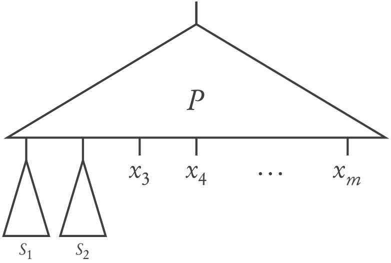

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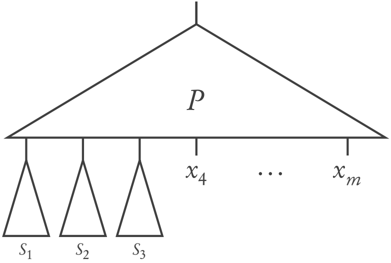

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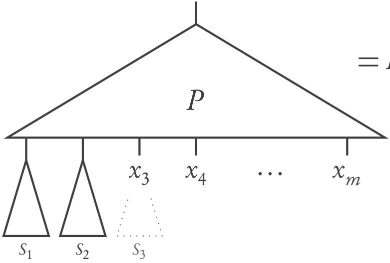

$$= \tilde{P}(x_3, \mathbf{x}_{\text{rest}}, \bar{y}\,|_{S_3}, \bar{y}\,|_{\text{rest}})$$

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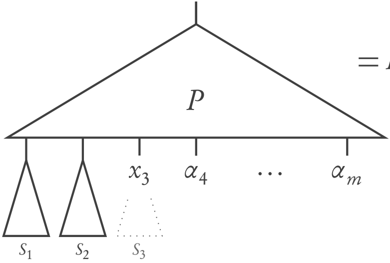

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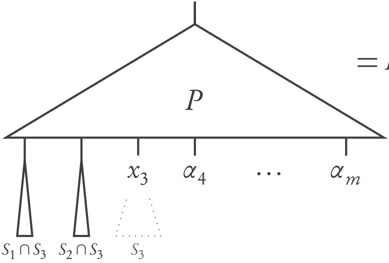

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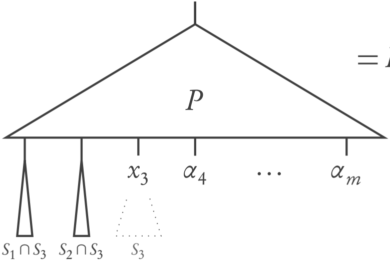

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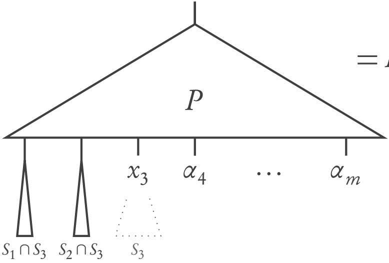

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Size $\leq s \cdot \left(rd \cdot d^{r-1}\right)$
Degree $\leq D \cdot dr$

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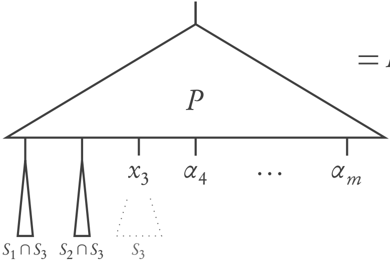

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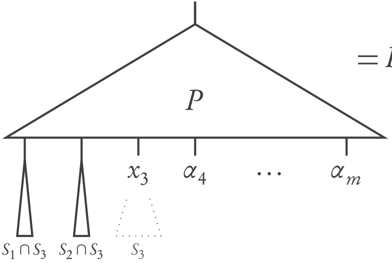

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[Kaltofen, Bürgisser]:
Factors have small circuits.

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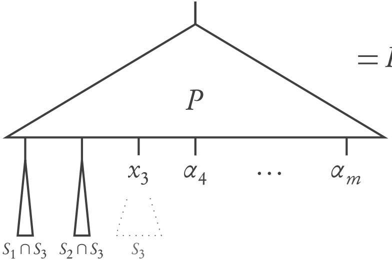

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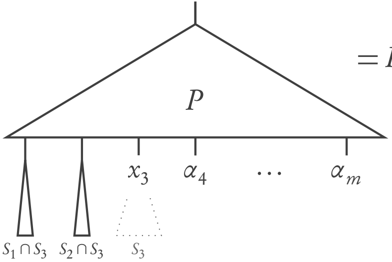

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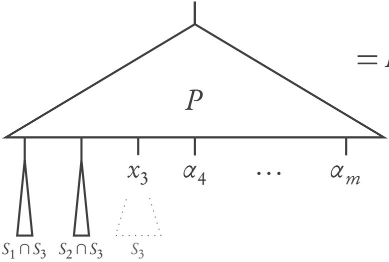

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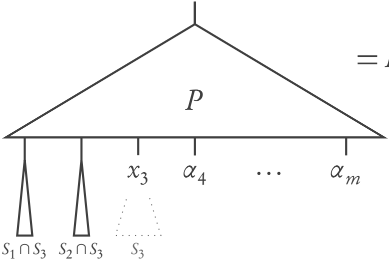

