Computational Social Choice 2026:
Assignment 2, due October 21

Please read the **modified** assignment policies on the course homepage before starting the assignment.

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1. Assume that there are n agents, a cake $C = [0, 1]$, and each agent has a normalised valuation density function f_i (thus $\int_0^1 f_i(x)\,dx = 1$). Your algorithms may use Robertson-Webb Cut and Eval queries.

   In class we saw an algorithm that makes $O(n \log n)$ RW queries and obtains a proportional allocation of cake. In this question we will obtain an approximately proportional allocation, but with fewer Cut queries.

   (a) (5 marks) Design a cake-cutting protocol for four agents that, using three Cut queries and a constant number (say at most 100) of Eval queries, guarantees each agent a share of at least $\frac{1}{6}$.

   (b) (15 marks) Design a protocol for $n \geq 3$ agents that, using $n - 1$ Cut queries and a poly(n) number of Eval queries, guarantees each agent a share of at least $\frac{1}{2n-2}$.
   Hint: A recursive protocol works, where in each step, given an interval $[s, t]$ and a set N' of agents, the protocol *uses a single Cut query* and multiple Eval queries to split the cake into $[s, x]$ and $[x, t]$ and the agents into N'_1 and N'_2. You need to determine how much value the agents in N'_1 should have for $[s, x]$ (and agents in N'_2 for $[x, t]$) for the guarantees to be satisfied recursively.

   This question is from an assignment given by Ulle Endriss.

For the remaining questions, you may assume a set N of n agents and a set M of m *indivisible* items. Valuations are monotone nondecreasing (all items are goods) unless specified otherwise, and $v_i(S) \in \mathbb{Z}$ for all $S \subseteq M$.

2. We say an allocation $A = (A_1, \ldots, A_n)$ is *equitable up to any item* (EQX) if for all agents i, j, if $v_i(A_i) < v_j(A_j)$ then for any good $g \in A_j$, $v_i(A_i) \geq v_j(A_j \setminus \{g\})$.

   (a) (8 marks) Show that an EQX allocation always exists.

   (b) (7 marks) Show that an EQX allocation can be obtained in pseudopolynomial time (i.e., in time poly$(n, m, v_{\max})$ where $v_{\max} = \max_i v_i(M)$).

3. Now assume that valuations are additive, and the goods are partitioned into k categories M_1, …, M_k so that each category has at most n items, and in any valid allocation, each agent gets at most one good from each category. That is, for all $l \in [k]$, $|M_l| \leq n$, and $A = (A_1, \ldots, A_n)$ is a valid allocation if for all $i \in N$, $l \in [k]$, $|A_i \cap M_l| \leq 1$.

   (a) (5 marks) Give a polynomial time algorithm to compute a valid EF1 allocation.

   (b) (5 marks) We next want to show next that when $k = 2$, there exists an allocation A that is (i) EF1, and (ii) a 1/2-approximation to the maximum Nash welfare. Let A^* be a valid allocation that maximizes the Nash welfare, and for any allocation A,

$$\mathrm{NW}(A) = \left(\prod_i v_i(A_i)\right)^{1/n}.$$

Assume $\mathrm{NW}(A^*) > 0$.

Let σ be a maximum weight matching in the bipartite graph (N, M), where each edge $\{i, j\}$ has weight $\log v_i(j)$. Let A' be the partial allocation where for every agent i, $A'_i = \{\sigma(i)\}$. Show that $\mathrm{NW}(A') \geq \mathrm{NW}(A^*)/2$.

(c) (5 marks) Now show that the partial allocation A' can be extended to a complete valid allocation A that is EF1, where $v_i(A_i) \geq v_i(A'_i)$ (and hence A is an EF1 allocation with $\mathrm{NW}(A) \geq \mathrm{NW}(A^*)/2$).

4. For an allocation A and $p \in \mathbb{R} \setminus \{0\}$, define the p-mean welfare $W_p(A)$ as

$$W_p(A) \;:=\; \left(\frac{1}{n}\sum_{i\in N}(v_i(A_i))^p\right)^{1/p}.$$

(a) (5 marks) Prove that $W_p(A)$ is monotonically nondecreasing in p. You may need to use Jensen's inequality: for a convex function f and nonnegative coefficients $a_1, \ldots, a_n$ that sum to 1,

$$\sum_i a_i f(x_i) \geq f\left(\sum_i a_i x_i\right).$$

(b) (5 marks) Prove that $\lim_{p\to 0} W_p(A) = \mathrm{NW}(A)$.

5. Consider the problem where there n agents with additive valuations over m *chores*: say $d_i(j)$ is agent i's *disutility* for chore j. The definition of the maximin share is similar, though each agent tries to minimize the maximum disutility:

$$\mathrm{MMS}_i^n(M) \;:=\; \min_{A\in\Pi_n} \max_{j\in[n]} d_i(A_j).$$

In this question we want to obtain a 2-approximate MMS allocation.

(a) (5 marks) As we did for goods, show that the problem can be reduced to obtaining a 2-approximate MMS allocation for an ordered instance.

(b) (10 marks) Now show that a simple round-robin algorithm obtains a 2-approximate MMS allocation for ordered instances.

6. In this question we want to show that if there are sufficiently many items, and valuations are drawn randomly, an EF allocation exists with high probability.

Suppose $m \geq 100\,(n+1)\ln n$. Agents have additive valuations, and the values $v_i(j)$ are drawn independently and uniformly at random from $[0, 1]$. We say an event happens *with high probability* if its probability tends to 1 as $n \to \infty$.

Let A be the allocation where each good is given to an agent who values it most. Ties occur with probability 0 and may be broken arbitrarily.

(a) (8 marks) Fix two distinct agents i, j. Show that

$$\mathbb{E}\big[v_i(A_i)\big] = \frac{m}{n+1} \qquad \text{and} \qquad \mathbb{E}\big[v_i(A_j)\big] = \frac{m}{2(n+1)}.$$

(b) (7 marks) Show that A is EF with high probability. The following Chernoff bound may be useful.

Theorem. *Let $Z_1, \ldots, Z_N$ be independent random variables taking values in $[0,1]$, let $S = \sum_k Z_k$ and $\mu_S = \mathbb{E}[S]$. Then for every $\delta \in (0,1]$,*

$$\Pr\left[S \leq (1-\delta)\mu_S\right] \leq e^{-\delta^2\mu_S/2}, \qquad \Pr\left[S \geq (1+\delta)\mu_S\right] \leq e^{-\delta^2\mu_S/3}.$$

7. (10 marks) Give up to 5 examples of different voting rules that are used in real-world situations. Explain each voting rule in 2-3 lines, and where it is used. You get a bonus point for every *unique* example, that is, an example not given by another other classmate.

8. A *tournament* is a graph that has a directed edge between every pair of vertices (for every pair of vertices u, v, the graph has exactly one of $\{(u,v),(v,u)\}$. Given a profile with m alternatives and an odd number n of candidates, there is a natural tournament, called the majority digraph, where vertices correspond to the m alternatives, and there is an edge (a,b) if $\mu(a,b) > 0$.

   In the following questions, assume m is even.

   (a) (6 marks) Show that every tournament on m vertices can be obtained as the majority digraph of a profile with at most $m(m-1)$ voters.

   (b) (3 marks) Prove that for even m, the complete graph K_m is the disjoint union of $m-1$ perfect matchings.

   (c) (6 marks) Use the above result to show that every tournament on m vertices can be obtained as the majority digraph of a profile with at most $2(m-1)$ voters.