The Log-Approximate-Rank Conjecture is False

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June 24, 2019

Communication Complexity

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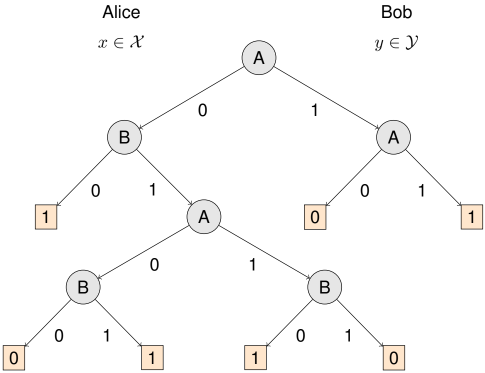
- In this talk, we focus on two parties (Alice and Bob) computing a Boolean function.

A Communication Protocol



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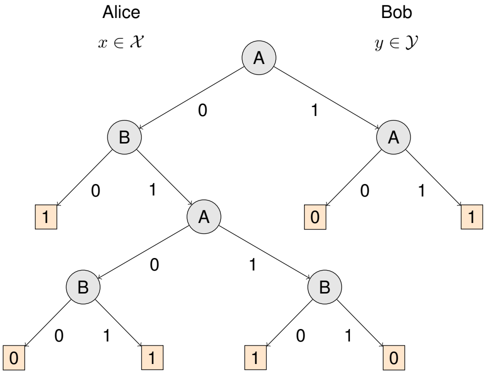

(x, y) is accepted

$\Leftrightarrow$

(x, y) reaches a 1-leaf.

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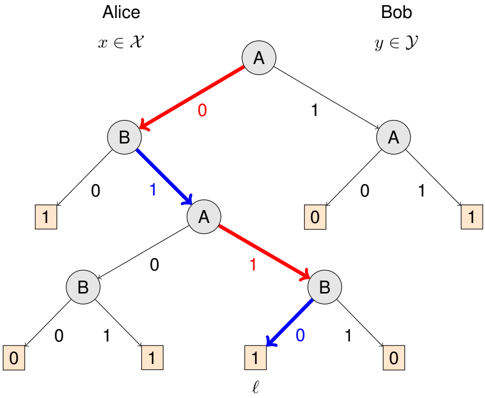

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Inputs that reach ℓ

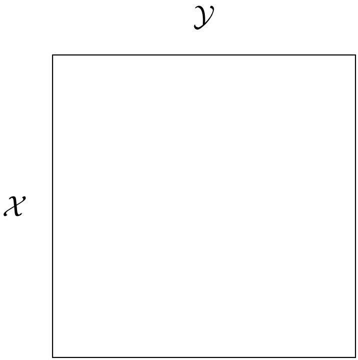
$=$

$\{x : x$ answers red$\}$

$\times$

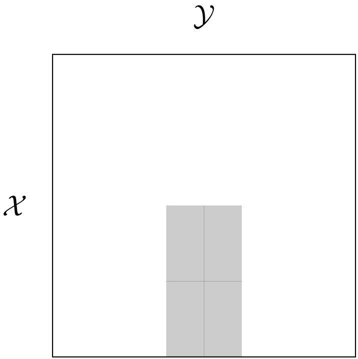
$\{y : y$ answers blue$\}$.

Rank



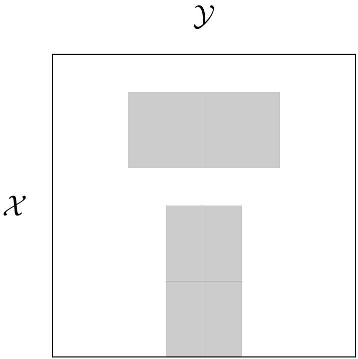
Building the truth table for the function computed by the protocol.

Rank



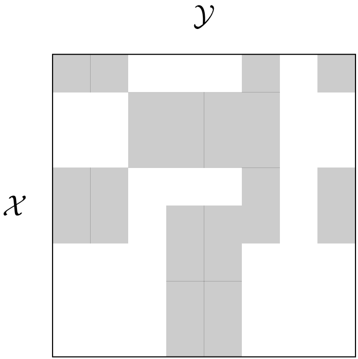
Inputs that reach leaf ℓ contribute a rank 1 matrix.

