Near-optimal Bootstrapping of Hitting Sets

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Dagstuhl
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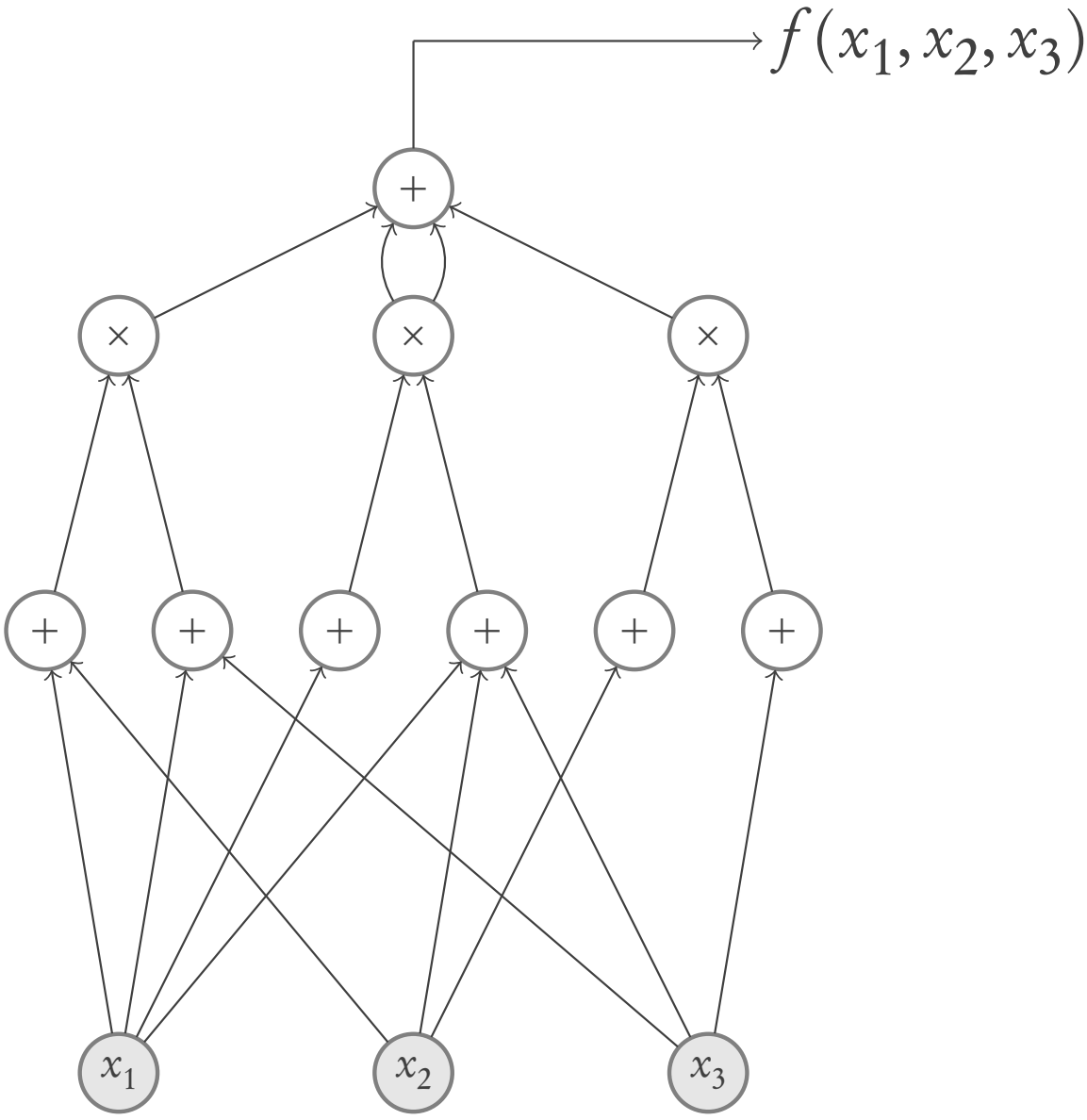
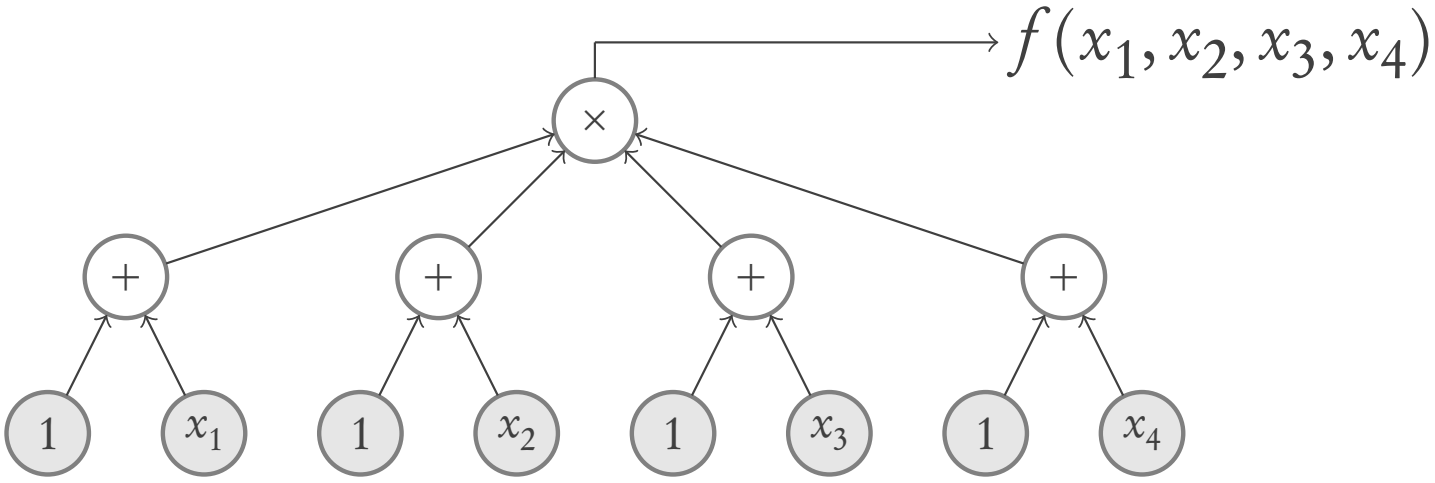
Algebraic Circuits



Algebraic Formulas



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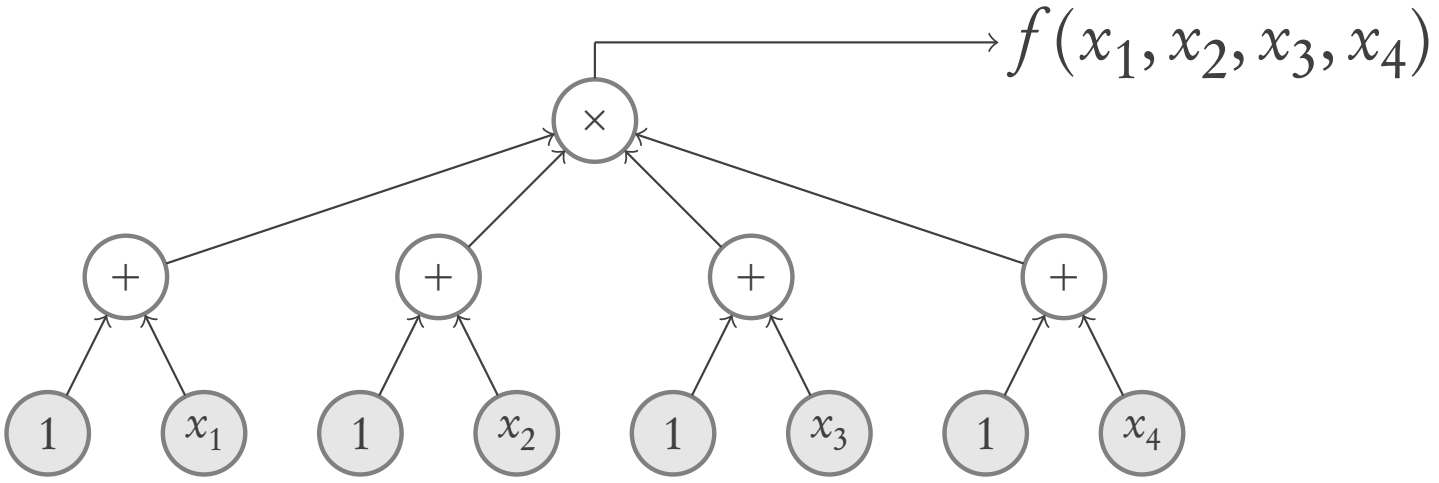

- A tree, made up of $+$ and $\times$ gates. Leaves containing variables or constants

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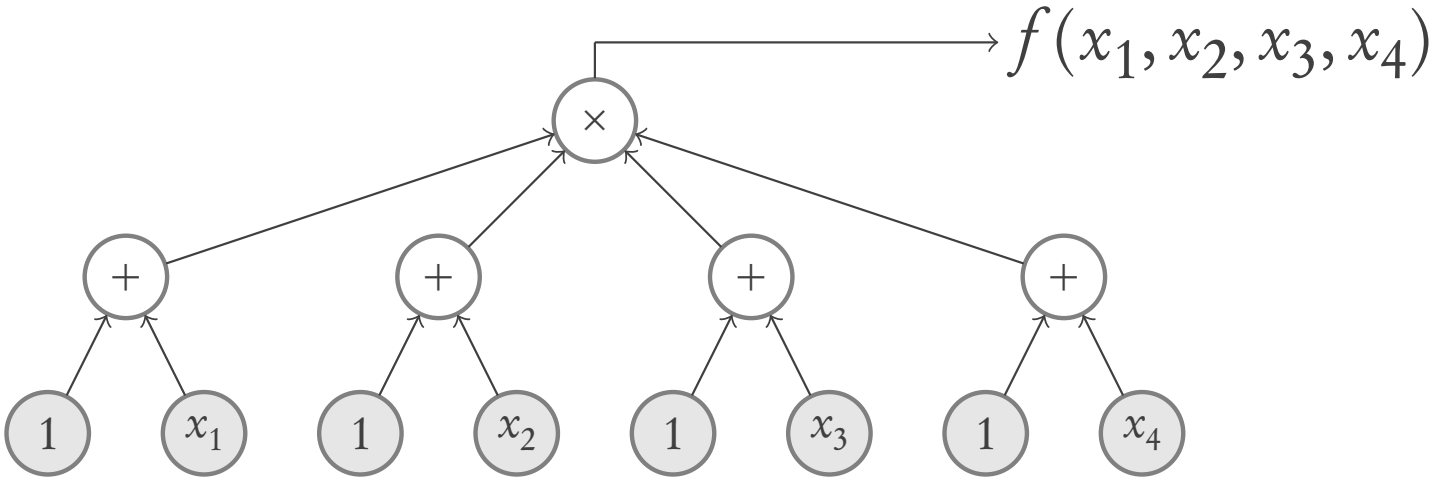

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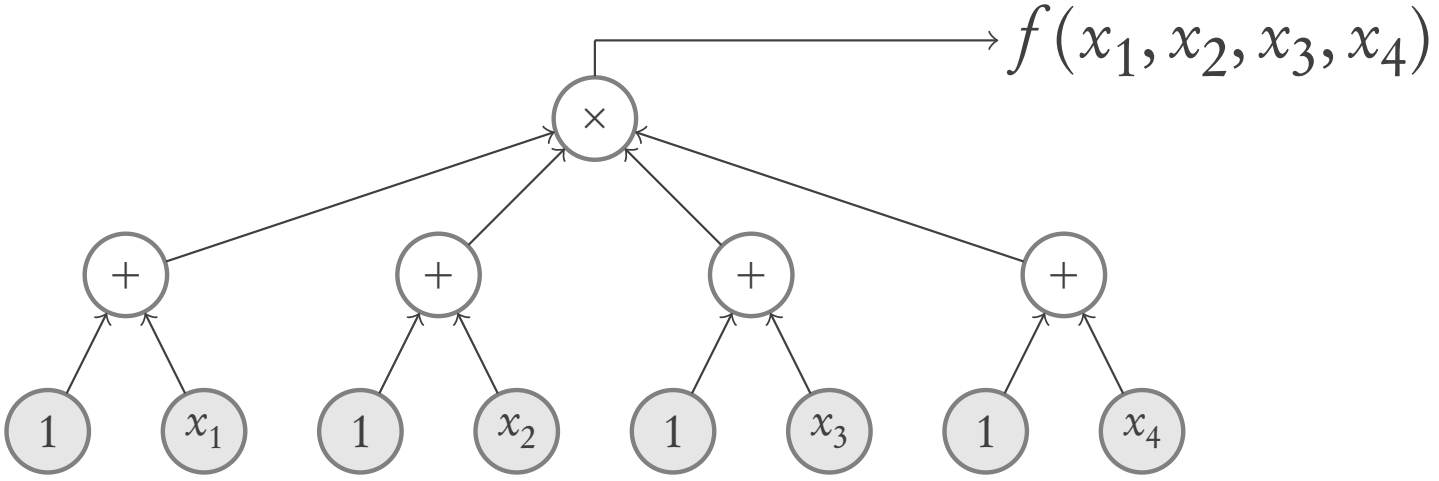

- A tree, made up of $+$ and $\times$ gates. Leaves containing variables or constants. Size = number of leaves
- $\text{Size}(f(g_1, \ldots, g_n)) \leq \text{Size}(f) \cdot \max_i(\text{Size}(g_i))$

