Computational Social Choice 2026:
Assignment 3, due November 11

Please read the **modified** assignment policies on the course homepage before starting the assignment.

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1. In class, what we actually proved was the Muller-Satterthwaite theorem: for $m \geq 3$ alternatives, any social choice function that is onto and satisfies strong (Maskin) monotonicity, must be a dictatorship. This implies Gibbard-Satterthwaite, since we showed strategyproofness implies strong monotonicity.

   (a) (6 marks) Prove the converse: strong monotonicity implies strategyproofness.

   (b) (6 marks) Define weak monotonicity for a social choice function f as follows: given a preference profile P, suppose $f(P) = a$. Define P' as the profile obtained when a single voter i moves a up in her ranking, keeping the relative rankings of the other candidates unchanged (in comparison to strong monotonicity, where we only require that the lower contour of a in P' is a superset of that in P). Then $f(P') = a$.

   Show that there exists a social choice function for $m \geq 3$, onto, weakly monotone, and non-dictatorial.

2. A profile is said to be *single-crossing* if the voters can be ordered so that for every pair of alternatives a, b, the voters preferring a to b form either a prefix or a suffix of that order. That is, for all a, b, the "crossing over" from $a \succ b$ to $b \succ a$ (or vice versa) happens at most once.

   (a) (6 marks) Prove that for odd n, the median voter's top choice is a Condorcet winner, and that majority is transitive.

   (b) (9 marks) Give a profile that is single-crossing but not single-peaked for any base order (3 voters and 3 candidates should suffice), and one that is single-peaked but not single-crossing (3 voters and 4 candidates should suffice).

3. (12 marks) Consider the following problem. Let the set of candidates for a voting instance $C = \{1, \ldots, 100\}^d$ (thus C is the set of all points in the d-dimensional grid of size 100 in each dimension). There are n voters, each of which is a point in C. For a voter $v = (v_1, \ldots, v_d)$, her *cost* for any candidate $c = (c_1, \ldots, c_d)$ is the $\mathbb{L}_1$ distance $\sum_{i=1}^{d} |v_i - c_i|$. Further, for a candidate c, define the *social cost* $\mathrm{SW}(c) = \sum_v \sum_i |v_i - c_i|$ as the sum of costs of the voters for c.

   Give a social choice function that is onto C, non-dictatorial, truthful, and minimizes the social cost.

4. This question requires you to be comfortable with writing down a linear program, its dual, and using complementary slackness and duality. If you are not comfortable with these, please let me know, I will give you a different question for the assignment.

   Consider a fair division instance with m divisible goods G, n agents N, and additive valuations v_{ij}. Assume that every agent values some good strictly positively, and every good is valued by some agent strictly positively. For an allocation $x = (x_{ij})_{i \in N,\, j \in G}$, let $u_i(x) := \sum_j v_{ij} x_{ij}$ be agent i's utility in x. Our goal is to obtain a characterization for PO allocations.

   Given an allocation x^*, let $u_i^* = u_i(x^*)$. Now consider the linear program that maximizes the utilitarian welfare over all feasible allocations, subject to agent i's utility being at least u_i^*. Now write down the dual of this LP.

(a) (9 marks) Use duality to prove that an allocation x is Pareto optimal iff there exists a weight vector $w \in \mathbb{R}^n_{>0}$ such that x maximizes $\sum_i w_i u_i(y)$ over all feasible allocations y.

(b) (6 marks) Use duality to prove that an allocation x is Pareto optimal iff there exist prices $p \in \mathbb{R}_{>0}$ such that if $x_{ij} > 0$, then j maximizes v_{ij}/p_j.

5. (8 marks) We are given a preference profile P with n voters and m candidates. Assume for this question that the number of voters n is even. We have seen that there may not be a Condorcet winner, and the majority graph may have cycles. To relax the requirement of a single Condorcet winner, we define a *Condorcet winning set* as a set of candidates S that beats every candidate not in S. That is, for every $c \notin S$, there exists $a \in S$ such that $\mu(a, c) > 0$. Show that there always exists a Condorcet winning set of size at most $1 + \lceil \log_2 m \rceil$.

6. (8 marks) Given a preference profile P with n voters and m candidates, show that for sufficiently large dimension d, there is an embedding of the voters and candidates in $\mathbb{R}^d$ so that the preference profile P is consistent with the L_2 distance metric.

7. (a) (5 marks) Show that any deterministic voting rule has distortion $\Omega(m^2)$. In particular, show a single preference profile so that no matter which candidate is chosen, there are consistent utilities for which the chosen candidate has welfare at most $O(1/m^2)$ that of the optimal candidate.

   (b) (5 marks) Show that plurality has distortion $O(m^2)$ (and hence, has asymptotically the best distortion among deterministic voting rules).

   (c) (5 marks) Show that in the metric case, plurality has distortion $O(m)$.

8. (10 marks) Let T be a strongly connected tournament (i.e., for all u, $v \in V$, there is a directed path from u to v). Prove hat T has a Hamiltonian cycle (a simple cycle through all the vertices).

9. (15 marks) Consider the following randomized voting rule, called *random dictatorship*: given a preference profile P, pick a voter i uniformly at random, and return voter i's top-ranked candidate. Show that in the metric setting, random dictatorship has distortion $3 - (2/n)$, where n is the number of voters.

   Any bound strictly better than 3 will get full marks, while a constant bound on the distortion gets partial marks.

The following questions on linear programming are for your practice only, and are not meant to be turned in.

1. Write down the dual for the following linear program:

$$\max \; 3x_1 + 4x_2 - x_3$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + x_2 + x_3 - x_4 && \geq 4 \\ & 2x_1 - x_2 + 2x_4 && \leq 2 \\ & 2x_1 + 2x_2 - 3x_4 && = 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

2. Give an example of a linear program where both the primal and the dual are infeasible.

3. Give an example of a linear program that is "self-dual", that is, the primal and the dual LPs are identical.

4. Given a graph $G = (V, E)$, with weight w_e on each edge and $m = |E|$, define a fractional matching as a vector $x \in [0, 1]^m$ over the edges so that for every vertex $v \in V$, $\sum_{e \in \delta(v)} x_e \leq 1$, where $\delta(V)$ is the set of edges incident on v. Write a linear program to find a maximum-weight fractional matching in G, and write its dual.