Rank



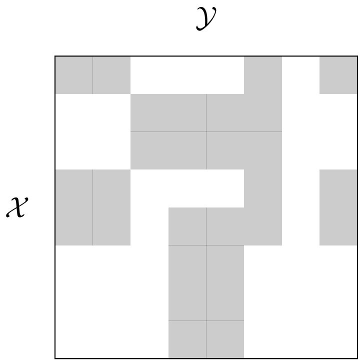
Inputs that reach leaves ℓ_1 or ℓ_2 form a rank ≤ 2 matrix.

Rank



Inputs that reach any 1 leaf form a rank $\leq 2^c$ matrix.

Rank



Cost c protocol for F

$$\Longrightarrow$$

M_F has rank $\leq 2^c$.

Protocol-Rank Equivalence?

Conjecture (Lovász Saks '88)

$\exists$ *constant* α *s.t.* $D(F) \leq \log^{\alpha} \mathsf{rank}(F)$

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Fun fact: LRC is True if you restrict the rank decomposition to be nonnegative.

A Randomized Communication Protocol

Alice

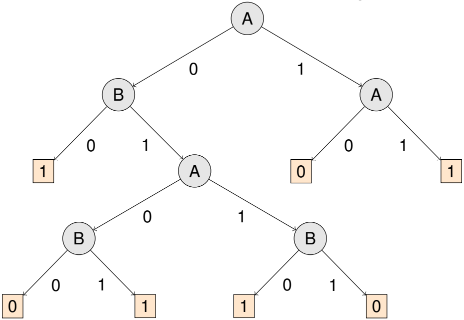
$x \in \mathcal{X}$

r_A, a string of random bits

Bob

$y \in \mathcal{Y}$

r_B, a string of random bits



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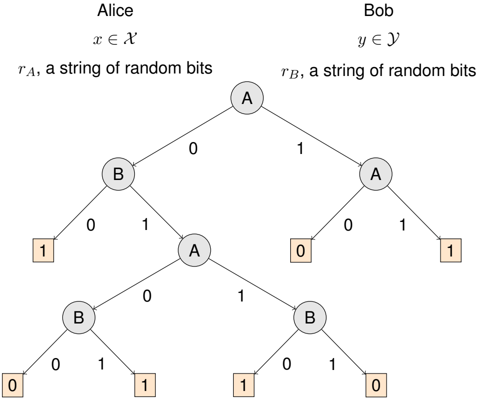

$$\Pr[(x, y) \text{ is accepted}] = \Pr[(x, y) \text{ reaches a } 1\text{-leaf}].$$

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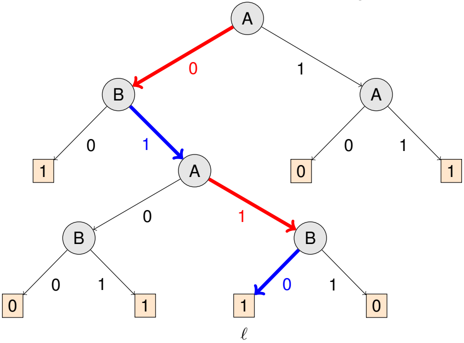

$$\Pr[(x, y) \text{ is accepted}]$$
$$=$$
$$\Pr[(x, y) \text{ reaches a } 1\text{-leaf}].$$

$$\Pr[(x, y) \text{ reaches } \ell]$$
$$=$$
$$\Pr_{r_A}[x \text{ answers red}]$$
$$\times$$
$$\Pr_{r_B}[y \text{ answers blue}].$$

Small Approximate Rank

$\mathrm{Pr}_{r_B}[y$ answers blue$]$

$\mathrm{Pr}_{r_A}[x$ answers red$]$	0	.5	0	.6
.5		.25		.3
.8		.4		.48
0				
0				

$\mathrm{Pr}[(x, y)$ reaches $\ell]$ is a rank 1 matrix.

Small Approximate Rank

$\Pr_{r_B}[y \text{ answers blue}]$

$\Pr_{r_A}[x \text{ answers red}]$

	0	.5	0	.6
.5		.25		.3
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$\Pr[(x, y) \text{ reaches } \ell]$ is a rank 1 matrix.