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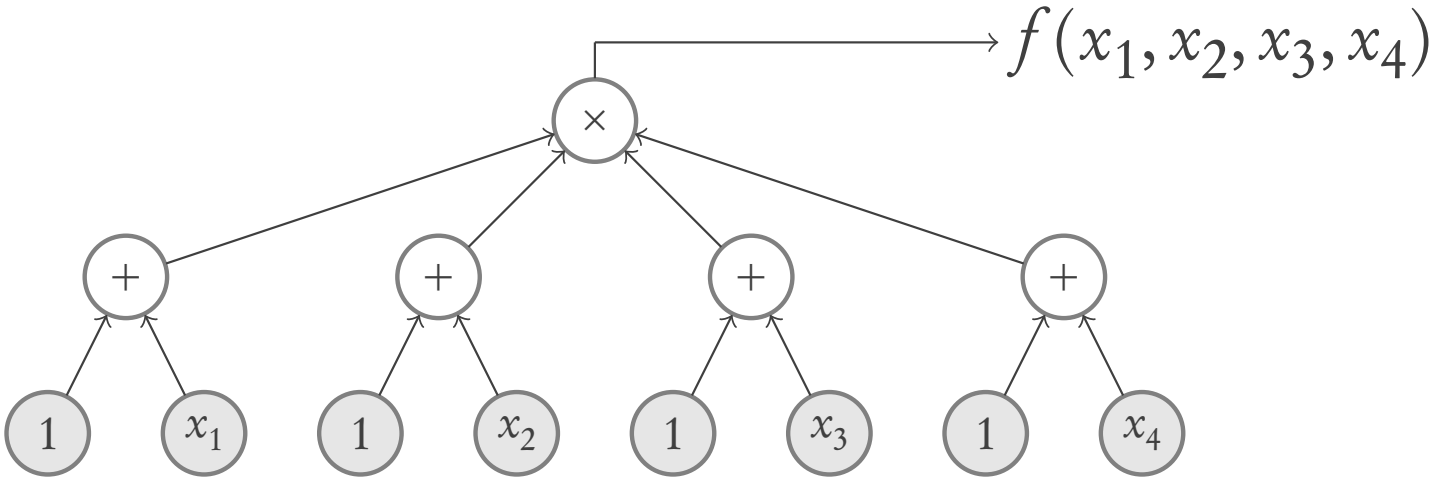
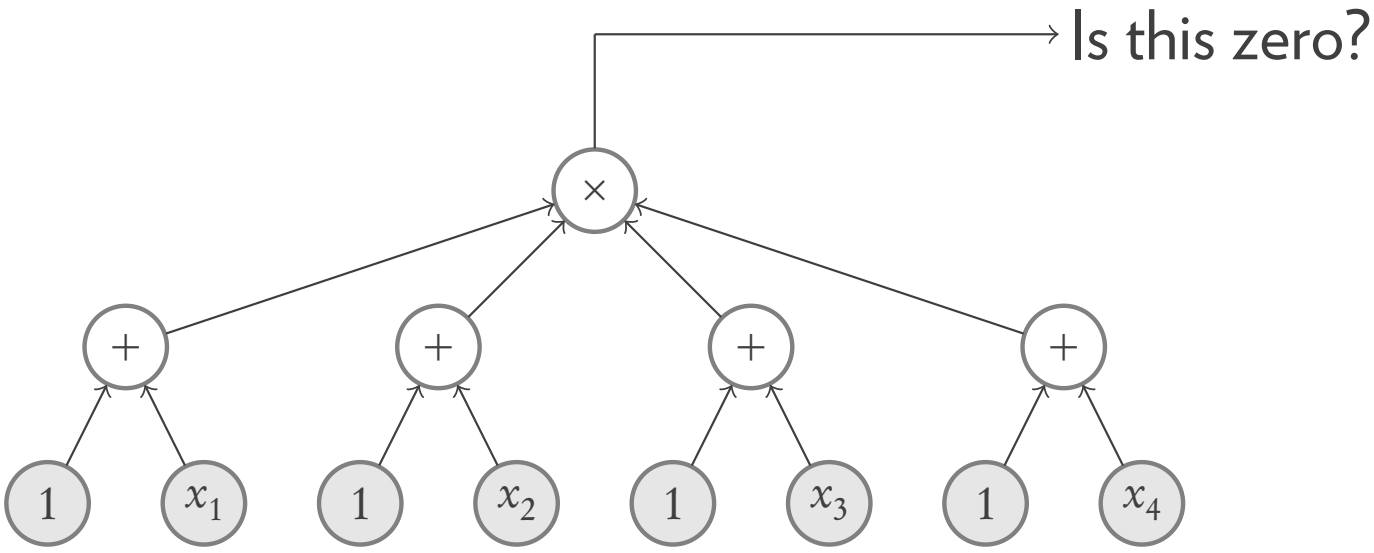


- A tree, made up of $+$ and $\times$ gates. Leaves containing variables or constants. Size = number of leaves
- $\text{Size}(f(g_1, \ldots, g_n)) \leq \text{Size}(f) \cdot \max_i(\text{Size}(g_i))$
- $\text{Formula}(n, d, s)$: n-variate, degree $\leq d$ polynomials computable by size s formulas. (note: $d \leq s$)

Polynomial Identity Testing



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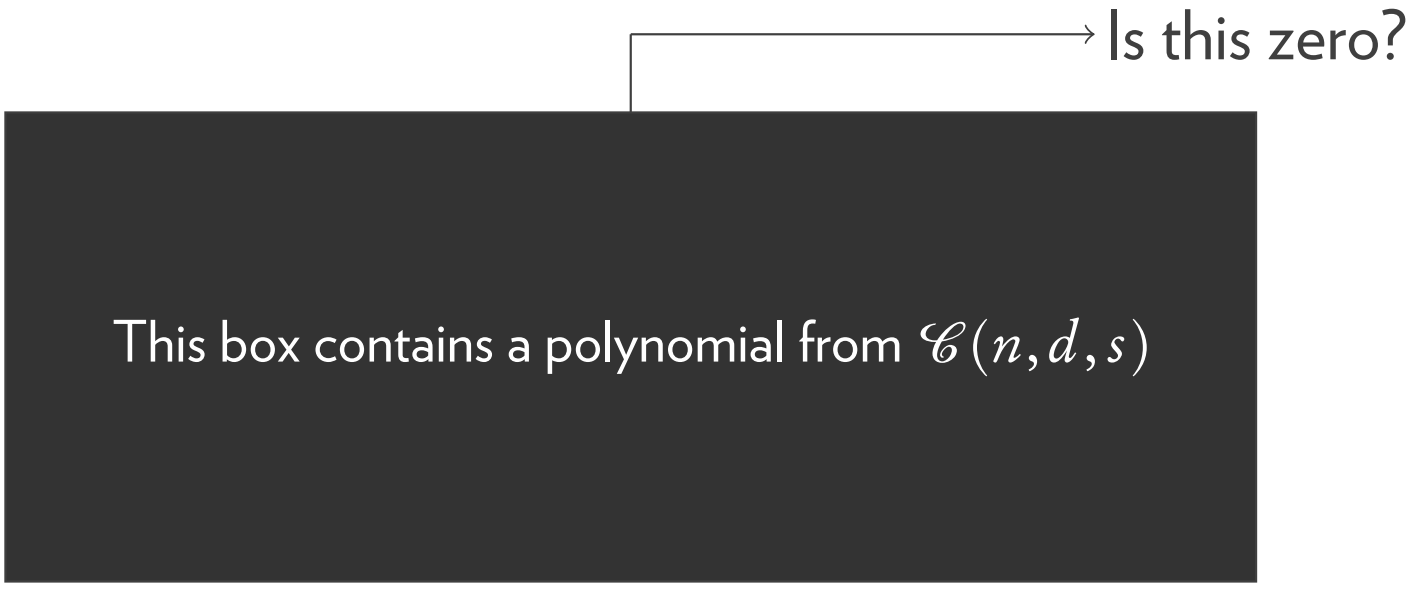

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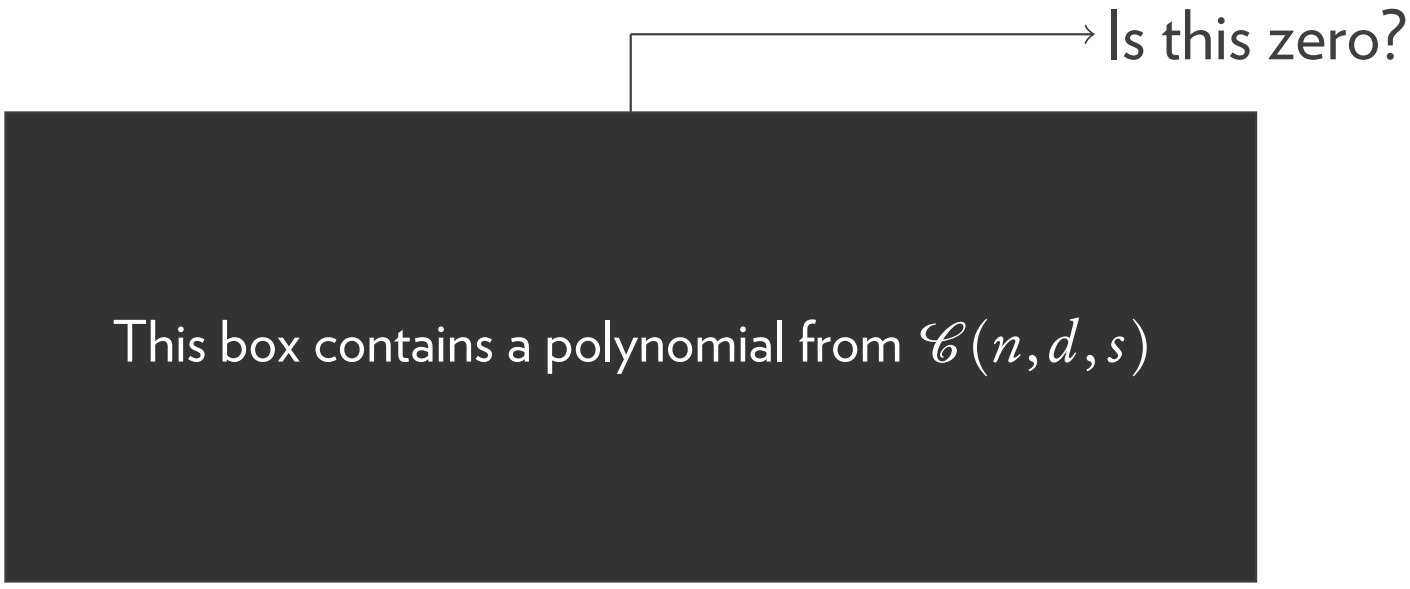

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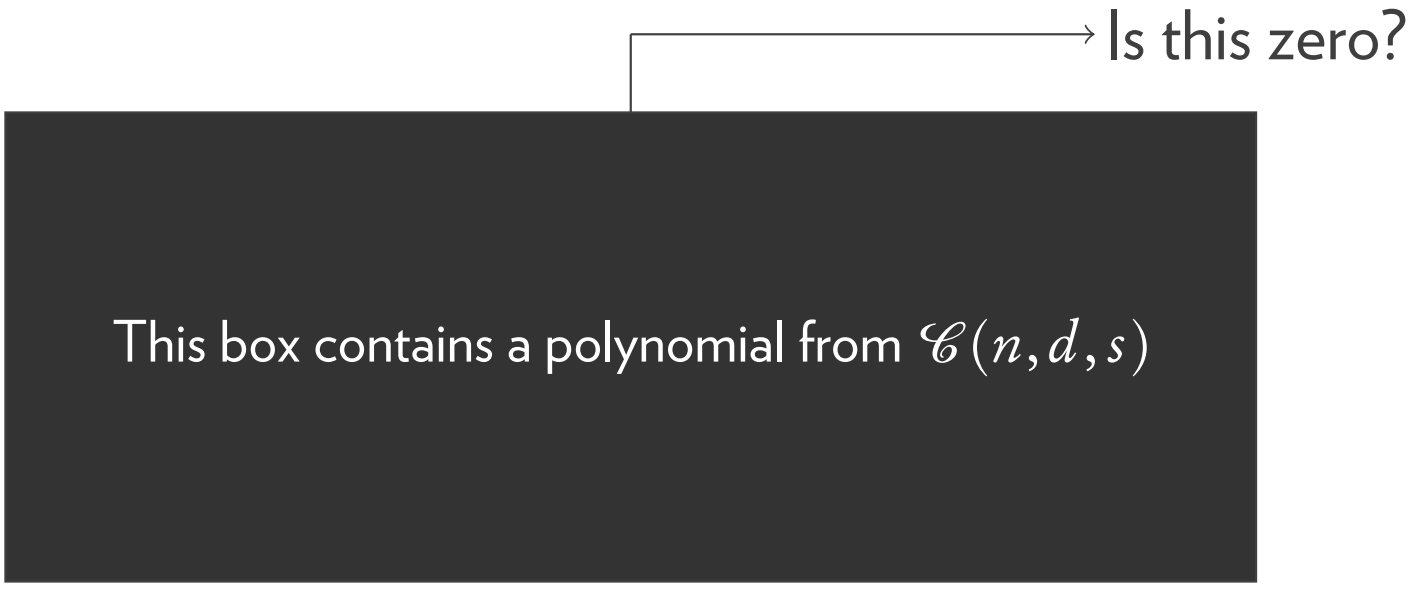

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Equivalent to constructing a hitting set H:

> *For every nonzero $P \in \mathscr{C}(n,d,s)$, there is some $\overline{a} \in H$ such that $P(\overline{a}) \neq 0$.*

Hitting Sets

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Counting argument

There are non-explicit hitting sets of $\mathrm{poly}(s)$ size for $\mathscr{C}(n, d, s)$.

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Lemma ([Ore, DeMillo-Lipton, Schwartz-Zippel])

If $S \subseteq \mathbb{F}$ with $|S| \geq s+1$, then S^n is a hitting set for $\mathscr{C}(n,d,s)$.

