Computational Social Choice 2026: Assignment 1, due Sept. 30

Please read the assignment policies on the course homepage before starting the assignment.

For the following cake division questions, you can assume that there are n agents, a cake $C = [0, 1]$, and each agent has a normalised valuation density function f_i (thus $\int_0^1 f_i(x)\,dx = 1$). The density functions are bounded, so there exists Λ so that $f_i(x) \leq \Lambda$. Your algorithms may use Robertson-Webb Cut and Eval queries.

If useful, you may also assume that $f_i(x) > 0$ for all $i \in N$, $x \in C$.

1. We say an allocation $A = (A_1, \ldots, A_n)$ is *equitable* if for all agents i, j, $v_i(A_i) = v_j(A_j)$.

   (a) (10 marks) Using Sperner's lemma, show that for *every* ordering of the agents, there exists a connected equitable allocation of the cake that is equitable, where agent i gets the ith interval.

   (b) (10 marks) Given $\epsilon > 0$, give an algorithm that runs in time polynomial in n, $\log \Lambda$, and $\log(1/\epsilon)$, and outputs an ϵ-approximate equitable allocation A, where for all agents i, j, $v_i(A_i) \geq v_j(A_j) - \epsilon$.

2. (5 marks) Give an example to show that a CEEI allocation of cake may not be Pareto optimal.

For the following questions, you may assume that there are m divisible goods, and $v_{ij} \geq 0$ is agent i's value for good j.

3. (10 marks) For divisible goods, prove directly that a maximum Nash welfare allocation is envy free, without going through market equilibrium. That is, let $x^* = (x^*_{ij})$ be an allocation maximizing the Nash welfare. If x^* is not EF, show a contradiction.

4. (10 marks) Consider the weighted envy-freeness problem: for every agent $i \in N$ we are additionally given a weight $w_i > 0$. An allocation A is weighted envy-free if for all agents i, j,

$$\frac{v_i(A_i)}{w_i} \geq \frac{v_i(A_j)}{w_j}\,.$$

   Modify the proof of existence of envy-free and PO allocations, to show that a weighted envy-free and PO allocation also exists.

5. Consider an instance with two agents. For an allocation $A = (A_1, A_2)$ (possibly fractional), we define the utilitarian welfare $UW(A) = v_1(A_1) + v_2(A_2)$. For each good $j \in G$, define $r(j) = \frac{v_{1j}}{v_{2j}}$, where $r(j) = 0$ if $v_{1j} = v_{2j} = 0$, and $r(j) = \infty$ if $v_{1j} > v_{2j} = 0$. For an instance $\mathcal{I}$, define $OPT(\mathcal{I}) = \max_A UW(A)$, where the maximum is over all 2-agent allocations.

   (a) (2 marks) Give a characterization of allocations that maximize the utilitarian welfare, in terms of the ratio $r(\cdot)$.

   (b) (3 marks) Give a characterization of proportional allocations with maximum utilitarian welfare, in terms of the ratio $r(\cdot)$.

   Define the *price of fairness* of proportional allocations as the ratio

$$\sup_{\mathcal{I}} \frac{OPT(\mathcal{I})}{\max\{UW(A) : A \text{ is a proportional allocation in } \mathcal{I}\}}\,.$$

(c) (5 marks) Show that, given any instance $\mathcal{I}$ with 2 agents and m goods, there exists an instance with 2 agents and just 3 goods with price of fairness at least that of the original instance.

(d) (10 marks) Show that for two agents, the price of fairness of proportional allocations is $8 - 4\sqrt{3}$.

6. (10 marks) For an iterated barycentric subdivision of the standard simplex, show there exists a barycentric labelling as well. That is, if T^0 is the standard simplex and T^i is the triangulation obtained by the barycentric subdivision of each elementary simplex in T^{i-1}, show that there exists a labelling $\beta : V(T^i) \to [n]$ so that for each elementary simplex, each vertex has a different label.

7. (10 marks) Show that for divisible goods, there is a rational allocation that maximizes the Nash welfare (i.e., each x_{ij} is rational). You can assume all v_{ij}s are integral.

   You may need to look again at the dual we wrote down in class, and use the facts that (i) the KKT conditions characterize optimal solutions, and (ii) a linear program with rational coefficients always has a rational optimal solution.