$\Pr[(x, y) \text{ is accepted}]$ is a rank $\leq 2^c$ matrix.

1	1	0	0
0	1	0	0
0	0	1	0
0	0	0	1

M_F

.8	.9	.1	.2
0	.9	.1	.1
0	.1	.8	0
.1	0	0	1

$M_{\text{Pr of accepting}}$

1	1	0	0
0	1	0	0
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.8	.9	.1	.2
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Rank $\leq 2^c$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

M_F

Approx. Rank $\leq 2^c$

$$\begin{bmatrix} .8 & .9 & .1 & .2 \\ 0 & .9 & .1 & .1 \\ 0 & .1 & .8 & 0 \\ .1 & 0 & 0 & 1 \end{bmatrix}$$

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$M_{\text{Pr of accepting}}$

Rank $\leq 2^c$

$$\log \mathsf{rank}_{1/3}(F) \leq c.$$

Protocol-Rank Equivalence?

Conjecture (ForgeGod '05, Lee Shraibman '07)

$$\exists \textit{ constant } \beta \textit{ s.t. } R(F) \leq \log^{\beta} \mathsf{rank}_{1/3}(F)$$

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For a randomized protocol, the number of bits exchanged in the worst case, R(f), is conjectured to be polynomially related to the following absurd formula:

min{rank(M'_f): $M'_f \in \mathbb{R}^{2^n \times 2^n}$, $(M_f - M'_f)_\infty \leq 1/3$}.

Figure: Screenshot from “Communication complexity - Wikipedia” (Dec ‘05)

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Implies the LRC! [Gavinsky Lovett '13]

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Set Disjointness shows that $\beta \geq 2$. [Kalyanasundaram Schnitger '92, Razborov '92]

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[Göös Jayram Pitassi Watson '17] showed that $\beta \geq 4$.

Protocol-Rank Non-Equivalence

Theorem (Chattopadhyay Mande S '19)

There is a function $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ *such that* $\mathsf{rank}_{1/3}(F) \leq O(n^2)$, *but* $R(F) \geq \Omega(\sqrt{n})$.

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The Function

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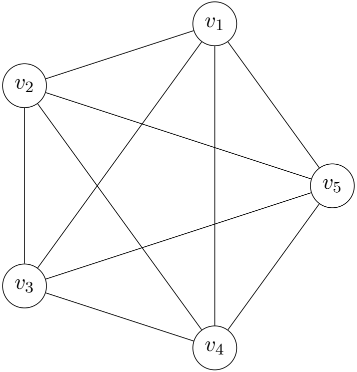

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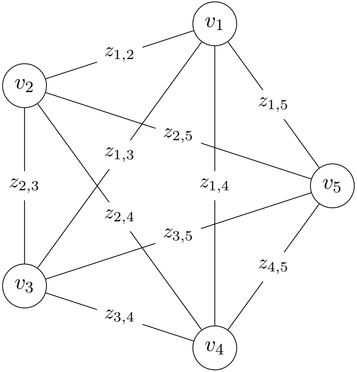

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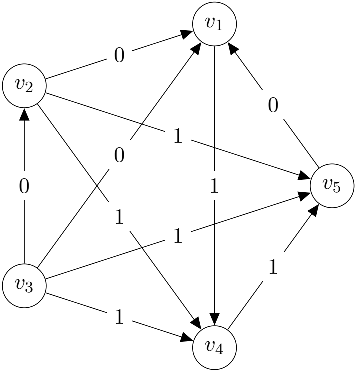

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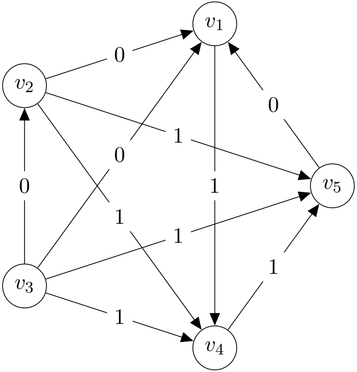
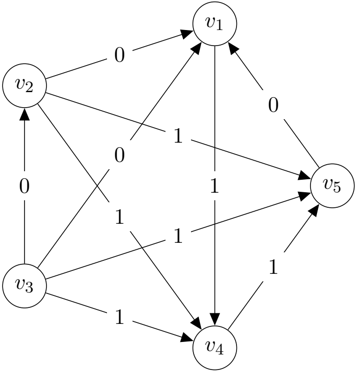


$\mathsf{SINK}(z) = 1$ iff there is a sink in the graph G_z.