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Question: Are there small explicit hitting sets for $\mathscr{C}(n,d,s)$?

Improving not-too-bad hitting sets

Theorem ([Agrawal-Ghosh-Saxena 2018])

Say n large enough.

Suppose, for each $s \geq n$, there is an explicit hitting set for $\mathrm{Circuits}(n, s, s)$ of size at most

$$(s+1)^{n^{0.49}}.$$

(Trivial hitting set size: $(s+1)^n$)

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Improving almost-trivial hitting sets

Theorem ([Kumar-S-Tengse])

Say n large enough.

Suppose, for each $s \geq n$, there is an explicit hitting set for $\mathrm{Circuits}(n, s, s)$ of size at most

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Then there is an explicit hitting set for $\mathscr{C}(n, s, s)$ of size at most

$$s^{\mathrm{tiny}(s)}.$$

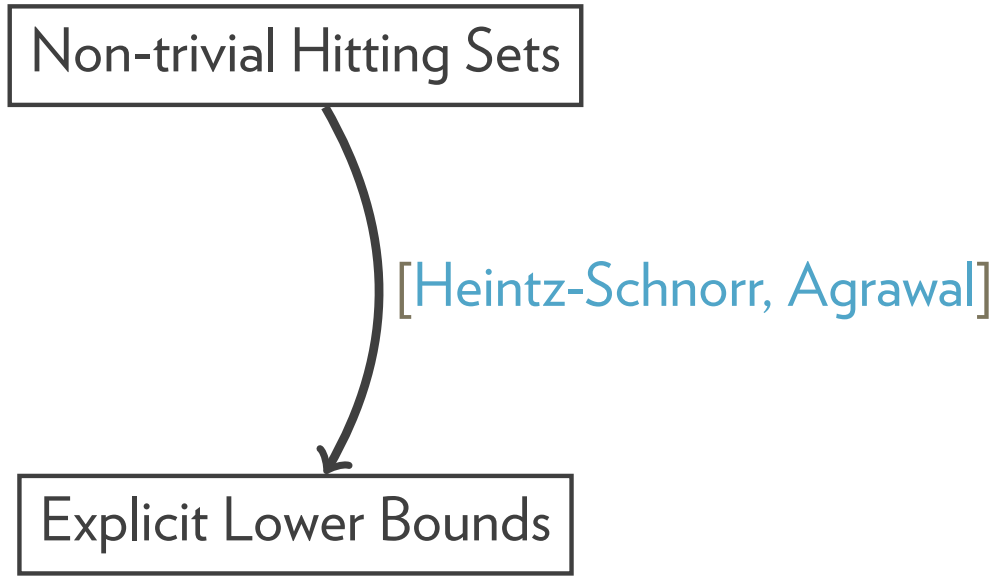
(where $\mathscr{C}$ is any class well-behaved under sums, projections and compositions)

A *very* high-level overview

Non-trivial Hitting Sets

Explicit Lower Bounds

A *very* high-level overview



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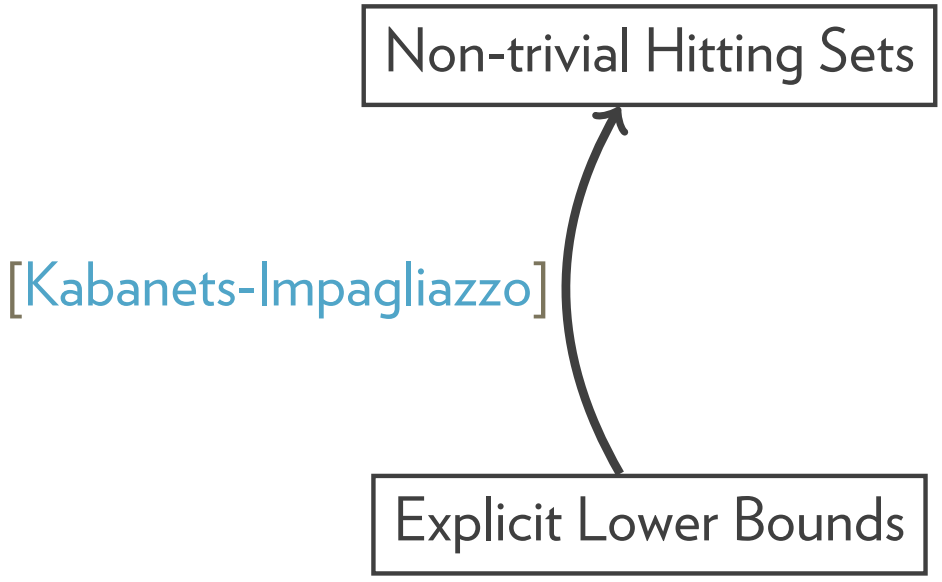

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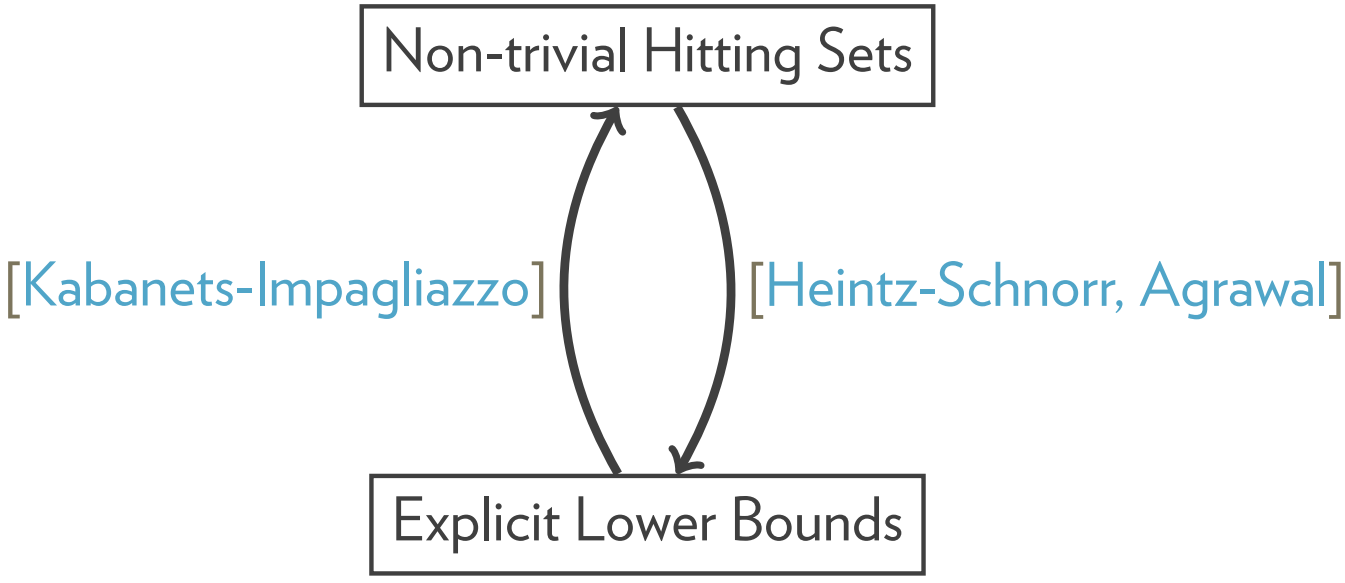

From a non-trivial hitting set, get a lower bound. Use that to get a *better* hitting set. And so on ...

A high-level overview

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For all $s \geq n_0$:
$\mathrm{PIT}(n_0, s, s)$: $s^{n_0 - 0.01}$

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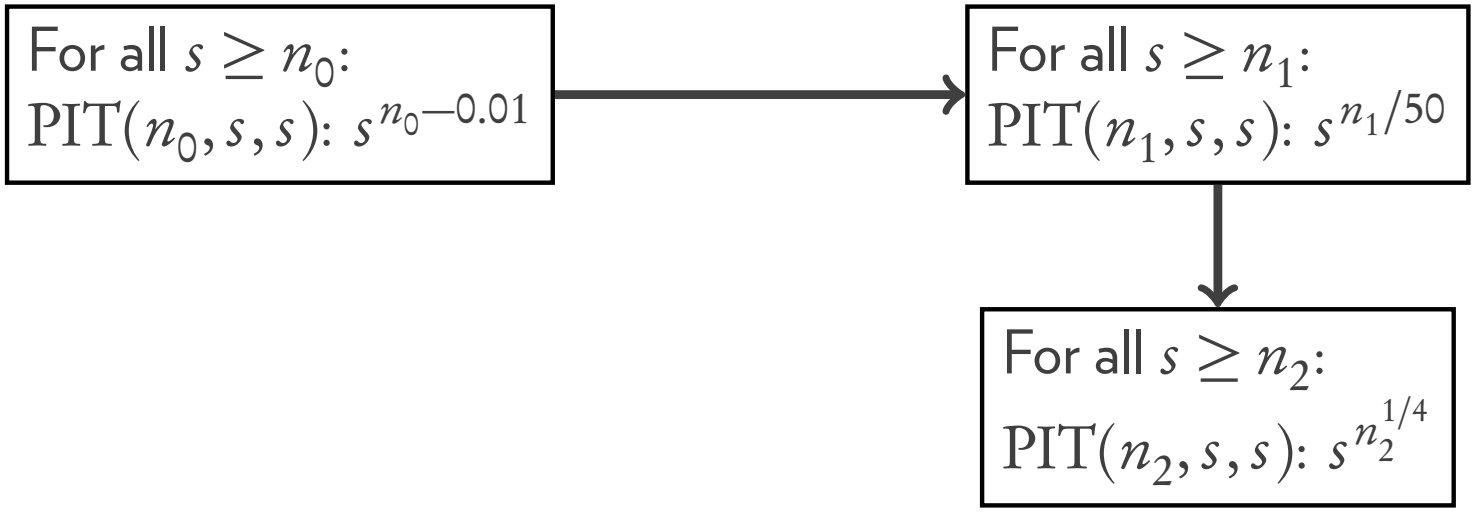

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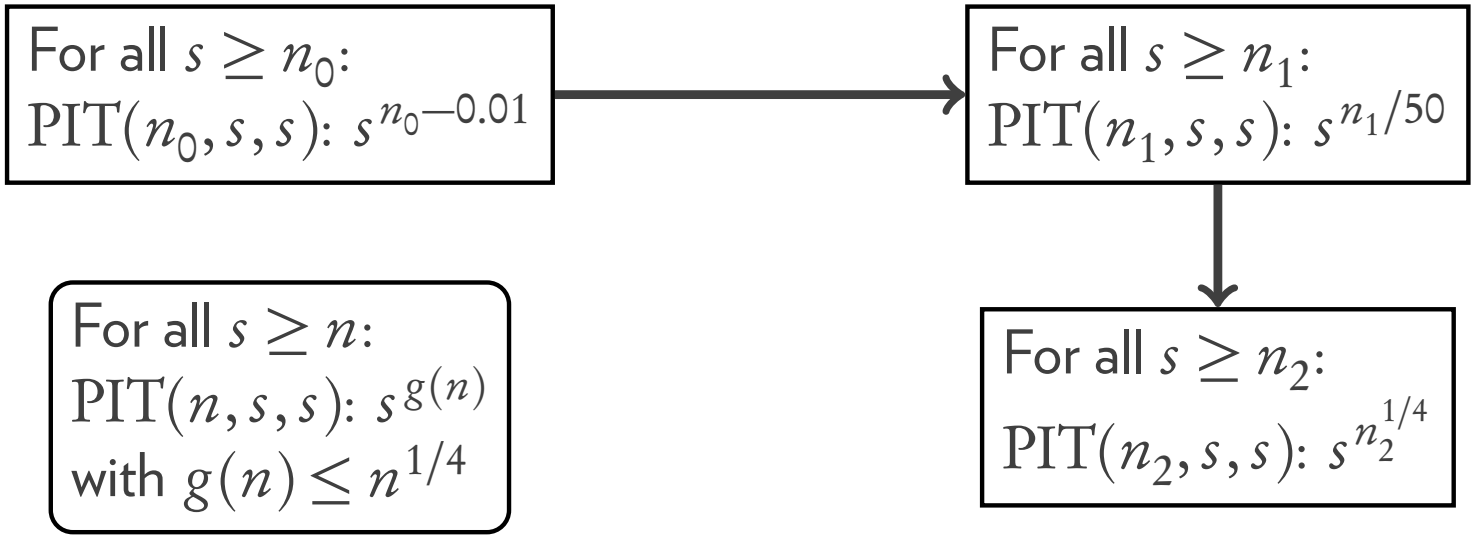

