Lecture 4

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Recap: Fisher market for goods:
- each agent $i \in N$ has a valuation/utility fn $f_i : C \to \mathbb{R}_+$
- there is a price vector $p : G \to \mathbb{R}_+$
- each agent has 1 $ to spend
Each agent chooses a piece $S \subseteq [0,1]$ so that $p(S) \le 1$ & S maximizes $v_i(S)$
demand set of i: $\{S \subseteq [0,1] : p(S) \le 1,\ S$ maximizes $v_i(S)$ subject to this$\}$

Thm: S is in i's demand set iff $\exists$ an eq'm partition $(A_1, \dots, A_n)$ such that:
1) A_i maximizes $v_i(\cdot)$;
2) $\forall j : v_i(A_i)/p(A_i) \ge v_i(A_j)/p(A_j)$;
3) $v_i(A_i)/p(A_i) < v_i(A_j)/p(A_j) \Rightarrow A_j = \emptyset$,
and either $v_i = 0$, or $p(S) = 1$.
(prove yourself)

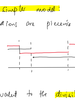
A Competitive Equilibrium from Equal Income (CEEI) is a price vector p & an allocation $A = (A_1, \dots, A_n)$ such that:
1) A partitions the cake
2) Each A_i is in i's demand set, given prices p
3) $p(A_i) = 1$.

Lemma 1: A CEEI is EF.

Lemma 2: If agents are non-satiable (i.e. $v_i(x) > 0$ $\forall x \in [0,1]$) then a CEEI is PO.

Today: how to find a CEEI.

A simple model:
Valuations are piecewise constant.



Equivalent to the divisible goods model:
m agents N, m goods, each agent has value v_{ij} for good j.
Assume $\forall i \ \exists j : v_{ij} > 0$, & $\forall j \ \exists i : v_{ij} > 0$.
Goods can be allocated fractionally. That is, an allocation $x = (x_1, \dots, x_m)$
where each x_i is a vector $x_{i1}, \dots, x_{in}$
& $\forall j \in [n]$ it is the case that $\sum_{i \in N} x_{ij} = 1$ (each good is completely allocated).

Note that we don't need EM agents now, can look directly at competitive equilibrium (prices).

An EF allocation is trivial now.
What about an EF+PO allocation?

A welfare objective:
Consider the following welfare objectives: (here allocation $x = (x_1, \dots, x_n)$ is arbitrary)
1) $\sum_i v_i(x_i)$ (i.e. utilitarian, social welfare)
2) $\min_i v_i(x_i)$ (the egalitarian social welfare)
3) $\left(\prod_i v_i(x_i)\right)^{1/n}$ (the Nash welfare)

Of these, only the Nash welfare is scale invariant.

This problem: $\max \left(\prod_i v_i(x_i)\right)^{1/n}$ s.t. $\forall i,\ \sum_j v_{ij} \le 1$ & $\forall i, j\ x_{ij} \ge 0$
is called the **Eisenberg-Gale convex program**.

Theorem: An allocation that maximizes the Nash welfare is a CEEI.
Conversely, any CEEI when every agent is non-satiated (i.e. $v_i(A_i) < 1$) maximizes Nash.

Note that it is immediate that the Nash maximizing allocation is PO.

Alternatively, we can write it as:
$$\max \sum_i \log v_i(x_i)$$
$$\text{s.t. } \forall j \ \sum_{i \in N} x_{ij} = 1 \qquad \forall i$$
$$\forall i,j \quad x_{ij} \ge 0 \qquad \forall i,j$$

This is a convex program: linear constraints, concave maximization objective. Hence $\exists$ a maximum. Can be solved to arbitrary precision in polynomial time.

Claim: $\exists$ a unique maximum system of optimal utilities $(u^*, \dots, u^*)$ (the allocation itself may not be unique).
(prove yourself)

Main $\Rightarrow$ CEET: with claim $\exists$ prices $\{p_j\}_{j \in G}$ s.t. $\forall i$,
1) $\sum_{j \in G} p_j x_{ij} = 1$,
2) $x_i \in \arg\max \left\{ \frac{v_i(S)}{\sum_{j \in S} p_j} \le 1 \right\}$

A brief primer on convex duality:

A convex program: $\min\ f(x)$ s.t. $g_i(x) \le 0 \quad \forall i$, f, g_i are convex.
$\forall i, x : g_i(x) \le 0 \quad \forall i$
(can be solved in polynomial time to arbitrary precision)

$L(x, \lambda) = f(x) + \sum_i \lambda_i g_i(x)$, $\lambda \ge 0$

In words: for any feasible x (i.e. $g_i(x) \le 0 \ \forall i$), $\lambda \ge 0$, $L(x, \lambda) \le f(x)$

Now define the dual: $h(\lambda) = \inf_x L(x, \lambda)$
Then $h(\lambda) \le L(y, \lambda) \le f(y)$ WEAK DUALITY

Note so far we have not used convexity, it got weak duality.

