Routing Games between Selfish Users

Umang Bhaskar

September 27, 2008

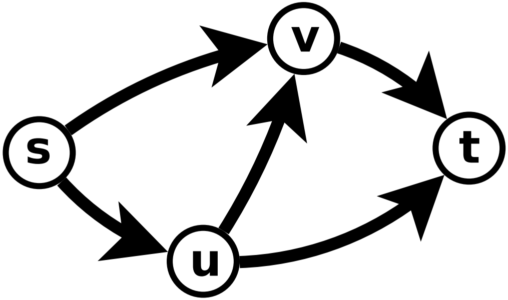
With Lisa Fleischer, Darrell Hoy, Chien-Chung Huang

Motivation

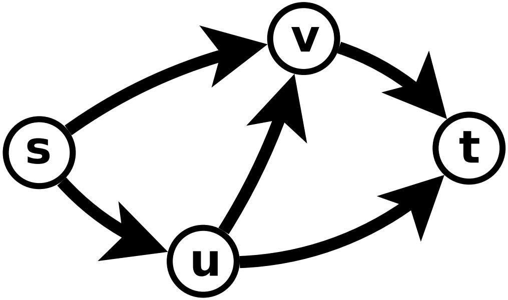
Motivation

- What's the best way from point a to point b?

Networks

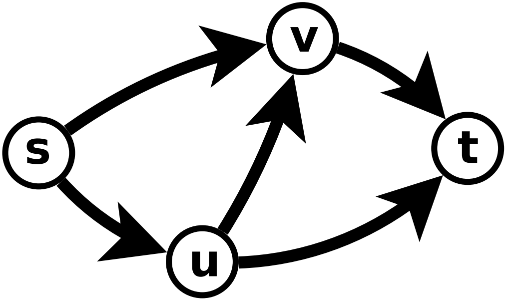


Networks



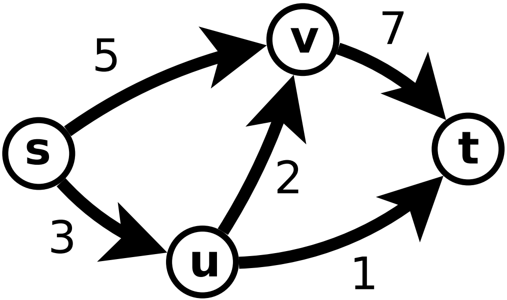
- Directed graph

Networks



- Directed graph
- Source s, destination t

Flows



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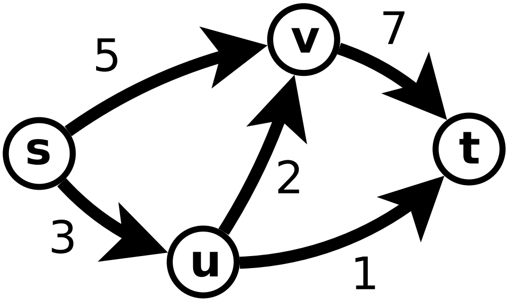

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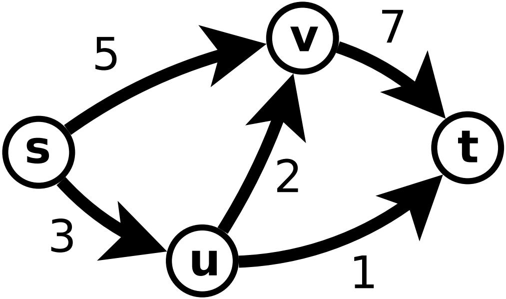
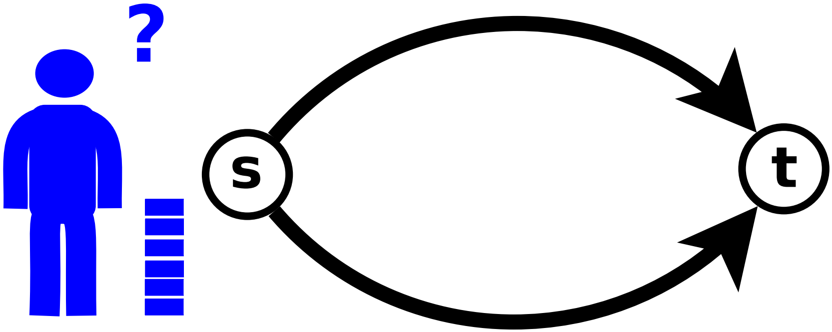
Flows