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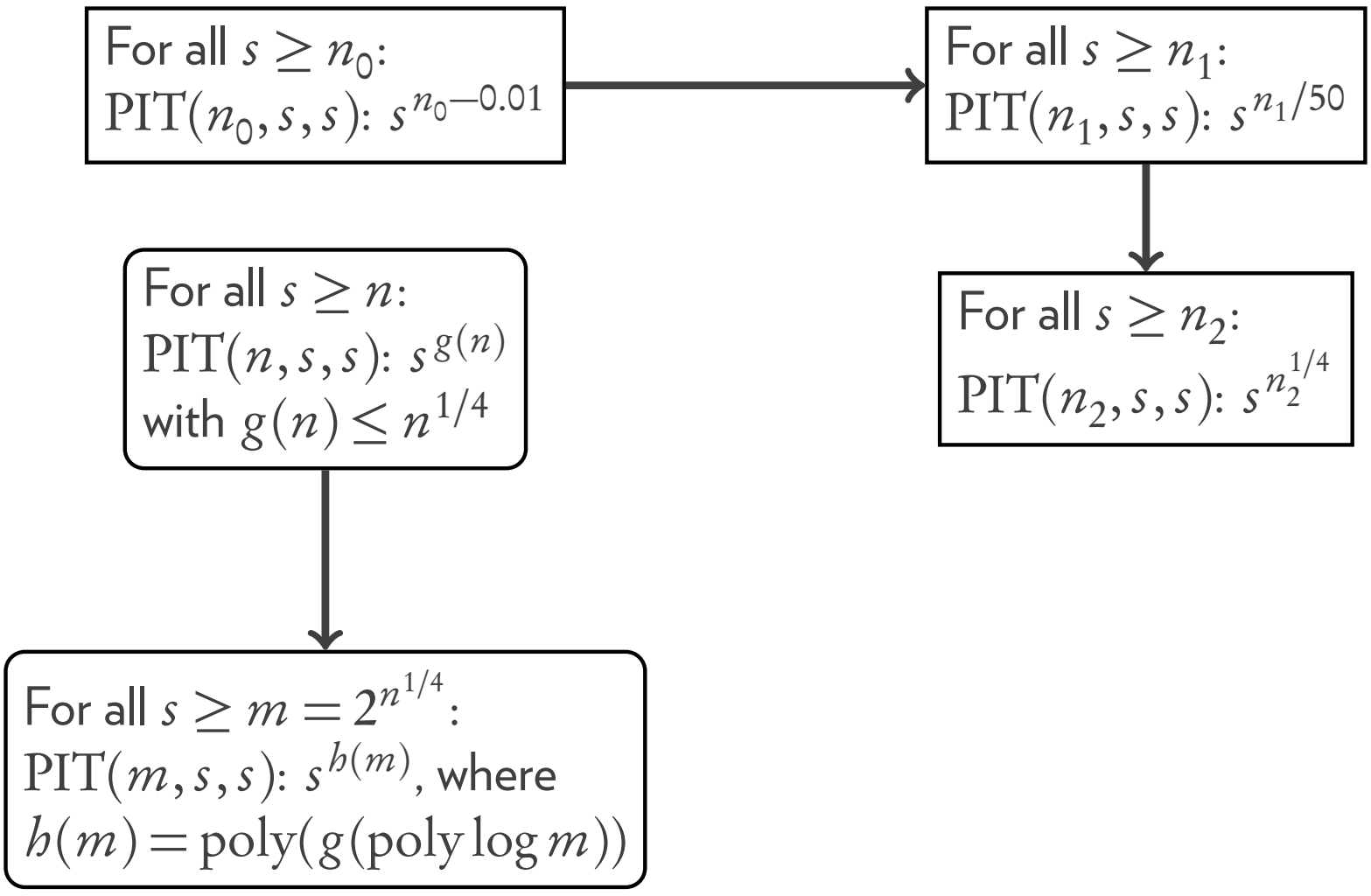

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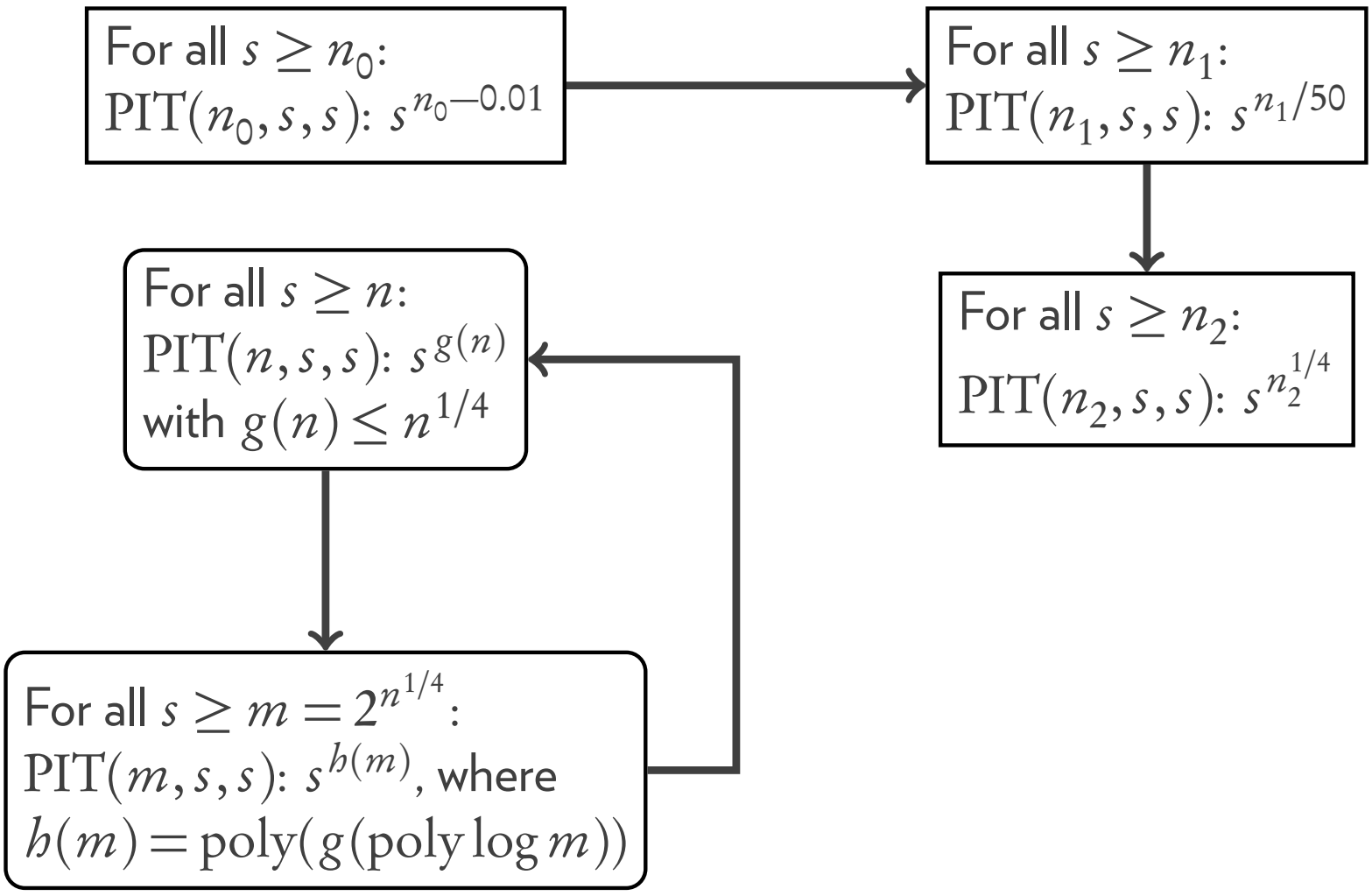

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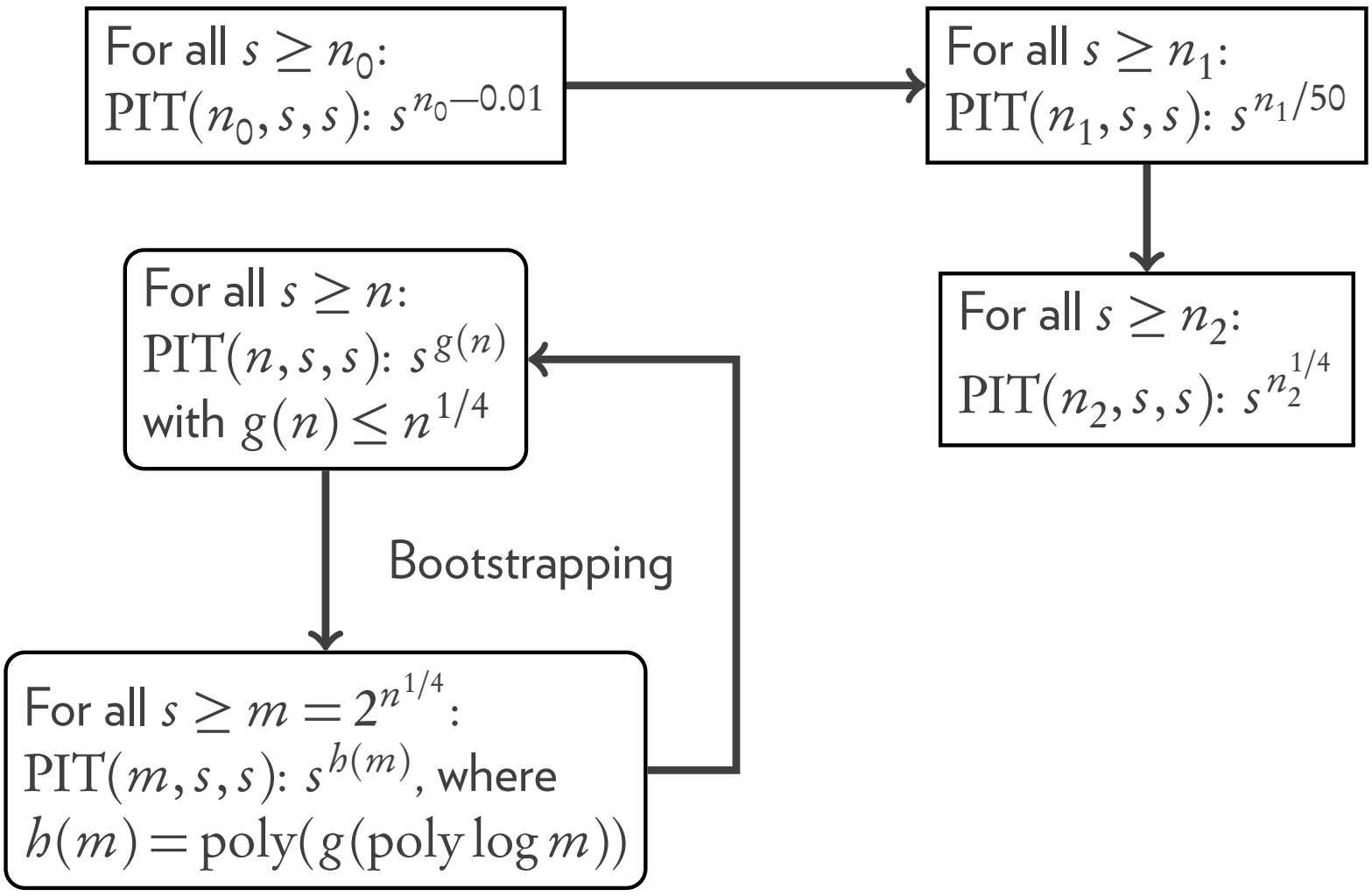

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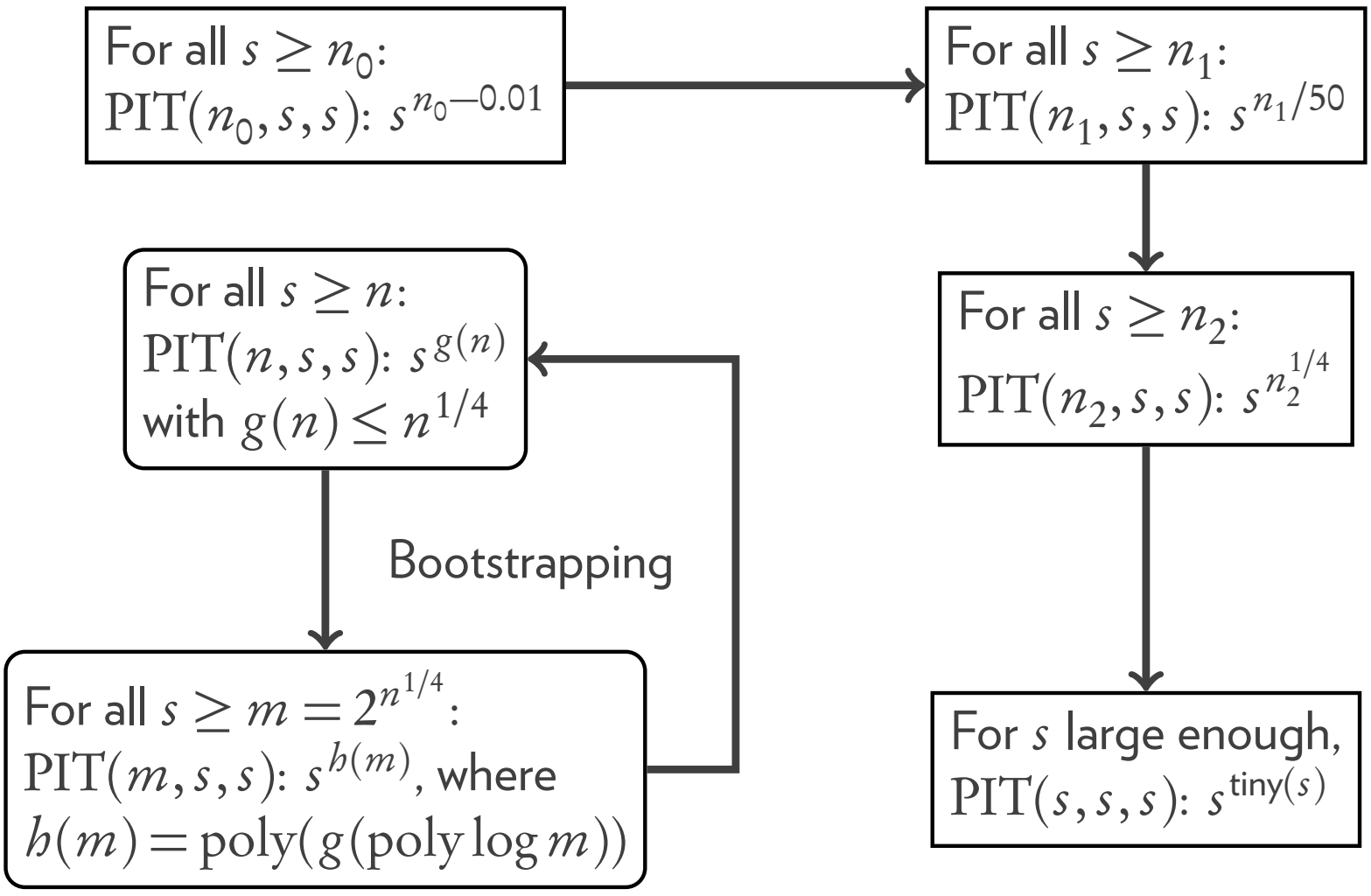

Preliminaries:

Hardness vs Randomness

for algebraic models

Lower bounds from hitting sets

H is a hitting set for $\mathscr{C}(n,d,s)$ if

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Observation

If P is a nonzero polynomial that *vanishes* on H, then P cannot be a member of $\mathscr{C}(n,d,s)$.