The Function

$$F := \mathsf{SINK} \circ \mathsf{XOR} : \{0,1\}^{\binom{m}{2}} \times \{0,1\}^{\binom{m}{2}} \to \{0,1\}$$



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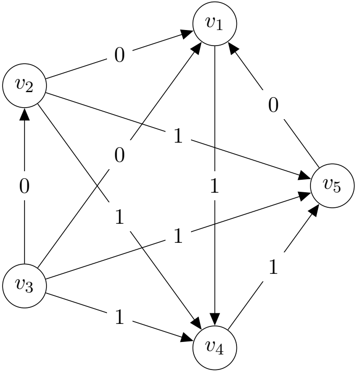

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Alice

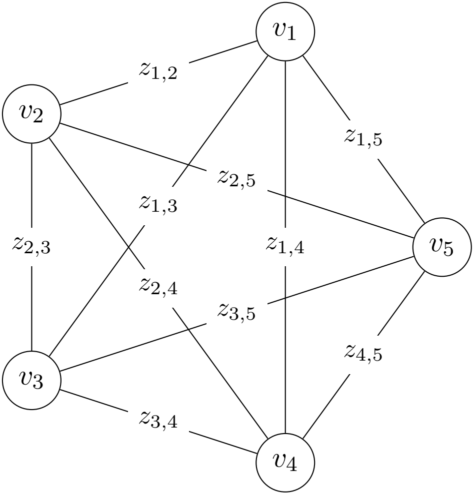
$x \in \{0,1\}^{\binom{m}{2}}$

$z = x \oplus y$

Bob

$y \in \{0,1\}^{\binom{m}{2}}$

Small Approximate Rank



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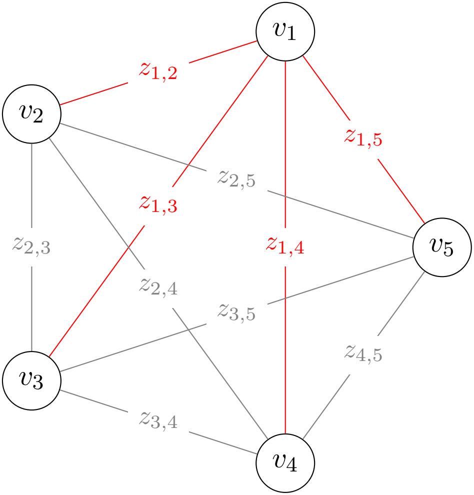

Whether or not v_1 is a sink is decided by the red variables, z_{v_1}.

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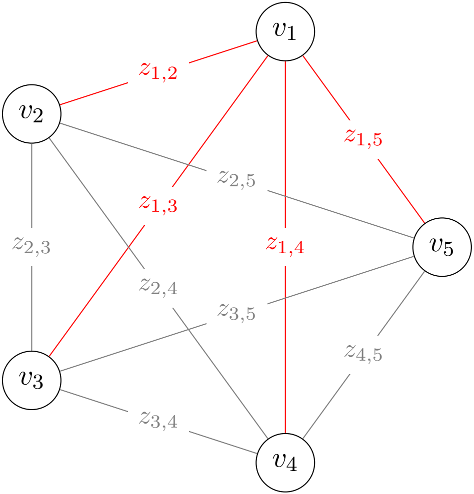

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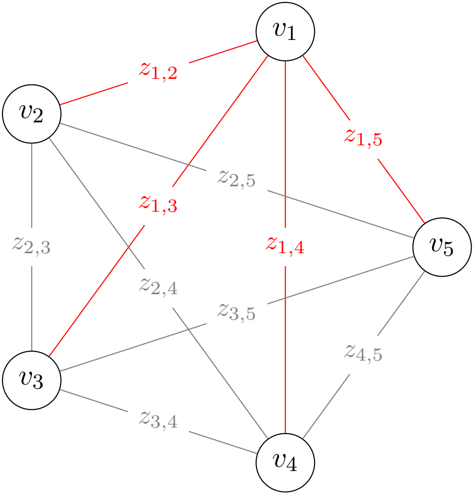

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$M_{v_1 \text{ is a sink}}$ has small approximate rank.

