Algorithmic Game Theory

Assignment 1: due September 21, 2022

Assignment policies:

1. While you may dicuss the problems with others, you must write up the solution by yourself, in your own words.
2. Please write in your submission the people with whom you discussed the problems, as well as any references you used.
3. Late submissions will not be accepted, unless agreed to by me **prior** to the last date for submission.
4. Please write clearly and legibly, and include how you arrived at the solution!

Question 1 [5]: Think of a social situation involving two people that can be modeled by a bimatrix game (e.g., Prisoners' Dilemma and Battle of the Sexes). Describe the situation and model it as a bimatrix game, explaining the rationale for the payoffs. Compute the equilibrium in this game.

Question 2 [5]: Show that, in an 2-player game, the strategies remaining for any player after iterated removal of strictly dominated strategies is independent of the order of removal.

Question 3 [5]: Given a bimatrix game, let $s^* = (s_1^*, s_2^*)$ be a pure Nash equilibrium. Let S_1, S_2 be the pure strategies remaining for the players after iterated removal of strictly dominated strategies. Show that $s_i^* \in S_i$ for $i \in \{1, 2\}$.

Further, show that if $S_i = \{t_i\}$, then (t_1, t_2) is a pure Nash equilibrium. That is, if there is a single strategy left for each player after removal of strictly dominated strategies, then the strategies remaining form a pure Nash equilibrium.

Question 4 [10]: Consider a 2-player game where the payoff matrices R, C are of size $n \times n$, and each entry of R and C is chosen independently and uniformly at random from the interval $[0, 1]$. Show that for large n, the probability that this game has a Nash equilibrium in pure strategies is at least $(1 - 1/e)$. Keep in mind that the events "(i, j) is a pure Nash equilibrium" and "(i, j') is a pure Nash equilibrium" are not independent.

Question 5 [10]: Players 1 and 2 are playing a zero-sum game with payoff matrix A for player 1, while simultaneously players 1 and 3 are playing a zero-sum game with payoff matrix B for player 1. Player 1 is constrained to use the same strategy in both games, and his payoff is the sum of his payoffs from the two games he plays. Give an efficient algorithm for computing equilibrium strategies for the three players.

You may need to take another look at LP duality to address this problem.

Question 6 [10]: Consider the polytope $Rx \leq 1$, where R is an $n \times n$ matrix with each entry in $[0, 1]$. Show that, for small enough ϵ, the polytope

$$x \geq 0$$
$$Rx \leq \begin{bmatrix} 1 \\ 1+\epsilon \\ 1+\epsilon^2 \\ \dots \\ 1+\epsilon^{n-1} \end{bmatrix}$$

is non-degenerate (it is sufficient to show that at any vertex, at most n constraints are tight).

Question 7 [10]: For a 2-player game with payoff matrices R and C of size $m \times n$, the strategy profile (x, y) is an ϵ-well supported Nash equilibrium for $\epsilon \geq 0$ if:

$$\forall i \in [m], x_i > 0 \Rightarrow (Ry)_i \geq \max_k (Ry)_k - \epsilon\,, \text{ and}$$
$$\forall j \in [n], y_j > 0 \Rightarrow (x^T C)_j \geq \max_k (x^T C)_k - \epsilon$$

That is, the equilibrium strategies are supported on ϵ-best responses.

Show that there exists a polynomial $p(\cdot)$ such that, given a two-player game (R, C) of bit-complexity b, and a $\frac{1}{2^{p(b)}}$-well supported Nash equilibrium, an exact equilibrium can be computed in polynomial time.

You may use the fact that an exact solution to a linear program can be obtained in time polynomial in the bit-complexity of the input linear program.

Question 8 [10]: Use the above two questions to suitably modify the Lemke-Howson algorithm to obtain an exact equilibrium, even if the polytope $P = \{x \geq 0,\ Rx \leq 1\}$ is degenerate.

Question 9 [10]: Recall the reduction shown in class from computing equilibria in a bimatrix game to computing equilibria in a symmetric game. Use it to show that there is a one-to-one mapping between equilibria in a bimatrix game and symmetric equilibria in the symmetric game.