- Conservation: Flow in = flow out except s and t
- "Value" of flow

The Problem of Routing



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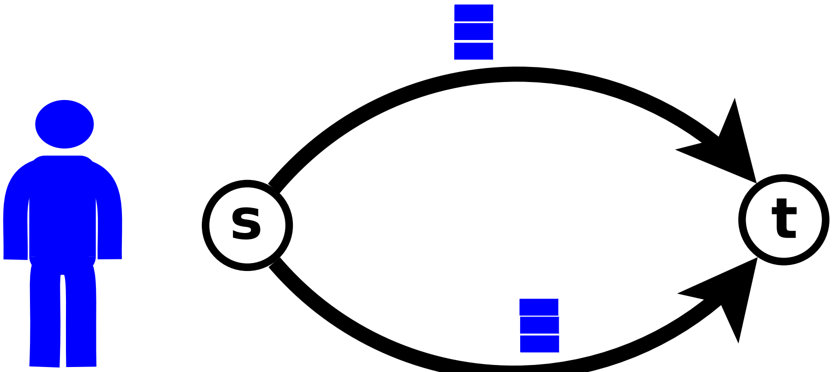
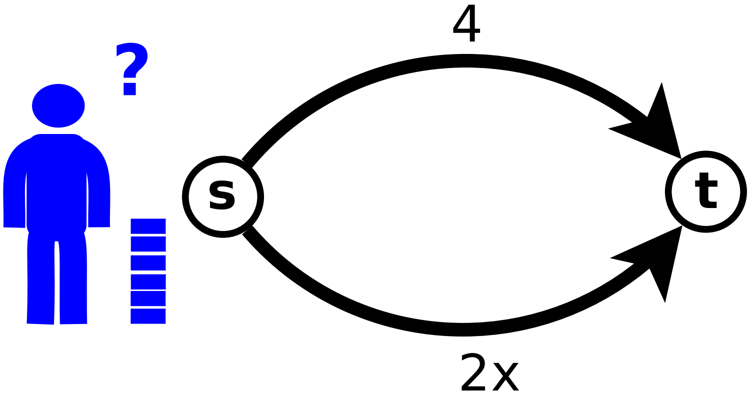
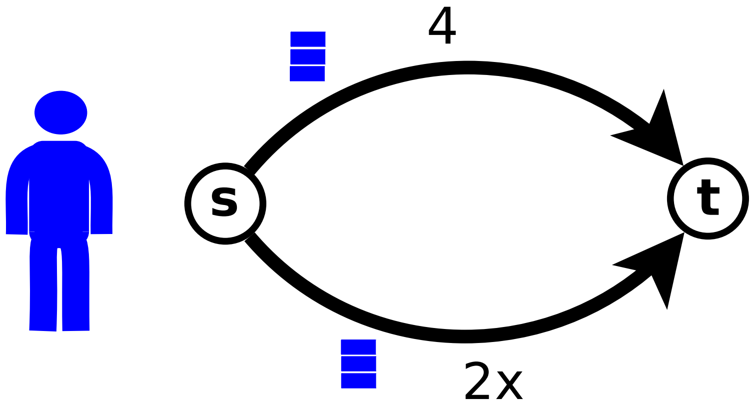