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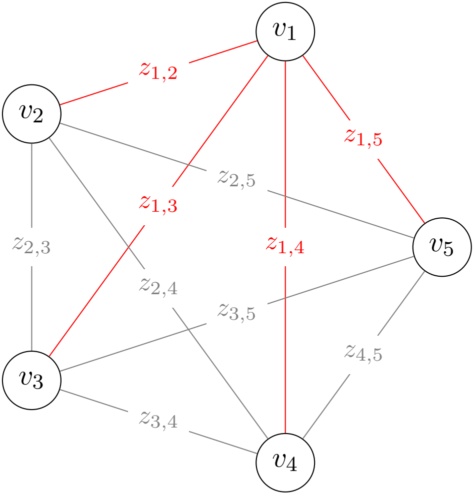

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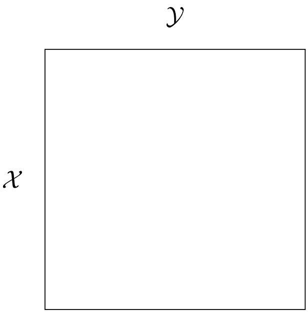
$M_F = \sum M_{v_i \text{ is a sink}}$ has small approximate rank.

Randomized Communication Lower Bound



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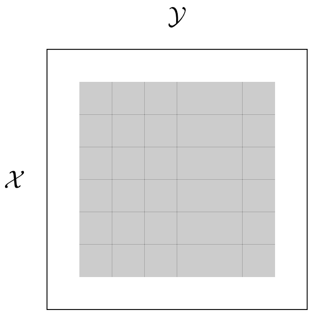
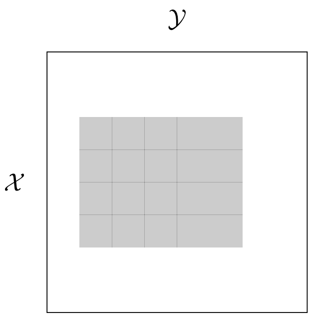


- A rectangle “biased” against v_1 being a sink must be small. (Follows from [Gavinsky ‘16].)

Randomized Communication Lower Bound



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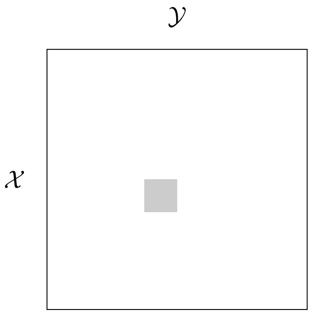

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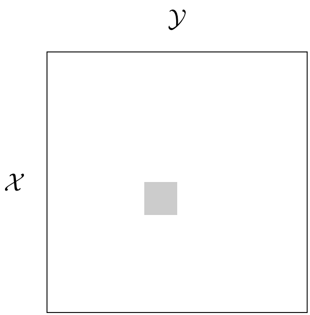

