Algorithmic Game Theory

Assignment 2: due March 11, 2025

Assignment policies:

1. While you may dicuss the problems with others, you must write up the solution by yourself, in your own words.
2. Please write in your submission the people with whom you discussed the problems, as well as any references you used.
3. Late submissions will not be accepted, unless agreed to by me **prior** to the last date for submission.
4. Please write clearly and legibly, and include how you arrived at the solution!

Question 1 [10]: Consider the bimatrix game (A, B) where A, $B \in \mathbb{R}^{n \times n}$ and, further, $\text{rank}(A) = \text{rank}(B) = k$. Give an algorithm that computes a Nash equilibrium in this game in time $\text{poly}(n^{O(k)}, |A|, |B|)$ where, as before, $|x|$ is the bit-complexity of x. You may need to use Caratheodory's theorem:

Theorem 1. *Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of points in $\mathbb{R}^k$, and y lie in the convex hull of S. Then there exists $\bar{S} \subseteq S$ of cardinality $k+1$, so that y lies in the convex hull of the points in $\bar{S}$.*

Question 2 [10]: Recall the single-agent regret-minimization problem with n pure strategies studied in class, for which we showed that the multiplicative weight algorithm with $\epsilon = \sqrt{\ln n/T}$ has regret $2\sqrt{\ln n/T}$. Modify the algorithm to remove the assumption that T is known to the algorithm, while maintaining a bound of $O(\sqrt{\ln n}/\sqrt{T})$ on the regret.

Question 3 [10]: We saw in class that any deterministic regret-minimization algorithm, that selects a point distribution p^t at each time t, has regret at least $1 - 1/n$. Consider the deterministic regret-minimization algorithm that at each time t, selects the pure strategy that has least cumulative cost so far. That is, $p^t(a) = 1$ for some $a \in \arg\min \sum_{\tau \leq t} c^\tau(a)$. Show the regret of this algorithm (called "Follow-the-Leader") is at most

$$\frac{(n-1)\text{OPT}}{T} + \frac{n}{T}.$$

Question 4 [10]: Recall the value of a zero-sum game: this was the payoff for the row player in any Nash equilibrium of the game. Show that, in fact, this extends to CCE of zero-sum games as well: any CCE has the same payoff for the row-player.

Question 5 [5]: In a 3-player zero sum game, for any pure strategy profile s, $\sum_{i=1}^{3} u_i(s) = 0$. Either give an efficient algorithm for computing an MNE in a 3-player zero-sum game, or prove that computing a MNE in a 3-player zero-sum game is PPAD-hard.

Question 6 [10]: Given a 2-player game (R, C), prove that the following problems are either in P or are NP-complete:

- Determine if there exists an MNE (x^*, y^*) where both players play each pure strategy with positive probability (i.e., $x_i^* > 0$ for all i, and $y_j^* > 0$ for all j).
- Determine if there exists an MNE (x^*, y^*) where $x_1^* = 1$ (i.e., a given pure strategy is played with probability 1).

Question 7 [10]: Players 1 and 2 choose an element of the set $\{1, \cdots, K\}$. If the players choose the same number, then player 2 pays 1 rupee to player 1; otherwise no payment is made. Find all pure and mixed strategy Nash equilibrium of this game.