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Theorem ([Heintz-Schnorr, Agrawal])

For any $k \leq n$ such that $k\,|H|^{1/k} \leq d$, we can find a nonzero k-variate polynomial Q of individual degree less than $|H|^{1/k}$ such that Q requires sizes more than s.

Hitting sets from lower bounds

Intuition

If $Q(x_1, \ldots, x_k)$ is a hard, then a small circuit P cannot distinguish between $(x_1, \ldots, x_k, x_{k+1})$ and $(x_1, \ldots, x_k, Q(\overline{x}))$.

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Theorem ([Nisan-Wigderson] (Informal))

If Q is hard-enough, then a small circuit P cannot distinguish between

$$(x_1, \ldots, x_m) \quad \textit{and} \quad (Q(\overline{y}_1), \ldots, Q(\overline{y}_m))$$

even if $\overline{y}_1, \ldots, \overline{y}_m$ are almost disjoint.

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Theorem ([Kabanets-Impagliazzo] (Informal))

If Q is hard-enough, then for any small algebraic circuit computing P, we have

$$P(x_1, \ldots, x_m) = 0 \iff P(Q(\overline{y}_1), \ldots, Q(\overline{y}_m)) = 0$$

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Aside: Combinatorial Designs

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Definition (Combinatorial designs)

$\{S_1, \ldots, S_m\} \subseteq [\ell]$ is a (ℓ, k, r)-design if $|S_i| = k$ and $\left|S_i \cap S_j\right| < r$.

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Fact

For all $\ell \geq k^2$ and $r \leq k$, we have explicit (ℓ, k, r)-designs with $m = \left(\frac{\ell}{k}\right)^r$.*

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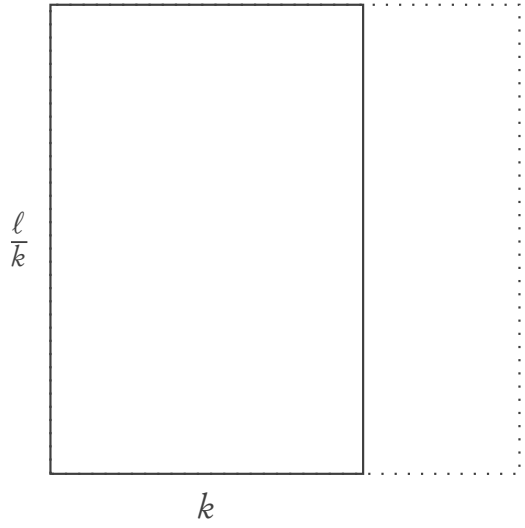

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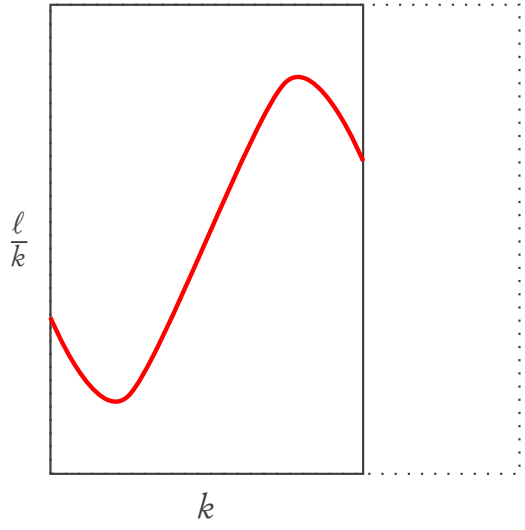

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For $p(z) \in \mathbb{F}[z]$ with $\deg(p) < r$,

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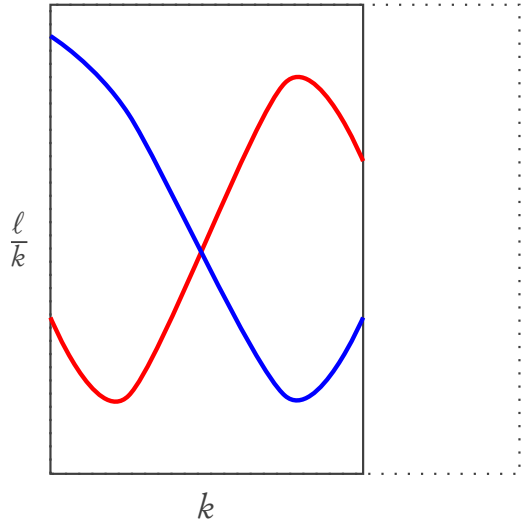

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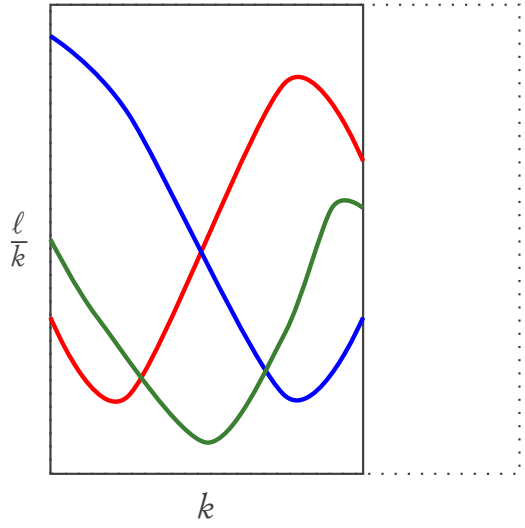

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Let $P(x_1, \ldots, x_m)$ is a nonzero polynomial of degree at most D that is computable by a size s circuit. Suppose Q is a k-variate polynomial of ind. degree $< d$ such that $P(Q[\![\ell, k, r]\!]) = 0$.
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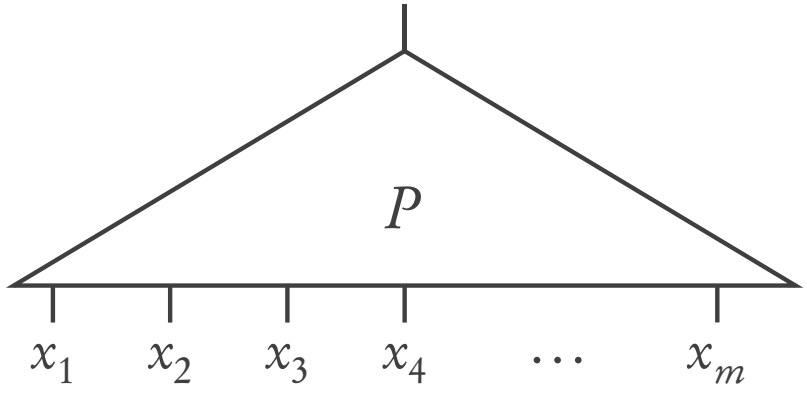

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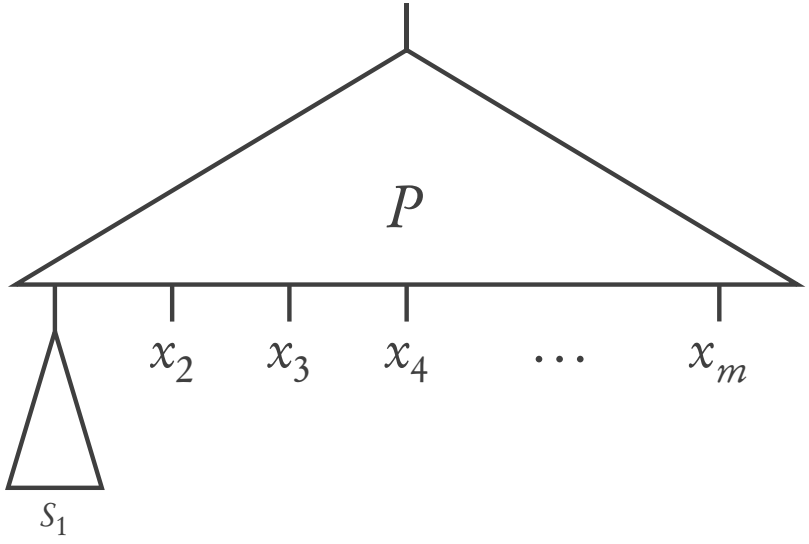

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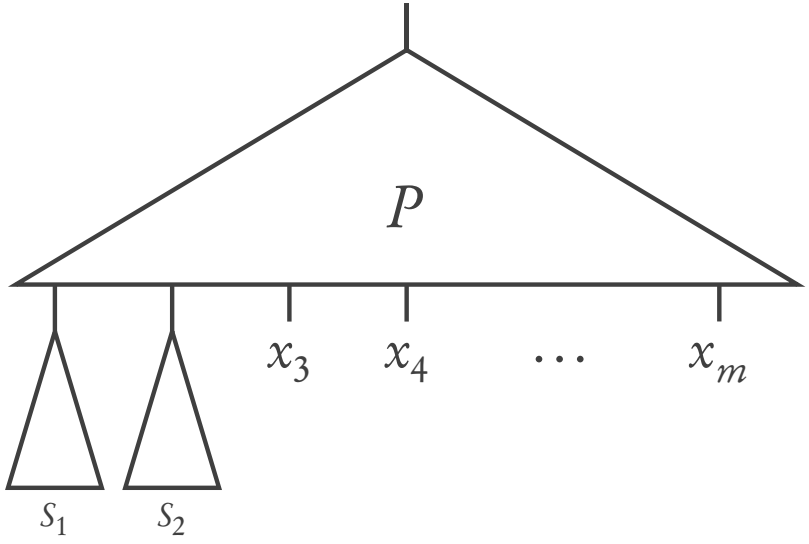

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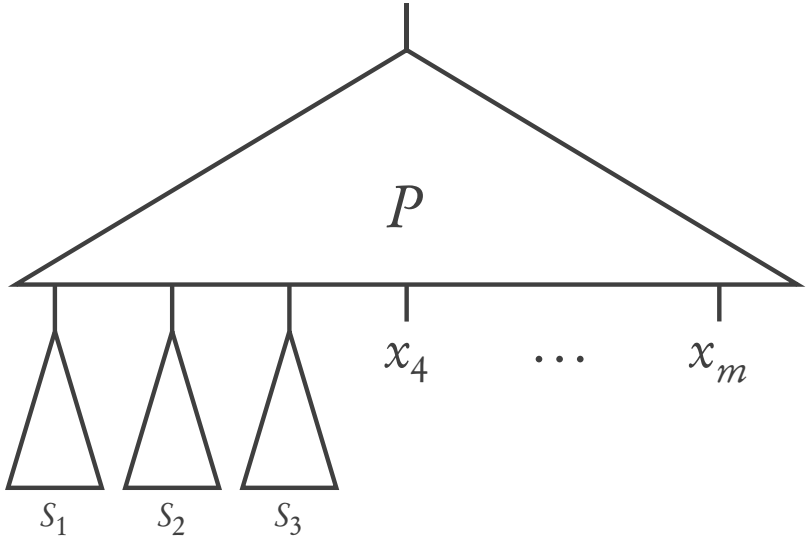

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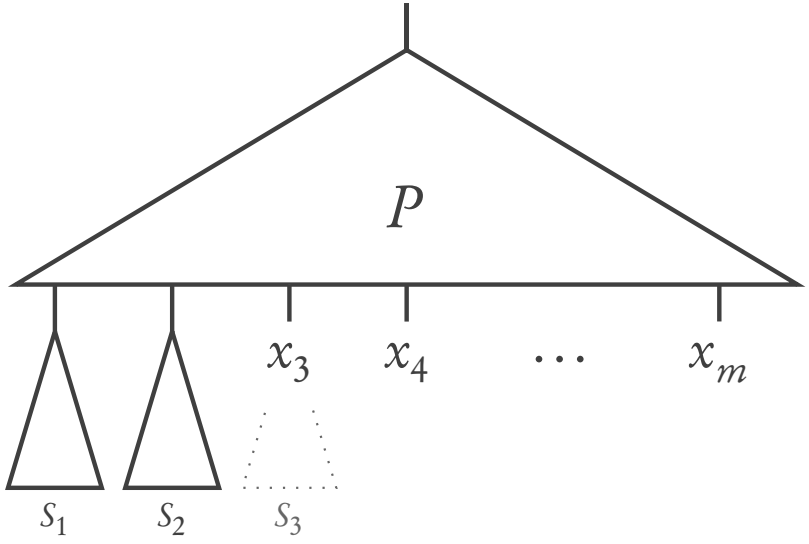

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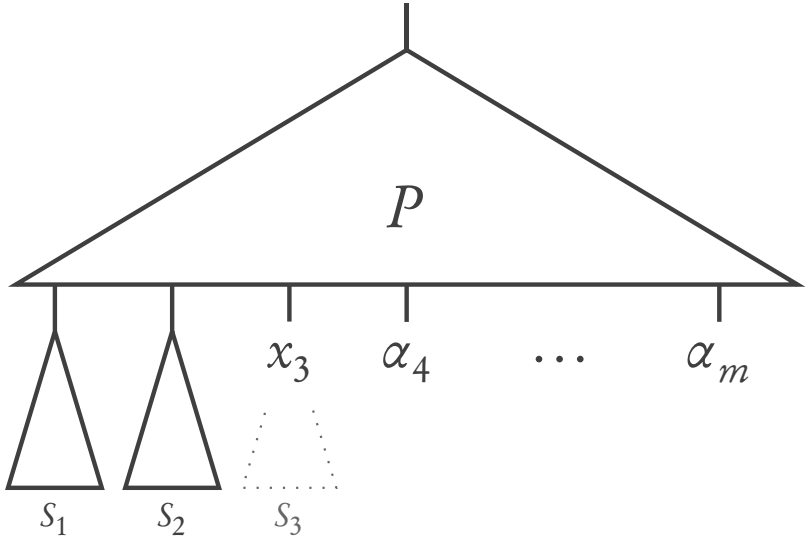