KKT Conditions (Karush-Kuhn-Tucker)
x^* is optimal for the convex program iff $\exists$ multiplier / dual variables λ_i^* s.t.:
1) $\nabla_x L(x, \lambda) = 0$ at x^*, λ^* (stationarity)
2) x^* is feasible ($g_i(x^*) \le 0 \ \forall i$)
3) $\lambda^* \ge 0$
4) $\forall i$, $\lambda_i^* g_i(x^*) = 0$ (complementary slackness)

Example: $\min f(x) = x + (x-2)^2$, $x \in [0,1]$
$-x \le 0 \quad \forall L$
$x - 1 \le 0 \quad \forall x$
$L(x, \lambda, \mu) = x + (x-2)^2 - \lambda x + \mu(x-1)$
$\frac{\partial L}{\partial x} = 1 + 2(x-2) - \lambda + \mu = 0 \Rightarrow 2x = \lambda - \mu + 3 \Rightarrow x = \frac{1}{2}(\lambda - \mu + 3)$
$\lambda, \mu \ge 0$, $x \in [0,1]$, $x, L = x + (x-2)^2$
if $\lambda > 0 \Rightarrow x = 0 \Rightarrow \lambda = 0$ ×
if $\mu > 0 \Rightarrow x = 1 \Rightarrow \mu = 0 \Rightarrow x = \frac{1}{2}(3 - \mu)$, so $\mu < 1$ ✓
so by KKT conds, an optimal sol'n is $x = \frac{1}{2}$, $\lambda = 0$, $\mu = 0$.

LEI conds above for the Eisenberg-Gale program:
$$L(\{p_j\}, u) = \sum_i \log v_i(x_i) - \sum_j p_j \left(\sum_i x_{ij} - 1 \right) + \sum_j \mu_j x_{ij} \quad , \ \forall j,\ p_j \ge 0$$

By KKT conds there, x^* is optimal iff $\exists\ p^*, \mu^* \ge 0$ s.t.:
1) x^* is feasible ($x^* \ge 0$, $\forall j \ \sum_{i \in N} x_{ij}^* \le 1$)
2) $p_j^* \cdot \mu_j^* \ge 0$
3) $\forall i,j$: $\frac{\partial L}{\partial x_{ij}} \Big|_{x^*, p^*, \mu^*} = 0$
4) $\forall j$: $p_j^* \left(\sum_i x_{ij}^* - 1 \right) = 0$ $\forall j$
5) $\mu_j^* x_{ij}^* = 0$ $\forall i,j$

Theorem: Let x^* be an optimal sol'n to the E-G convex program, & let p^*, μ^* be the KKT dual variables.
Then: 1) $\sum_j p_j^* x_{ij}^* = 1$
2) x_i^* maximizes $v_i(x_i)$ subject to 1)

Proof:
$$\left. \frac{\partial L}{\partial x_{ij}} \right|_{x^*, p^*, \mu^*} = 0 \Rightarrow \frac{v_i'(x_i^*)}{v_i(x_i^*)} - p_j^* + \mu_j^* = 0$$
Then $\forall j$, $\quad p_j^* = \frac{v_i'}{v_i(x_i^*)} + \mu_j^*$
$$\Rightarrow \sum_j p_j^* x_{ij}^* = \sum_j \frac{v_i' x_{ij}^*}{v_i(x_i^*)} + \sum_j \mu_j^* x_{ij}^*$$
$$= 1 \cdot \frac{v_i'}{v_i(x_i^*)} \sum_j x_{ij}^*$$
... but $\sum_j x_{ij}^* > 0 \Rightarrow \mu_j^* = 0$
Hence $\forall i$, $\sum_j p_j^* x_{ij}^* = 1$ — ①

Secondly, let $x_i' = v_i(z_i)$. Then
$$\forall j \quad \frac{v_i'}{v_i(x_i^*)} - p_j^* - \mu_j^* \le 0$$
or $\forall j$, $\frac{v_i'}{v_i(x_i^*)} \le p_j^*$ or $\infty \ge \frac{v_i'}{p_j^*}$
Suppose $\exists j : p_j^* = 0$. Then by stationarity,
$\forall i$, $\frac{v_i'}{v_i(x_i^*)} - p_j^* - \mu_j^* = 0$
or $\frac{v_i'}{v_i(x_i^*)} = -\mu_j^* \le 0$
Then $\forall i$, $x_{ij}^* = 0$, which is unbounded for logarithm.
Thus, $\alpha_i^* \ge \frac{v_i'}{p_j^*}$ $\forall i,j$
Further,
$\forall j$ $p_j^* > 0$, $\mu_j^* = 0$
$\Rightarrow \frac{v_i'}{v_i(x_i^*)} = p_j^*$ or $\alpha_i^* = \frac{v_i'}{p_j^*}$
Thus, x_i^* maximizes $v_i(x_i)$ subject to $\sum_j p_j^* x_{ij} \le 1$ ■

Claim: There is a rational path to the E-G convex program. (left as exercise)

Q: Can it be that for some $j \in G$, $\sum_i x_{ij}^* < 1$? What are prices then? $= 0$?

Q: Given optimal x^*, compute prices?