Algorithmic Game Theory

Assignment 3: due March 28, 2025

Assignment policies:

1. While you may dicuss the problems with others, you must write up the solution by yourself, in your own words.

2. Please write in your submission the people with whom you discussed the problems, as well as any references you used.

3. Late submissions will not be accepted, unless agreed to by me **prior** to the last date for submission.

4. Please write clearly and legibly, and include how you arrived at the solution!

Question 1 [5]: Prove that for atomic network congestion games with affine cost functions on the edges, the Price of Stability is at most 2.

Question 2 [10]: Consider the *weighted* version of a network atomic congestion game, where each player i has weight w_i, played on parallel edges. The cost of each edge $c_e(x_e(s))$ for a pure strategy profile s is now an increasing function of the total weight $x_e(s) = \sum_{i:s_i=e} w_i$ of the players that use the edge. As before, each player's strategy set is the set of parallel edges, and a player's cost is the cost of the edge the player uses. Show that this game has a pure Nash equilibrium.

Question 3 [10]: For the facility location game we described in class, show by example that the bound of 2 we obtained on the Price of Anarchy is tight.

Question 4 [10]: We studied load-balancing games in class where the machines may have different rates. If the machines are *identical*, then every machine has the same rate $r_j = 1$. In this case, give an example with m machines and $2m$ jobs where the PoA for PNE is $(2 - \frac{2}{m+1})$.

This is question 20.2 from the AGT book, which also gives an example with 2 identical machines where the PoA is 4/3, and a proof that $(2 - \frac{2}{m+1})$ is an upper bound on the PoA of PNE for identical machines.

Question 5 [5]: Given an $n \times n$ zero-sum game, give an efficient algorithm to find an equilibrium (x^*, y^*) where the $|\text{supp}(x^*)|$ is largest, i.e., among all equilibria, x^* places positive probability on the largest number of pure strategies.

Question 6(a) [7]: Consider the following game (known as a *public goods game*). Agents are vertices in an undirected graph $G = (V, E)$. For an agent i, let the neighbourhood $N_i := \{i\} \cup \{j \in V : \{i, j\} \in E\}$. Each agent $i \in V$ must choose an effort $e_i \in \mathbb{R}_+$. Given a strategy profile $e = (e_1, \ldots, e_n)$, the utility of player i is given by

$$U_i(e) = f\left(\sum_{j \in N_i} e_j\right) - ce_i$$

where $f(\cdot)$ is a strictly concave function, and $c > 0$ is the cost for putting in effort.

Give an algorithm for computing a pure Nash equilibrium in this game.

Question 6(b) [2]: What is the exact Price of Anarchy for the public goods game?

Question 6(c) [6]: Can you show that the public goods game does *not* have an exact potential function? As a hint, consider a simple example where two players play their best responses in succession. In one case, player 1 moves first, followed by player 2; while in the other case, player 2 moves first, followed by player 1. Construct the two cases so that together, they show that no function $\Phi(e)$ satisfies the requirement to be an exact potential.