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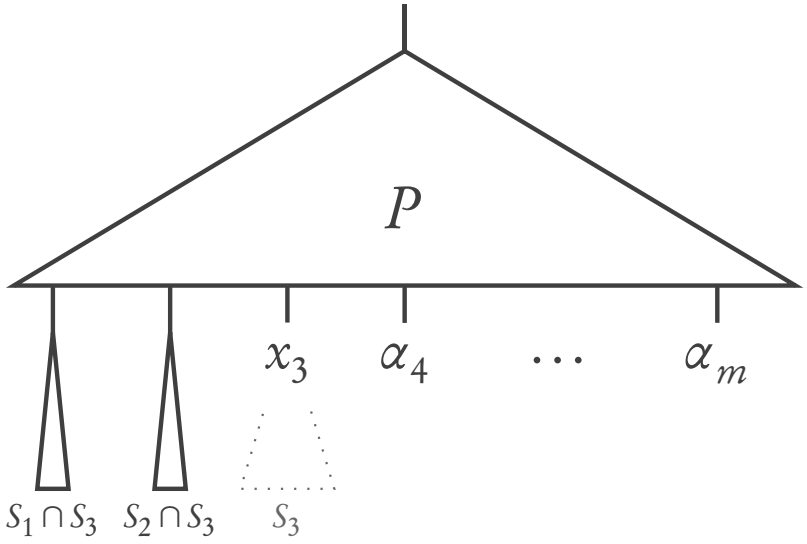

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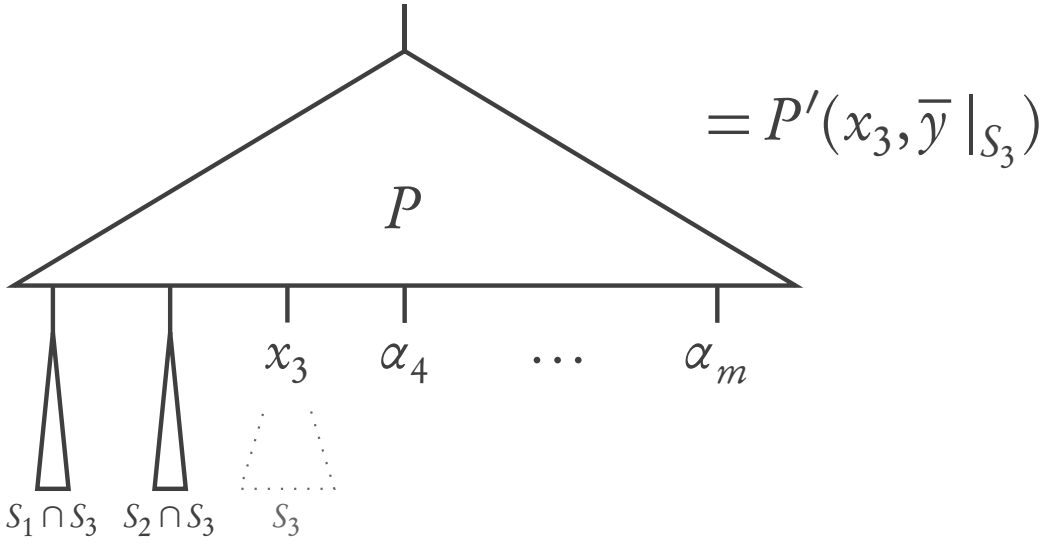

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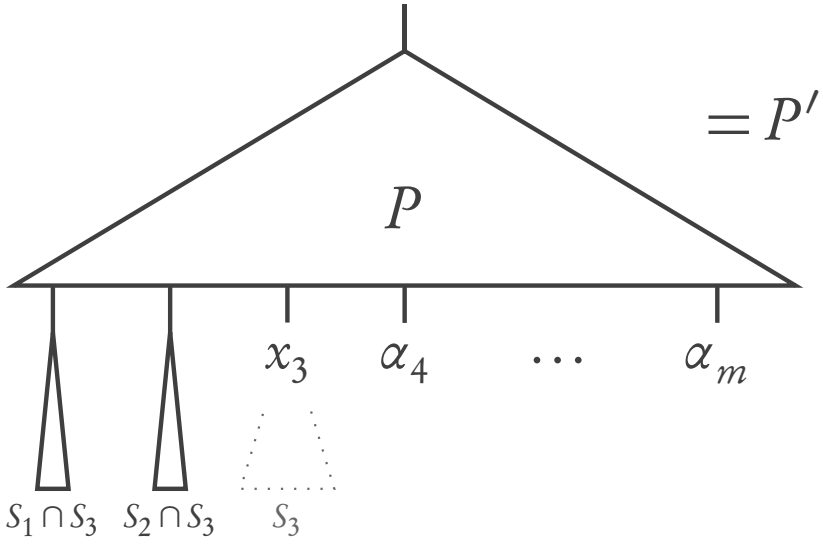

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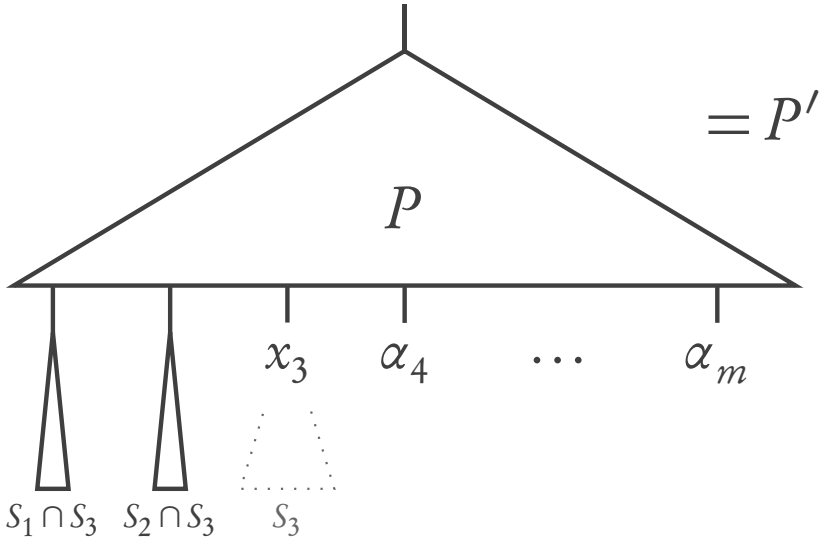

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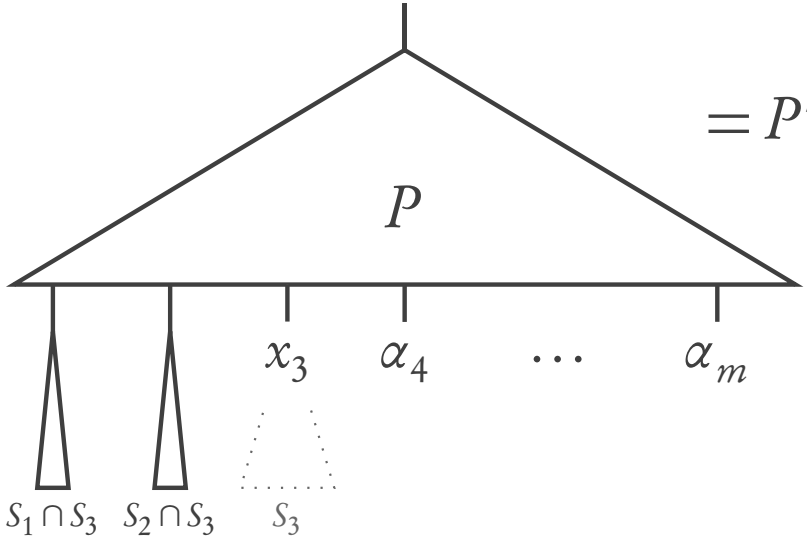

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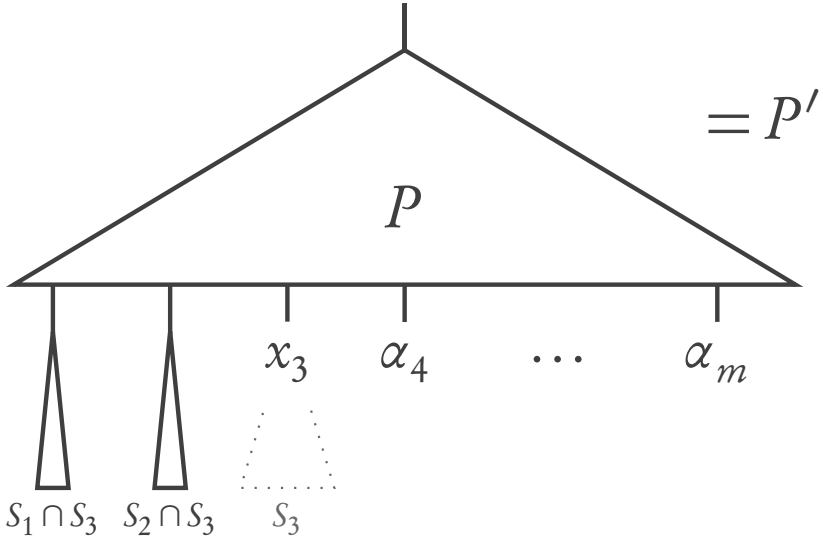

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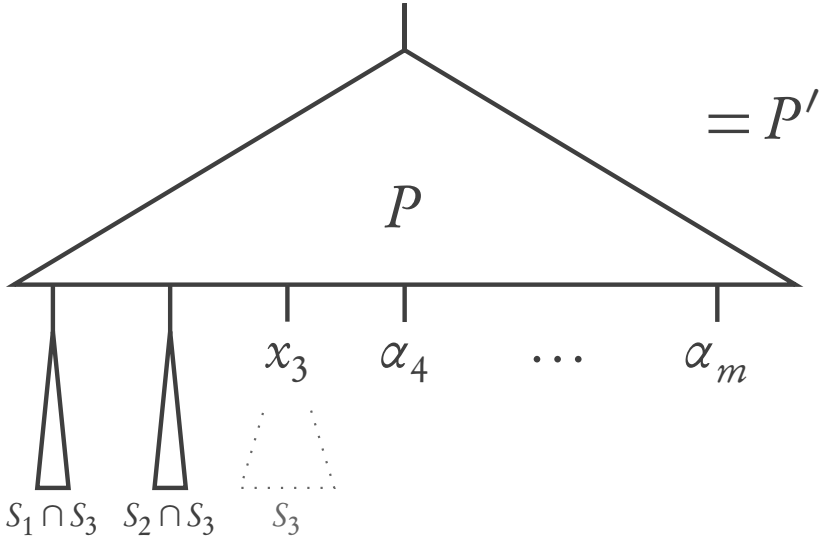

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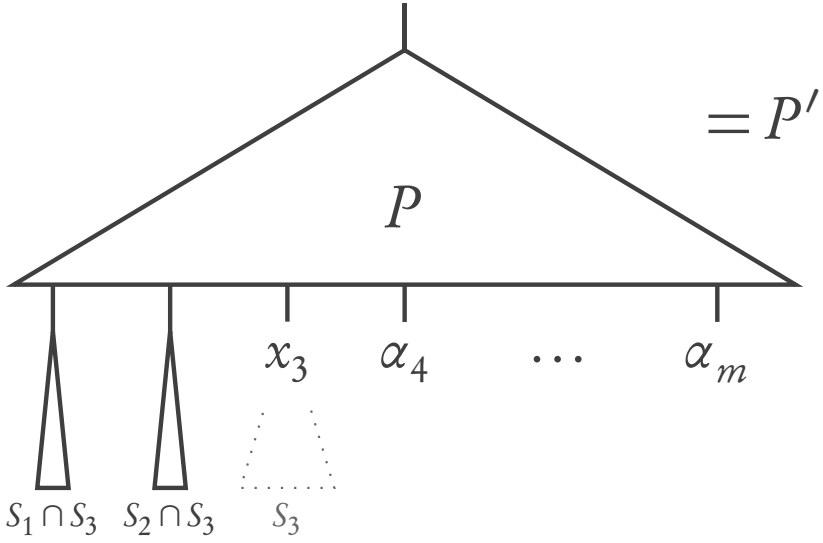

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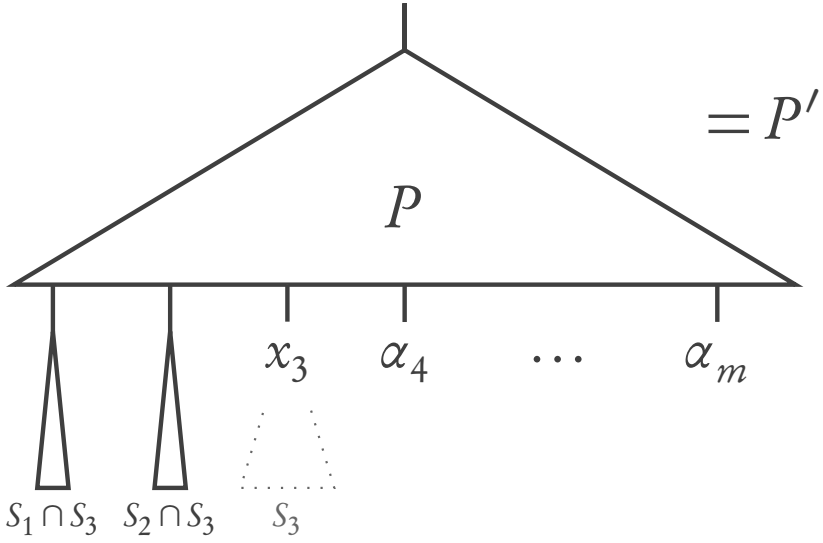

