

Assignment 3: LP-Based Algorithms

April 10, 2026

Total points: 100. Each problem carries 10 points. Deadline: 30.04.26

1. Show that the maximal (inclusion-wise) nontrivial faces of a nonempty polyhedron P are precisely its facets.
2. An independent set S in a graph $G = (V, E)$ is a set of vertices such that no two vertices in S are adjacent. Let P denote the convex hull of the incidence vectors of all independent sets of G . It is clear that the inequalities

$$x_i + x_j \leq 1 \quad \text{for all } (i, j) \in E$$

are valid for P .

- (a) Give an example of a graph G for which P is *not* equal to

$$\{x \in \mathbb{R}^{|V|} : x_i + x_j \leq 1 \ \forall (i, j) \in E, \ x_i \geq 0 \ \forall i \in V\}.$$

- (b) Show that if G is bipartite, then P equals

$$\{x \in \mathbb{R}^{|V|} : x_i + x_j \leq 1 \ \forall (i, j) \in E, \ x_i \geq 0 \ \forall i \in V\}.$$

3. Let G be a graph. Define a set system $M = (V, \mathcal{I})$ as follows: the ground set is the vertex set V of G , and

$$\mathcal{I} = \{S \subseteq V : S \text{ is covered by some matching in } G\}.$$

Show that M is a matroid.

4. Let A_1, A_2, \dots, A_n be a family of sets (not necessarily disjoint). A *transversal* is a set

$$T = \{a_1, a_2, \dots, a_n\}$$

such that the elements a_i are distinct and $a_i \in A_i$ for all i . A *partial transversal* is a transversal for some subfamily $A_{i_1}, A_{i_2}, \dots, A_{i_k}$. Show that the family of all partial transversals forms a matroid on the ground set $E = \bigcup_i A_i$.

5. Show that finding a maximum-size independent set in the intersection of three matroids is NP-hard. You may use a reduction from the *Directed Hamiltonian Cycle* problem, which is NP-complete.

6. Suppose we are given an undirected graph $G = (V, E)$ with nonnegative edge costs, where each edge is assigned a color from $\{1, 2, \dots, k\}$. A spanning tree is called *colorful* if it contains at most one edge from each color class E_i . Design a polynomial-time algorithm to compute a minimum-cost colorful spanning tree, if one exists.
7. Let $G = (V, A)$ be a directed graph, where edge costs are given by $c : A \rightarrow \mathbb{Z}_{\geq 0}$ and $r \in V$ be the root vertex. Consider the following linear program for finding a minimum-cost collection of k arc-disjoint arborescences rooted at r :

$$\min \sum_{a \in A} c_a x_a$$

subject to

$$\sum_{a \in \delta^+(S)} x_a \geq k \quad \text{for all } S \subseteq V \text{ with } r \in S,$$

$$0 \leq x_a \leq 1 \quad \text{for all } a \in A.$$

Show that every optimal extreme-point (i.e. vertex) solution to this LP is integral, and hence yields an optimum solution to the minimum-cost k arc-disjoint arborescences problem.

8. Given an undirected graph $G = (V, E)$ with non-negative edge costs and an integer k , the *k-Edge-Connected Spanning Subgraph (k-ECSS)* problem asks for a minimum-cost subgraph of G that is k -edge-connected. Using the result of Problem 7, design a polynomial-time 2-approximation algorithm for the k -ECSS problem.
9. Let $G = (V, A)$ be a strongly connected directed graph with non-negative arc costs. Show that computing a minimum-cost strongly connected spanning subgraph of G is NP-hard. Further, design a polynomial-time 2-approximation algorithm for this problem.
10. Let $r : V \times V \rightarrow \mathbb{Z}_{\geq 0}$. Define $f : 2^V \rightarrow \mathbb{Z}_{\geq 0}$ as:

$$f(S) = \begin{cases} 0 & \text{if } S = \emptyset \text{ or } S = V, \\ \max_{u \in S, v \notin S} r(u, v) & \text{otherwise.} \end{cases}$$

Show that f is weakly supermodular. Recall that a function $g : 2^V \rightarrow \mathbb{Z}_{\geq 0}$ is *weakly supermodular* if $g(\emptyset) = g(V) = 0$ and for all $A, B \subseteq V$,

$$g(A) + g(B) \leq \max\{g(A \cap B) + g(A \cup B), g(A \setminus B) + g(B \setminus A)\}.$$