Routing with Delays



- Delay function: $f(x)$, where x is total flow on edge

Routing with Delays



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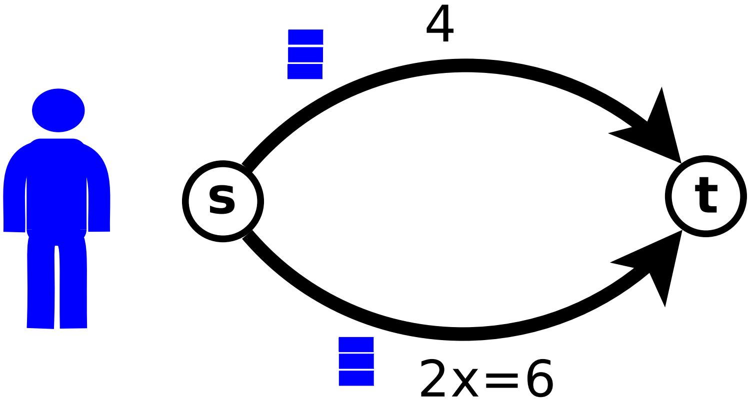

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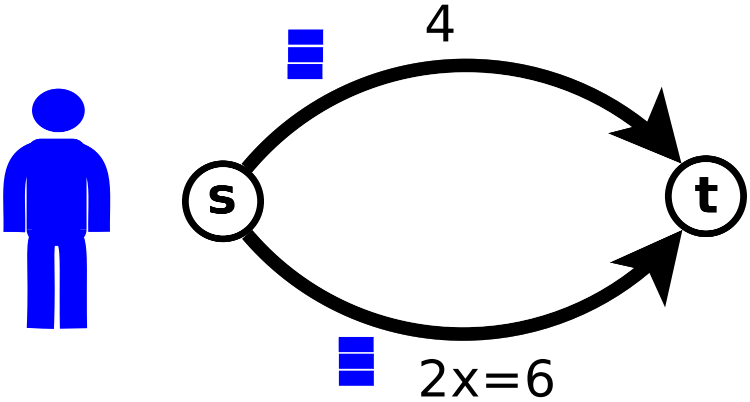

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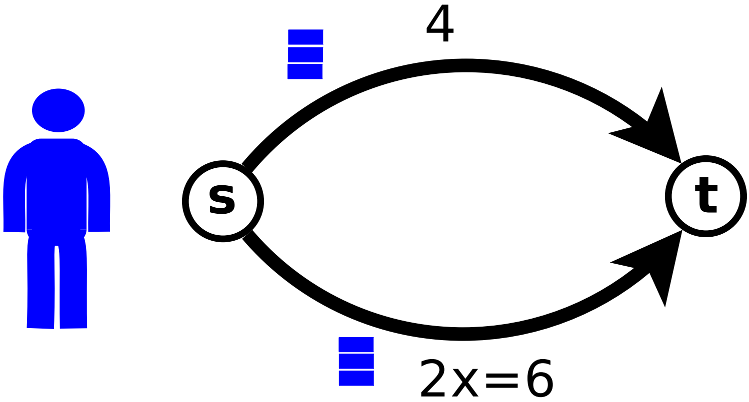
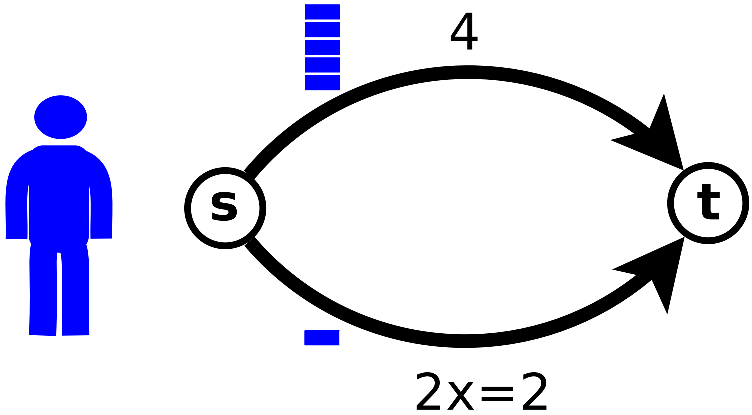
Routing with Delays



