Algorithms and Data Structures: Assignment 5, due Dec 6

Please read the assignment policies on the course homepage before starting the assignment.

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1. Reading: from CLRS, Sections 29.2 and 34.1
2. Linear Programming exercises, not to be turned in. If you're stuck, you can ask me for hints.
   (a) Give examples of linear programs where: (i) both the primal and dual are infeasible; (ii) both the primal and dual polyhedra are unbounded; (iii) the primal and dual are the same linear program (this is a self-dual linear program).
   (b) This is known as the *diet problem*. Suppose the only foods in the world are as follows:

Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Price (Rs / serving)	Limit (servings / day)
Oatmeal	28g	110	4	2	3	4
Chicken	100g	205	32	12	24	3
Eggs	2	160	13	54	13	2
Milk	237 cc	160	8	285	9	8
Chocolate	170g	420	4	22	20	2
Beans	260g	260	14	80	19	2

   A satisfactory diet must have at least 2000 kcal, 55g of protein, and 800 mg of calcium per day. The limits on the number of servings per day for each of the food items is as given. Write a linear program to calculate the least cost satisfactory diet.
   (c) Given a directed graph $G = (V, E)$ with non-negative weights w_e on the edges and two vertices $s, t \in V$, write a linear program for finding the min-weight s-t path. The feasible region of the linear program should include all s-t paths, and the optimal solution must be a min-weight s-t path. Can you say something interesting about this LP?
   (d) Consider a restaurant that is open seven days a week. The minimum number of workers needed on each particular day of the week is as follows:

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Number	14	13	15	16	19	18	11

   Every worker works five consecutive days, and then takes two days off, repeating this pattern indefinitely. How can we minimize the number of workers required to run this restaurant?
   (e) Write the following optimization problems as linear programs in canonical form (i.e., $\min c^T x$ s.t. $Ax \le b$, $x \ge 0$):
      i.
$$\begin{aligned} \max \sqrt{x_1^2 + 2x_2^2 + x_3^2} \\ x_1^2 + x_2^2 \le 3 \\ x_1^2 + x_2^2 + x_3^2 \le 6 \end{aligned}$$
      ii.
$$\begin{aligned} \min |x_1 - x_2| \\ x_1 + 2x_2 \ge 3 \\ 3x_1 + x_2 \le -2 \\ x_1 \le -1 \end{aligned}$$

3. (15 marks) A doubly-stochastic matrix is one where every entry is nonnegative, and the sum of entries in each row and each column is 1. A permutation matrix is a doubly-stochastic matrix where each entry is either 0 or 1. Show that every doubly-stochastic matrix can be obtained as a convex combination of permutation matrices.

4. (20 marks) Exercise 12.4-5, CLRS Page 303.

5. (20 marks) Show that if a system of linear equalities $Ax \leq b$ does not have a feasible solution, with $x \in \mathbb{R}^n$, then we can select $n + 1$ of the inequalities such that the resulting subsystem also does not have a solution.

6. (20 marks) Let $G = (V, E)$ be a directed graph where for each vertex v, the in-degree and out-degree of v are equal. Let s and t be two vertices of G and suppose that G contains k edge-disjoint paths from s to t. Does G have to contain k edge-disjoint paths from t to s as well? Give a proof or a counterexample.

7. (15 points) Consider the following scheduling problem: Given n jobs where job j requires p_j units of processing time, and m identical machines, find an assignment of jobs to machines so that the total processing time for any machine is minimized (i.e., the objective is to minimize the maximum processing time over machines). Show that a simple greedy algorithm is a 2-approximation algorithm for this problem.

8. (25 points) Exercise 26-6, CLRS Page 763.

9. Given an undirected graph $G = (V, E)$ with positive weight w_e on each edge, we want to find a cut $(X, V \setminus X)$ to maximize the weight of edges across the cut. We denote by $w(X) := \sum_{e \in \delta(X)} w(e)$ the weight of edges across the cut. The problem is known to be NP-hard. Our objective in this problem is to come up with a deterministic 2-approximation for this.

   (a) (10 points) Give a randomized algorithm that in expectation is 2-approximate (i.e., if $(X, V \setminus X)$ is the cut obtained by the algorithm, and $(X^*, V \setminus X^*)$ is the optimal cut, then $\mathbb{E}[w(X^*)]/w(X) \leq 2$.

   (b) (10 points) Use the method of conditional expectations to derandomize this algorithm, to obtain a deterministic 2-approximate algorithm.