Algorithmic Game Theory

Assignment 5: due May 5, 2025

Assignment policies:

1. While you may dicuss the problems with others, you must write up the solution by yourself, in your own words.

2. Please write in your submission the people with whom you discussed the problems, as well as any references you used.

3. Late submissions will not be accepted, unless agreed to by me **prior** to the last date for submission.

4. Please write clearly and legibly, and include how you arrived at the solution!

Question 1 [10]: Consider the sponsored search auction we discussed earlier, in the Bayesian setting: there are k advertisement slots, and each slot j has a (public) click-through rate (CTR) α_j. Each bidder's valuation is drawn iid from a known regular distribution F. Bidder i gets utility $v_i\alpha_j$ minus payments if it is allocated slot j. Describe the revenue-optimal auction, including payments, for this setting.

Question 2 [10]: Show that, in the Bayesian single-item single-parameter case, the revenue of Vickrey's second-price auction with n bidders is at least $(n-1)/n$ times that of the optimal auction.

Question 3 [10]: Consider the Bayesian single-parameter *public projects* environment, where the feasible allocation set $X = \{0^n, 1^n\}$, that is, either every bidder is allocated, or no bidder is (e.g., consider a public project as a park which everyone enjoys if it is built). Describe the revenue-optimal mechanism when bidder valuations are drawn iid from $U[0, 1]$.

Question 4 [5]: The algorithms we analysed in the last couple of lectures were all DSIC, had more than 2 outcomes, were non-dictatorial, and operated without payments. However, the Gibbard-Satterthwaite theorem we studied earlier stated that any mechanism that is onto a set of at least 3 outcomes and is DSIC, must be a dictatorship. How do we resolve the two statements? That is, what conditions of the G-S theorem do our mechanisms not satisfy, in order to avoid the impossibility result?

Question 5 [10]: Consider the following variant of what is known as the house allocation problem: There are n agents and n houses, which are unassigned initially. Each agent must be allocated a house, and has an ordered preference list (possibly with indifference) over the houses. An allocation A of houses to agents is said to be *Pareto-optimal* if every allocation A' that improves the allocation to an agent, also makes some agent worse off. Design a mechanism that is DSIC and produces an allocation that is Pareto-optimal. Note that the preferences may be weak, i.e., an agent may be indifferent between multiple houses.

Question 6 [10]: A finite n-player game is called an *identical interest game* if for any strategy profile $s = (s_1, \dots, s_n)$, all players have the same utility (i.e., $u_i(s) = u_j(s)$ for all agents i, j). Show that an appropriately-defined best-response dynamics always converges to a PNE in identical interest games. Give your definition of best-response dynamics for which you obtain convergence.

Question 7 [10]: Recall the network congestion games defined earlier. Similarly, we can define bottleneck congestion games as follows. We are given a directed graph $G = (V, E)$, and n players, so that each player i has a source s_i and a sink t_i. The strategy set $\mathcal{S}_i$ for player i is the set of all s_i-t_i paths. Each edge additionally has a strictly increasing cost function $c_e : [n] \to \mathbb{Z}_+$.

Given a strategy profile $s = (s_1, \dots, s_n)$, the cost to player i is $\max_{e \in s_i} c_e(n_e(s))$, where $n_e(s) := |\{i : e \in s_i\}$ is the number of players using edge e. That is, player i incurs the cost of the highest-cost edge used by the player.

Show that for a bottleneck congestion game there always exists a PNE.

Question 8 [10] Show that if all players have the same source s and the same sink t, a PNE in bottleneck congestion games can be computed in polynomial time (you may need to look up the literature on min-cost flows for this question).