LP-based Algorithms

Assignment 1: due February ~~20~~ 24, 2026

Assignment policies:

1. For assignments, you may refer to any books or notes, but not to any online resources.
2. You may discuss the problems with others in the class, but you must write up the solution by yourself, in your own words.
3. Please write in your submission the people with whom you discussed the problems, as well as any references you used.
4. Please write clearly and legibly, and include how you arrived at the solution!

The following questions are for your own practice, and **do not need to be turned in.**

Question 0.1 Consider the problem

$$\begin{array}{ll} \text{minimize} & \dfrac{c^T x + d}{f^T x + g + 1} \\ \\ \text{s.t.} & Ax \leq b \\ & f^T x + g \geq 0 \end{array}$$

where c, $f \in \mathbb{R}^n$, d, $g \in \mathbb{R}$, and $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Suppose we know that the optimal cost lies in the interval $[K, L]$. Give an algorithm that uses linear programming as a subroutine and for any $\epsilon > 0$, allows us to compute the optimal cost within desired accuracy ϵ with $O(\log(L - K)/\epsilon)$ calls to the subroutine.

Question 0.2 Write the following problem as a linear program: Given k points $z_1, \ldots, z_k$ in $\mathbb{R}^n$, find a point that minimizes the maximum L_1 distance from these points.

Question 0.3 Given an LP

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \end{array}$$

with bounded feasible region (i.e., the feasible region is a polytope), as well as an optimal solution x^*, give an algorithm to find an optimal extreme point (without having to solve the LP again).

Question 0.4 Given an undirected graph $G = (V, E)$ with edge lengths $l_e \geq 0$ for each edge, and two vertices s, t, write a linear program to find an s-t path of minimum length.

Two questions added below.

Question 0.5 Give an LP so that both the given LP and its dual are infeasible.

Question 0.6 Give an LP that is self-dual, that is, the primal and dual LPs are exactly the same.

The following problems are to be turned in by February ~~20th~~ 24th. ~~I will further add 3-4 problems by February 14th, so that the assignment consists of 8-9 problems.~~ 3 questions added on February 16th.

Question 1 [10]: Write the following problem as a linear program: Given k points $z_1, \ldots,$ z_k in $\mathbb{R}^n$ and $t \leq k$, find a point x and a set $S \subseteq [k]$ of size t (these are called outliers) so that x minimizes the maximum L_1 distance from the remaining points not in S. I.e., solve the following problem:

$$\min_{x \in \mathbb{R}^n} \min_{S \subseteq [k], |S|=t} \max_{i \notin S} \|z_i - x\|_1 .$$

Question 2 [10]: Consider a polyhedron $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ where $A \in \mathbb{R}^{m \times n}$. A ball with center y and radius r is defined as the set of all points within Euclidean distance r from y. Provide a linear program to find the ball with the largest possible radius entirely contained within the set P (the center of such a ball is called the Chebychev center of P).

You may use the fact that the distance between the point $z \in \mathbb{R}^n$ and the hyperplane $a^T x = b$ is $\dfrac{|a^T z - b|}{\sqrt{a^T a}}$ (though it's a good exercise to derive this expression).

Question 3 [10]: Consider the following linear program, with $\epsilon \in (0, 1/2)$:

$$\begin{aligned}
\text{minimize} \quad & -x_n \\
\epsilon \leq \quad & x_1 \quad \leq 1 \\
\epsilon x_{i-1} \leq \quad & x_i \quad \leq 1 - \epsilon x_{i-1} \qquad \text{for } 2 \leq i \leq n
\end{aligned}$$

Show that the polytope described has 2^n vertices, and give an ordering of these vertices so that the objective value strictly decreases when traversed in this order.

Question 4 [10]: Given a matrix A, suppose we wish to find a vector $x \in \mathbb{R}^n$ that satisfies $Ax = 0$ and $x \geq 0$, and such that the number of strictly positive components of x is maximized. Show that this can be accomplished by solving the following linear program:

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} y_i \\
\text{s.t.} \quad & A(y+z) = 0 \\
& y_i \leq 1 \quad \forall i \in [n] \\
& y, z \geq 0
\end{aligned}$$

Question 5 [10]: A mapping f is called *affine* if it is of the form $f(x) = Ax + b$ where A is a matrix and b is a vector. Let P and Q be polyhedra in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. We say that P and Q are *isomorphic* if there exist affine mappings $f : P \to Q$ and $g : Q \to P$ so that $g(f(x)) = x$ for all $x \in P$, and $f(g(y)) = y$ for all $y \in Q$.

If P and Q are isomorphic, show that there exists a one-to-one correspondence between their extreme points.

Question 6 [10]: Consider the following LP in standard form.

$$\begin{array}{ll} \min & c^T x \\ & \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

Suppose that there exists a variable x_j such that $x_j^* = 0$ in *all* optimal solutions x^*. Show that there exists an optimal dual solution y^* so that $A^T y^* < c_j$.

Question 7a [7]: Prove the following variant of Farkas' Lemma (also sometimes called Gordan's Theorem): Given matrix $A \in \mathbb{R}^{m \times n}$, exactly one of the following statements is true:

- $\exists x \neq 0$ such that $Ax = 0$ and $x \geq 0$
- $\exists y$ such that $A^T y < 0$

Question 7b [8]: Consider the following LP in standard form.

$$\begin{array}{ll} \min & c^T x \\ & \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

Show that if the LP has a unique feasible solution x^*, then there are an infinite number of feasible dual solutions.

Question 8 [10]: Given a matrix A with entries in $\{0, +1, -1\}$ so that any column has at most 2 non-zero entries, give a polynomial-time algorithm for determining if A is totally unimodular.