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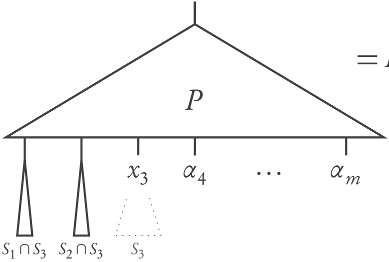

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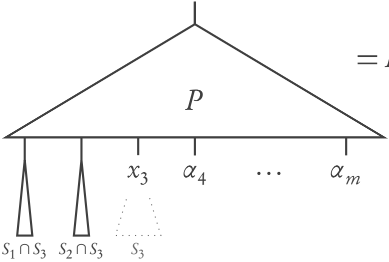

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From hitting sets for k-variate formulas, we obtain a hitting set for m-variate formulas.

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Unlike the boolean setting, we **can** find such polynomials of suitably large degree.
Thus, there is nothing stopping you from doing this again and again.

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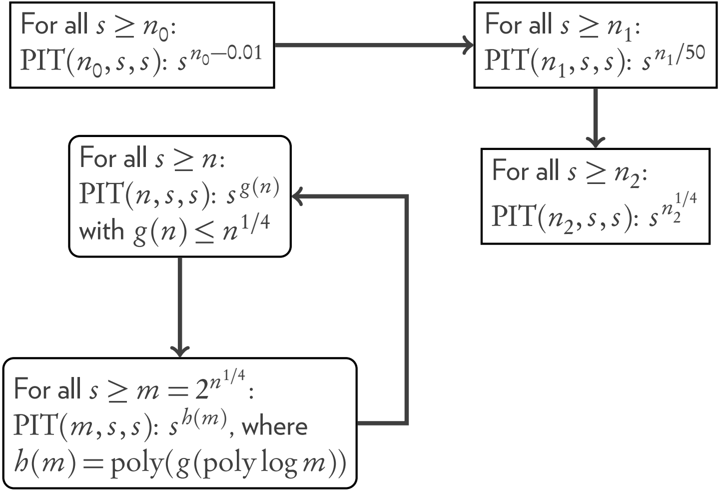

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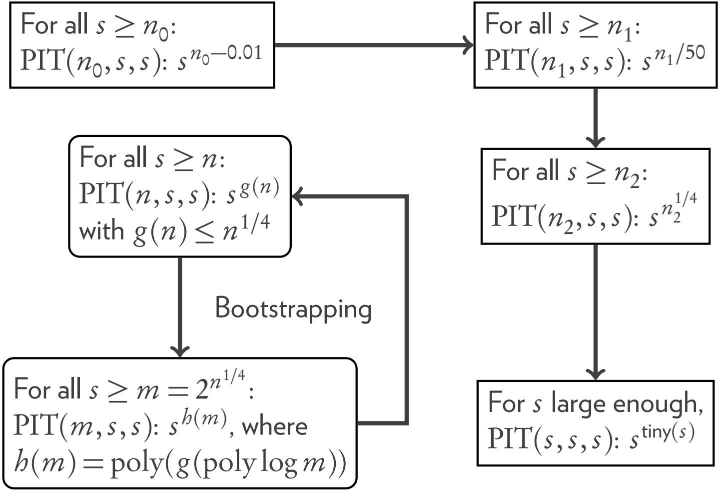
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Let n be large enough $(n > 10^{10})$. Suppose, for all $s \geq n$, there is an explicit hitting set for $\mathrm{Formula}(n, s, s)$ of size at most

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Let n be large enough $(n > 10^{10})$. *Suppose, for all $s \geq n$, there is an explicit hitting set for* $\mathrm{Formula}(n, s, s)$ *of size at most*

$$s^{g(n)}, \qquad \text{with } g(n) \leq \left(\frac{n^{1/4}}{10}\right).$$

Then, for $m = 2^{n^{1/4}}$ and all $s \geq m$, we have an explicit hitting set for $\mathrm{Formula}(m, s, s)$ *of size at most*

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Using the hypothesis again, we get a hitting set of size $s^{20(g(n))^2}$ for $\mathrm{Formula}(m,s,s)$. $\square$

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Claim: $0 \neq P \in \mathrm{Formula}(m,s,s) \implies P(Q[\![\ell,k,r]\!]) \neq 0$

Proof: If not, there is a nonzero multiple $\tilde{Q}$ of Q, whose degree is at most $s \cdot (rd) \leq s^{300n}$, computable by a formula of size

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Complexity to compute an $(r-1)$-variate polynomial of ideg d

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Hyp: $s^{n-0.01}$ hitting sets for $\mathscr{C}(n,s,s)$, for any $s \geq n$.

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$P(Q[\![\ell,k,r]\!])$ is a formula on $\ell = n^5$ variables of degree $s \cdot k \cdot s^{300n-3} \leq s^{300n}$.
[O-DL-S-Z] lemma: a hitting set of size $s^{300n \cdot n^5} \leq s^{m/50}$ for $\mathrm{Formula}(m,s,s)$. $\square$

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