- Delay function: $f(x)$, where x is total flow on edge
- Delay to player p = p's flow on edge $\times$ delay of edge
- Total delay $= 3 \times 4 + 3 \times 6 = 30$

Routing with Delays



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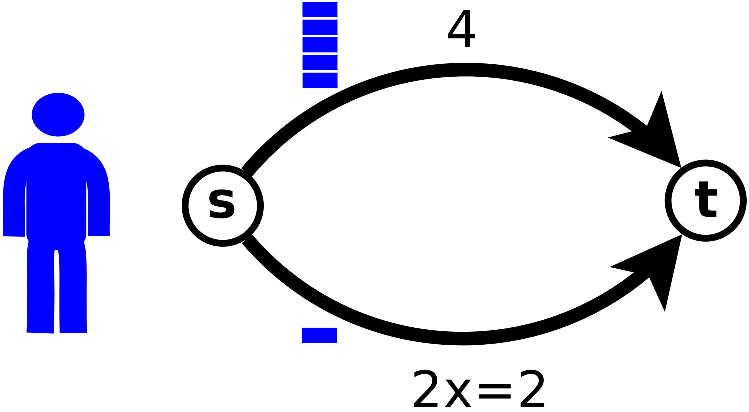
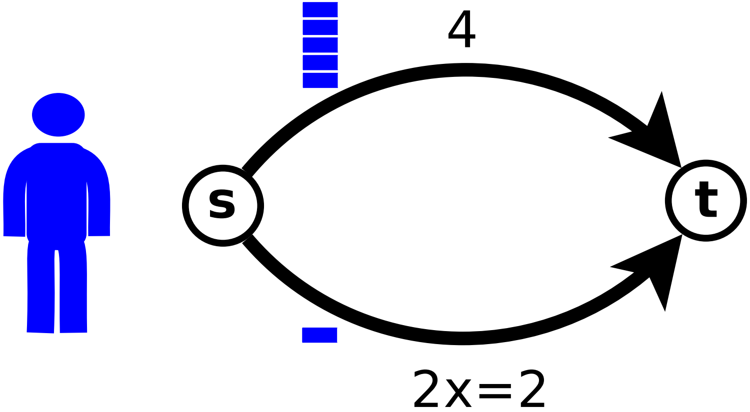


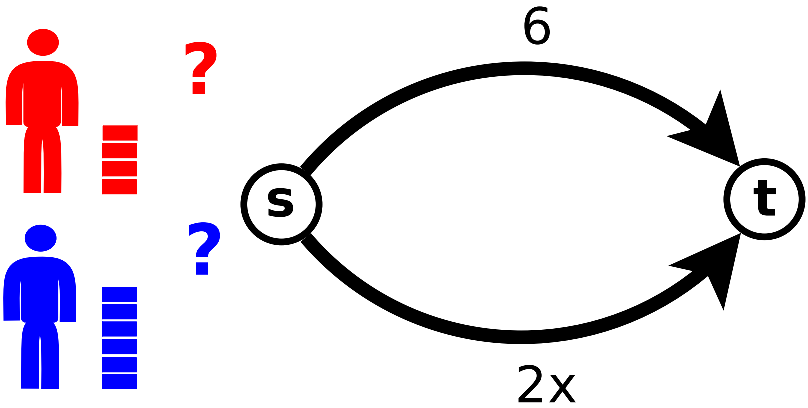
- Total delay $= 5 \times 4 + 1 \times 2 = 22$

