

# Assignment 2: LP-Based Algorithms

March 22, 2026

**Total points: 100. Each problem carries 10 points. Deadline: 07.04.26**

1. Show that the running time of the Hungarian method is  $\mathcal{O}(n^3)$ .
2. Let  $P$  be a bounded polyhedron with vertices  $\{v_1, v_2, \dots, v_n\}$ . Show that

$$P = \text{conv}\{v_1, v_2, \dots, v_n\}.$$

3. Formulate a linear program for finding a minimum-weight matching with *exactly*  $k$  edges. Use this LP to give a polynomial time algorithm for finding a minimum-weight matching with at most  $k$  edges.
4. Consider a bipartite graph  $G = (V, E)$  with bipartition  $(A, B)$ , where  $V = A \cup B$ . Suppose that for some subsets  $A_1 \subseteq A$  and  $B_1 \subseteq B$ , there exists:
  - a matching  $M_A$  covering all vertices in  $A_1$ , and
  - a matching  $M_B$  covering all vertices in  $B_1$ .

Prove that there exists a matching covering all vertices in  $A_1 \cup B_1$ .

5. Let  $G = (V, E)$  be a bipartite graph in which every vertex has degree  $k$ . Show that  $G$  has a perfect matching using both of the following methods:
  - Hall's Theorem
  - The polyhedral description of the perfect matching polytope for bipartite graphs
6. Let  $G = (V, E)$  be a graph and let  $b : V \rightarrow \mathbb{N}$ . A set of edges  $F \subseteq E$  is called a *b-matching* if  $\delta_F(v) \leq b(v)$  for all  $v \in V$ .<sup>1</sup> It is called a *perfect b-matching* if  $\delta_F(v) = b(v)$  for all  $v \in V$ .
  - Give a polyhedral description of the convex hull of all perfect  $b$ -matchings.
  - Use the above to derive a polyhedral description of the convex hull of all  $b$ -matchings.
7. Let  $G = (V, E)$  be a graph and let  $P$  denote its matching polytope (i.e., the convex hull of all matchings in  $G$ ). Given a candidate vector  $x$ , design a polynomial-time algorithm that:
  - returns **true** if  $x \in P$ , and

---

<sup>1</sup>That is, the number of edges in  $F$  incident to  $v$  is at most  $b(v)$ .

- returns a violated inequality if  $x \notin P$ .
8. Show that every 3-regular, 2-edge-connected graph has a perfect matching. A graph is  $d$ -regular if every vertex has degree exactly  $d$ , and it is  $k$ -edge-connected if every nontrivial edge cut has size at least  $k$ .
  9. Let  $G = (V, E)$  be an undirected graph with edge costs  $c : E \rightarrow \mathbb{R}_{\geq 0}$ . Let  $T \subseteq V$  be such that  $|T|$  is even. Show that the optimal value of the following linear program equals the cost of an optimal  $T$ -join:

$$\begin{aligned}
 \min \quad & c^T x \\
 \text{s.t.} \quad & x \in \mathbb{R}^{|E|}, \\
 & x \geq 0, \\
 & \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \subseteq V \text{ with } |S \cap T| \text{ odd.}
 \end{aligned}$$

Here,  $\delta(S)$  denotes the set of edges with exactly one endpoint in  $S$ .

10. Show how to find a negative cycle in an undirected graph (if one exists) by reducing the problem to an appropriate  $T$ -join problem.