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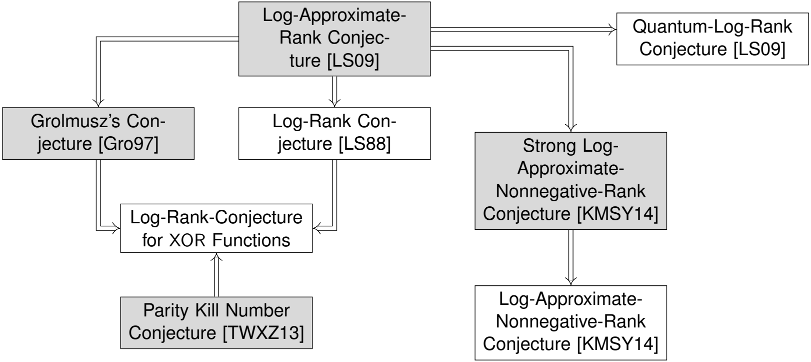
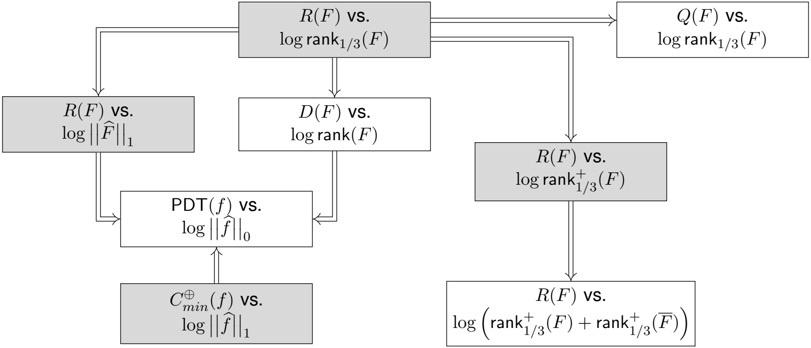
- A rectangle "biased" against sinks must be tiny.

Any randomized protocol for F must be costly.

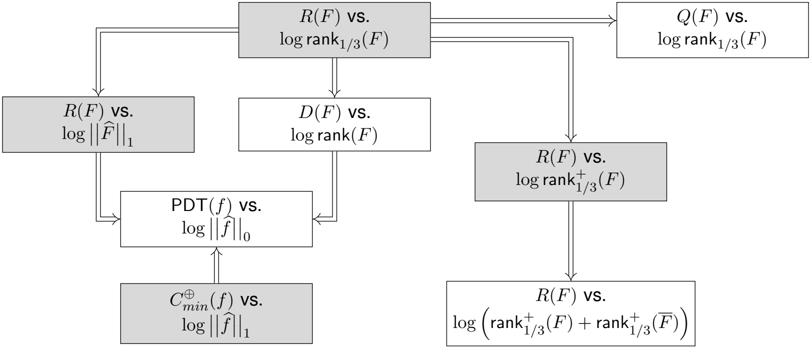
Other Sunken Conjectures



Other Sunken Conjectures



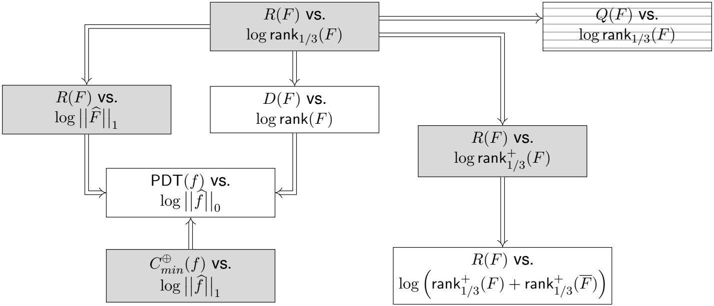
Other Sunken Conjectures



$\mathrm{rank}^{+}_{\epsilon}(\overline{\mathrm{SINK} \circ \mathrm{XOR}}) \geq 2^{\Omega(\sqrt{n})}$

Other Sunken Conjectures

[Anshu Boddu Touchette '18, Sinha & de Wolf '18]



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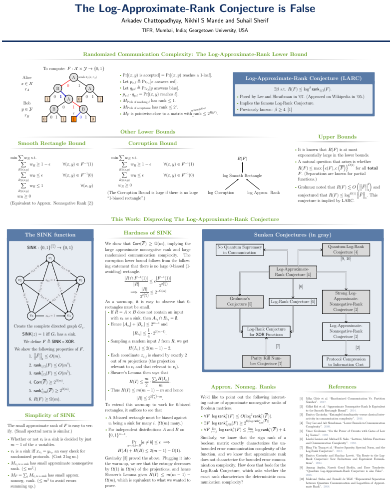
So what now?

- ~~Quantum vs Log Approximate Rank?~~
- Can the Log Approximate Nonnegative Rank Conjecture be similarly refuted?
- What other functions refute the LARC?

Thank You

Questions are welcome if time permits.
Find out more at the poster session!



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