Routing with Delays



- Total delay $= 5 \times 4 + 1 \times 2 = 22$
- Properties of delay functions

Routing with Multiple Users



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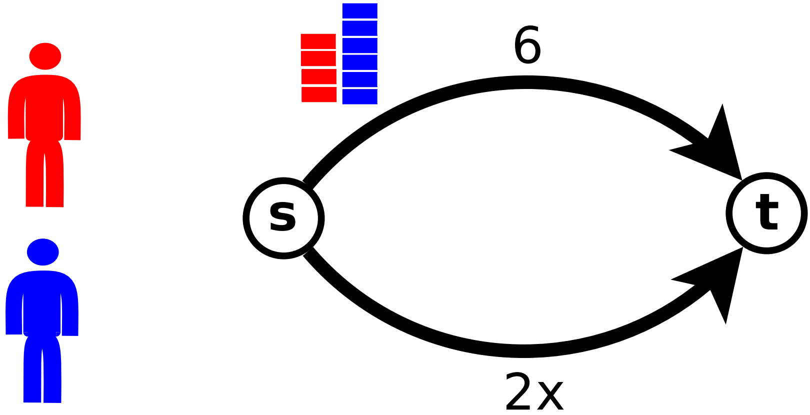

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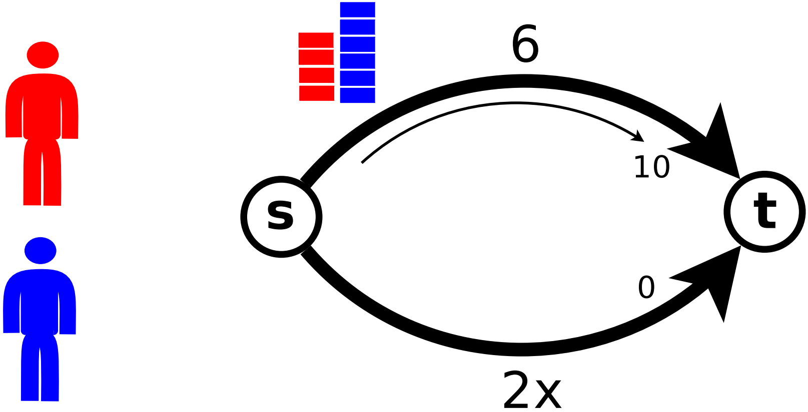

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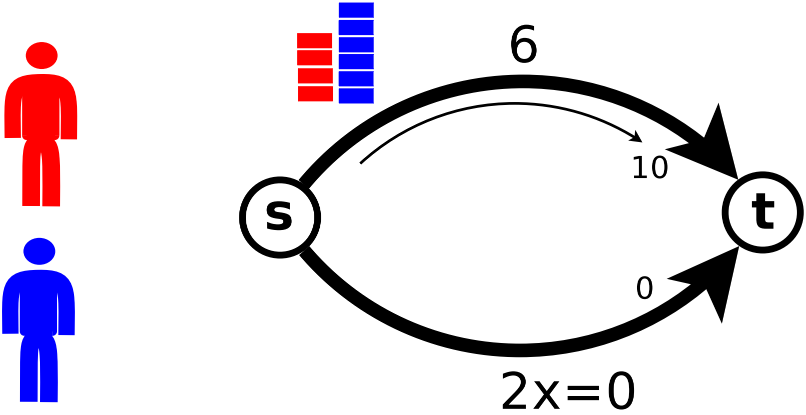

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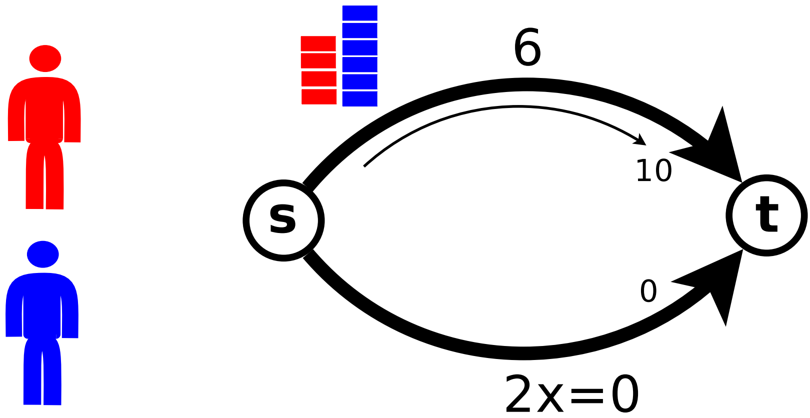