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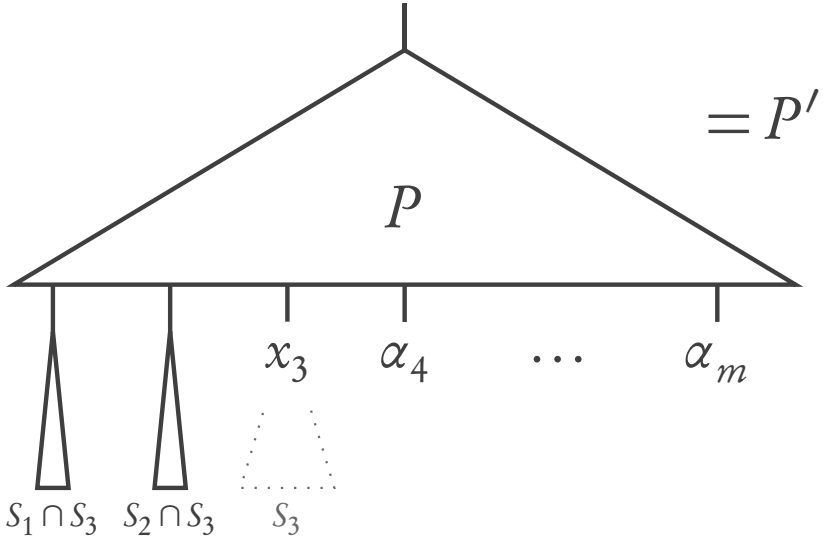

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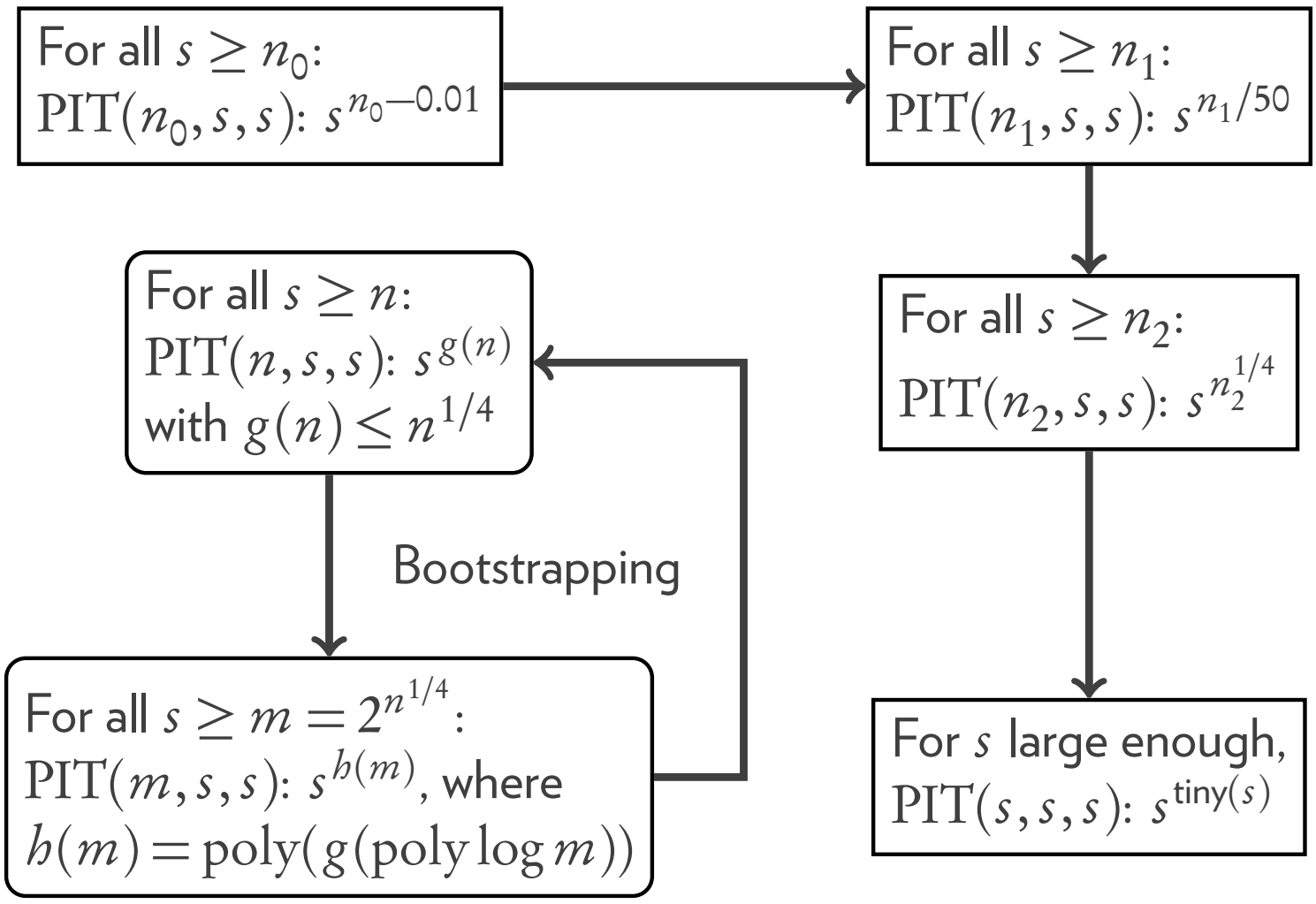
From hitting sets for k-variate formulas, we obtain a hitting set for m-variate formulas.

Plan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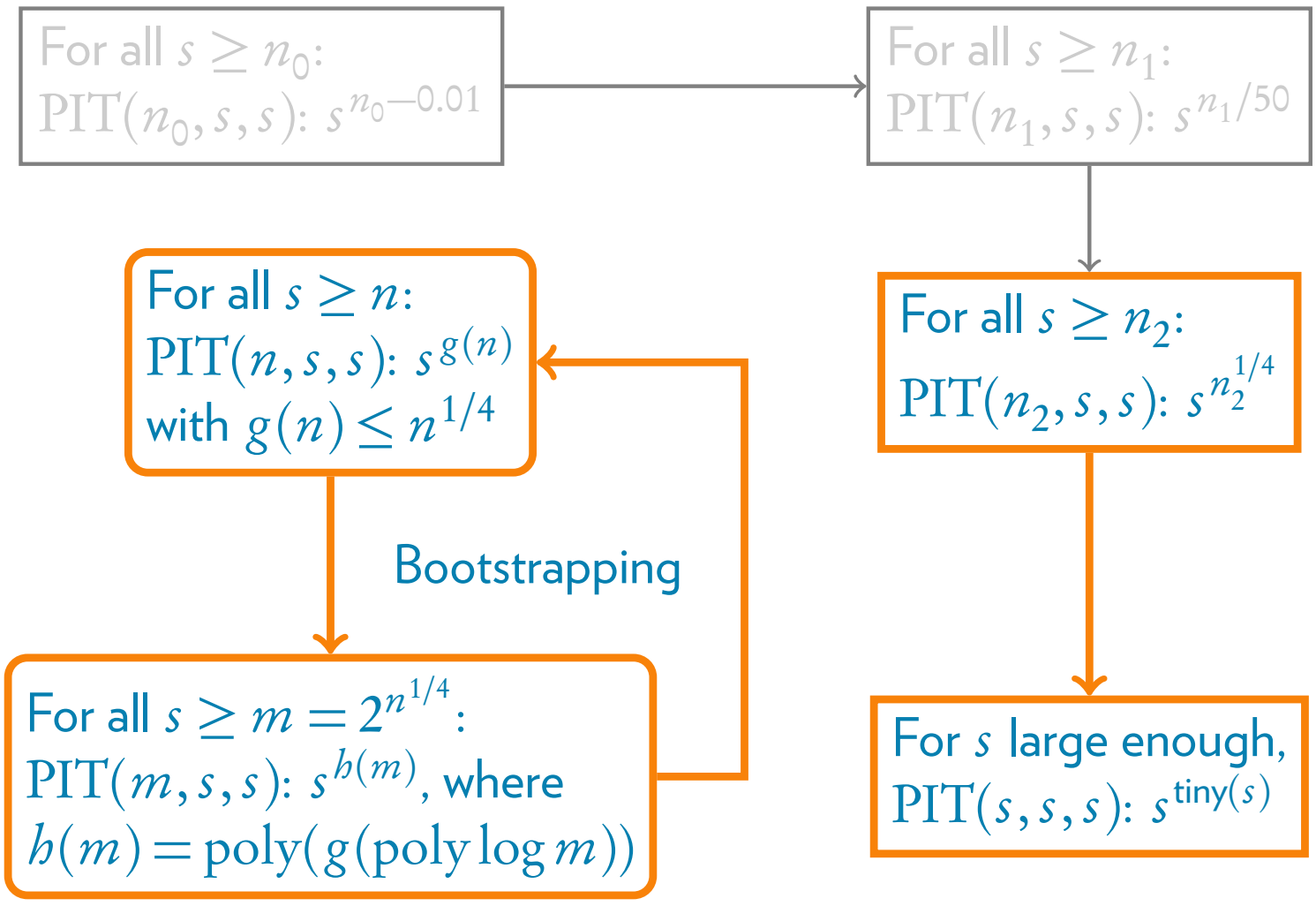
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Let n be large enough ($n > 10^{10}$). Suppose, for all $s \geq n$, there is an explicit hitting set for $\mathrm{Formula}(n, s, s)$ *of size at most*

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Using the hitting set H for $\mathrm{Formula}(n, s^5, s^5)$ of size $s^{5g(n)}$, find Q vanishing on H such that:

- Q is k-variate, and $\mathrm{ideg}(Q) < d := s^{5g(n)/k}$.
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Proof: If not, there is a nonzero multiple $\tilde{Q}$ of Q, whose degree is at most $s \cdot (rd) \leq s^5$, computable by a formula of size

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Proof of the bootstrapping lemma

Hyp: $s^{g(n)}$ hitting sets for $\mathscr{C}(n,s,s)$, for any $s \geq n$, with $g(n) \leq \frac{n^{1/4}}{10}$.

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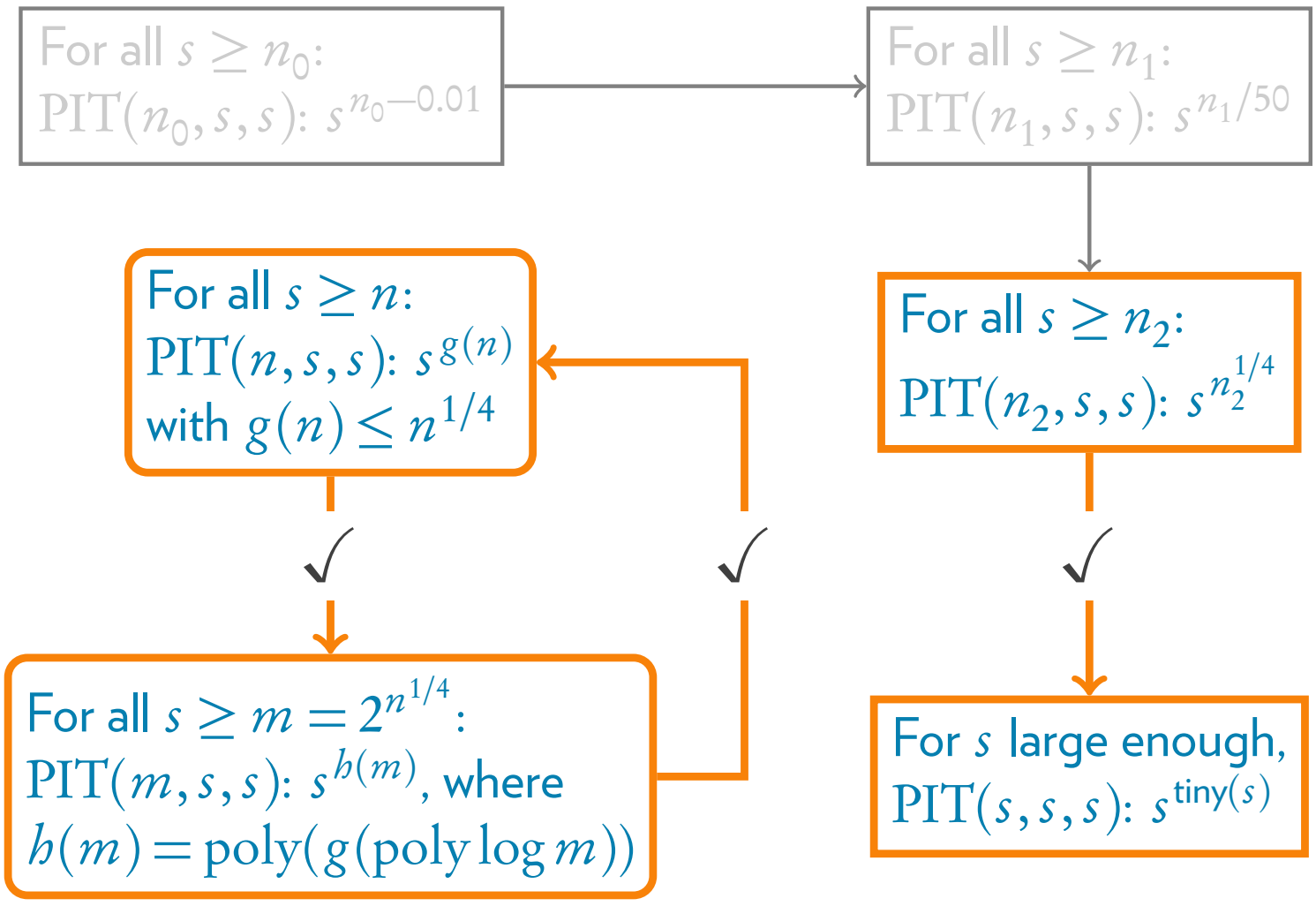
Using the hypothesis again, we get a hitting set of size $s^{20(g(n))^2}$ for $\mathrm{Formula}(m,s,s)$. □

Plan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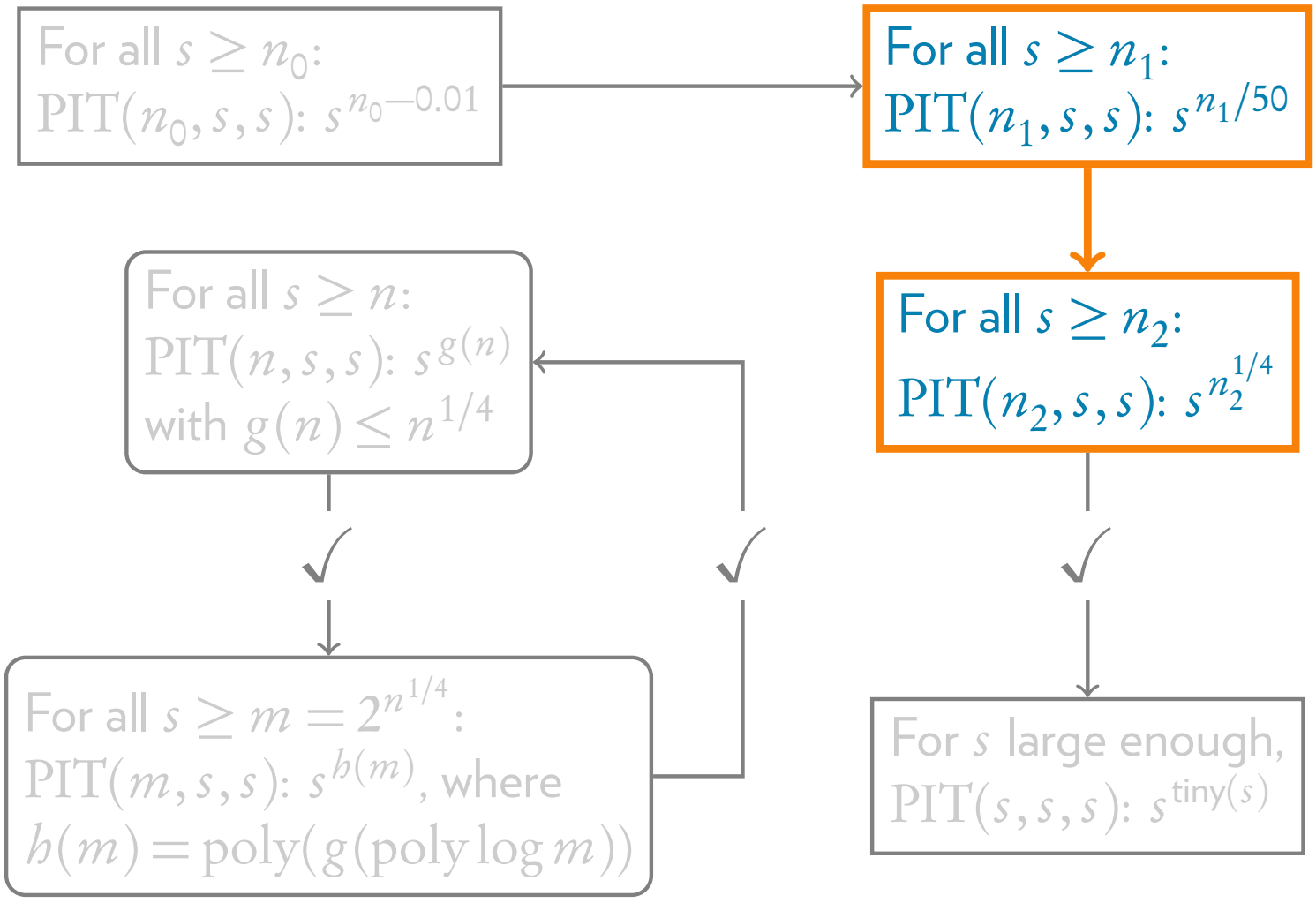
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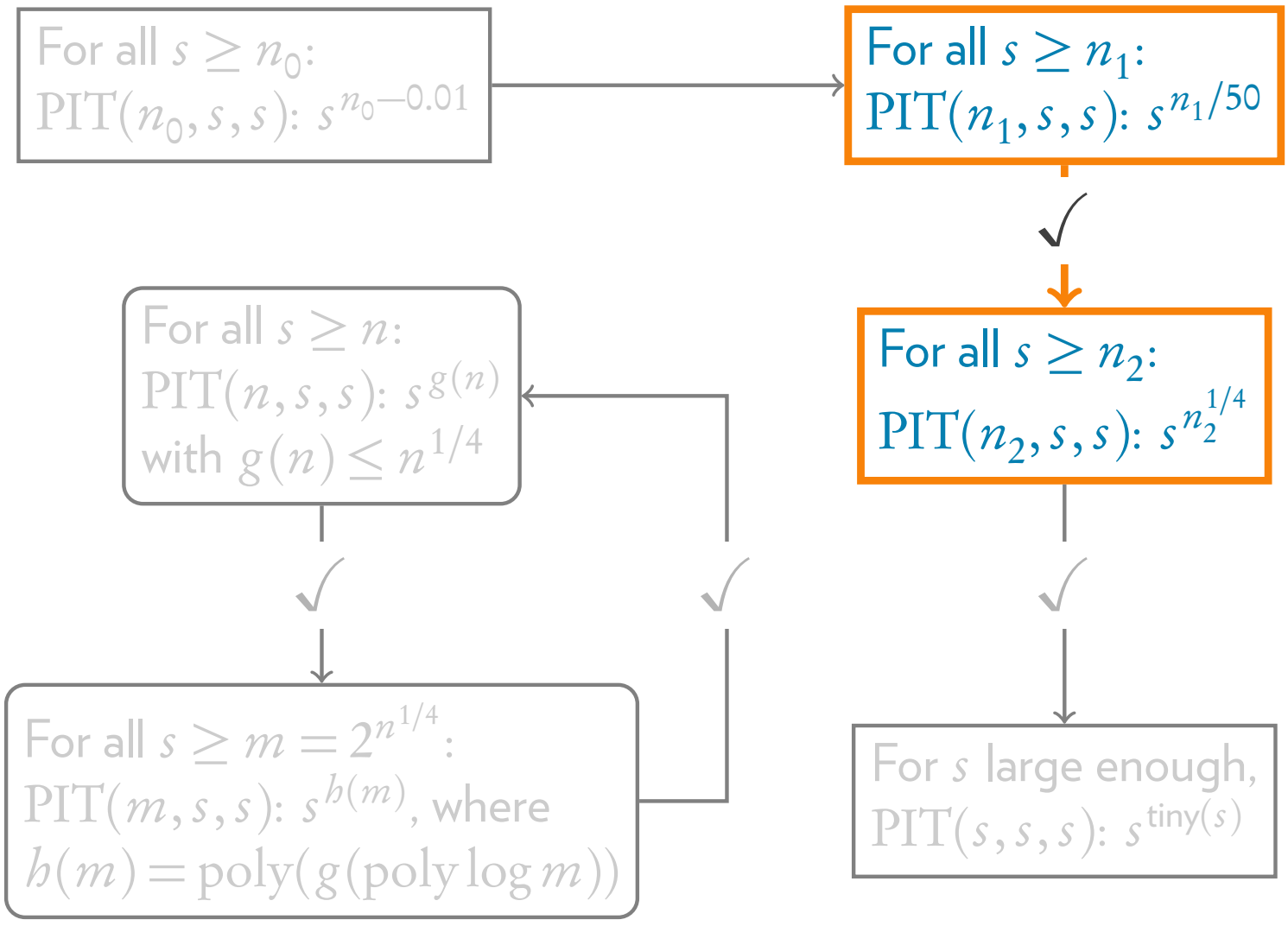
Plan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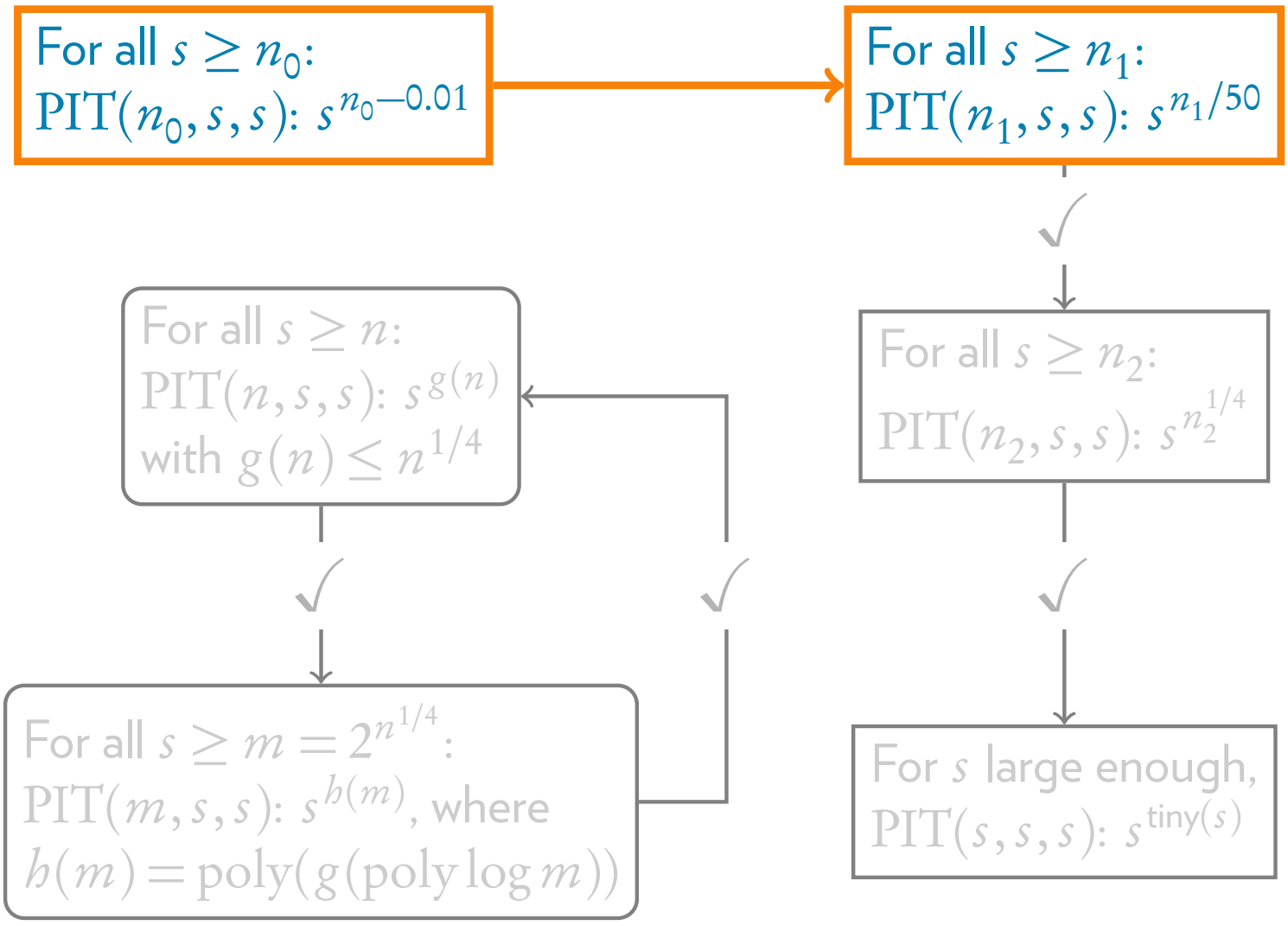
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Use the hitting set for $\mathrm{Formula}(n, s^{300n}, s^{300n})$ to get

- Q is k-variate, and $\mathrm{ideg}(Q) < d := s^{300(n-0.01)} = s^{300n-3}$.
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Complexity to compute an $(r-1)$-variate polynomial of ideg d

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Complexity to compute an univariate polynomial of degree d

Déjà vu

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Let $S_1, \ldots, S_m$ be an (ℓ, k, r)-design with $m = \left(\frac{\ell}{k}\right)^r = n^8$.

- Q is k-variate, and $\mathrm{ideg}(Q) < d := s^{300(n-0.01)} = s^{300n-3}$.
- $s^{300n} < \mathrm{FormulaSize}(Q) \leq s^{300n^2}$

Claim: $0 \neq P \in \mathrm{Formula}(m,s,s) \implies P(Q[\![\ell,k,r]\!]) \neq 0$

Proof: If not, there is a nonzero multiple $\tilde{Q}$ of Q, whose degree is at most $s \cdot (rd) \leq s^{300n}$, computable by a formula of size

$$s \cdot 10d \cdot (s+1)$$

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$$s \cdot 10d \cdot (s+1) \leq s^3 \cdot s^{300n-3} \leq s^{300n}$$ ...no way...

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$P(Q[\![\ell,k,r]\!])$ is a formula on $\ell = n^5$ variables of degree $s \cdot k \cdot s^{300n-3} \leq s^{300n}$.
[O-DL-S-Z] lemma: a hitting set of size $s^{300n \cdot n^5} \leq s^{m/50}$ for $\mathrm{Formula}(m,s,s)$.

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