- Delay to player p = p's flow on edge $\times$ delay of edge

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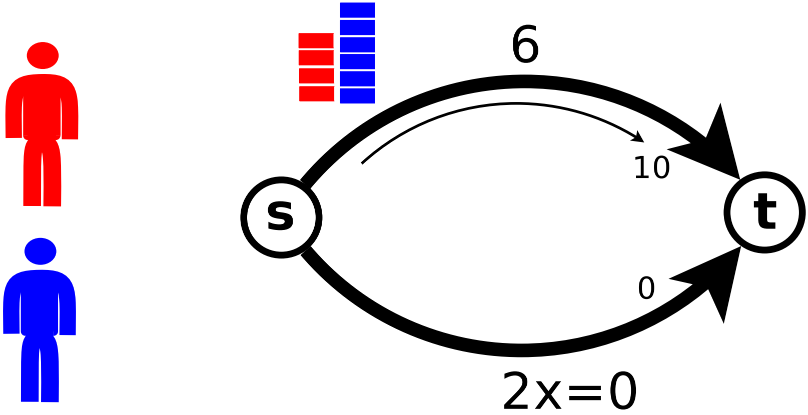
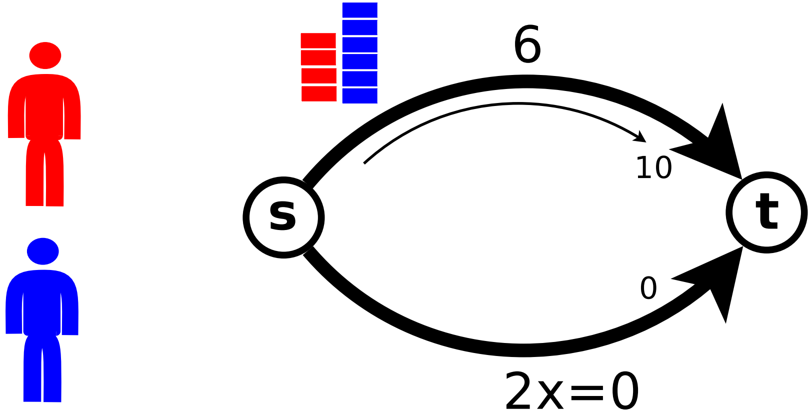


- Delay to player p = p's flow on edge $\times$ delay of edge
- Delay to red player $= 4 \times 6 = 24$

Routing with Multiple Users



- Delay to player p = p's flow on edge $\times$ delay of edge
- Delay to red player $= 4 \times 6 = 24$
- Delay to blue player $= 6 \times 6 = 36$

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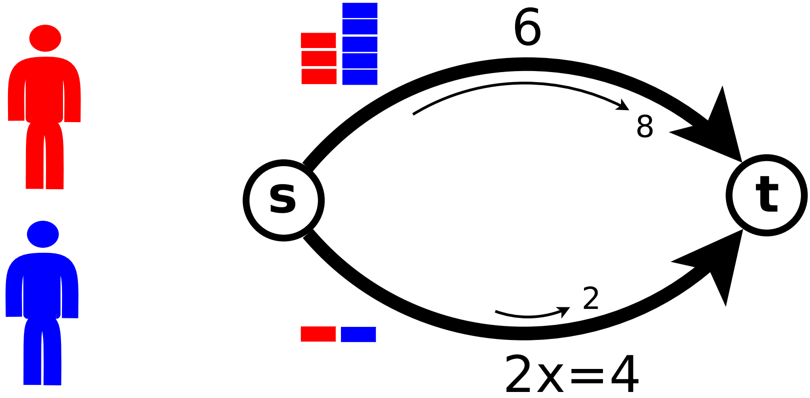

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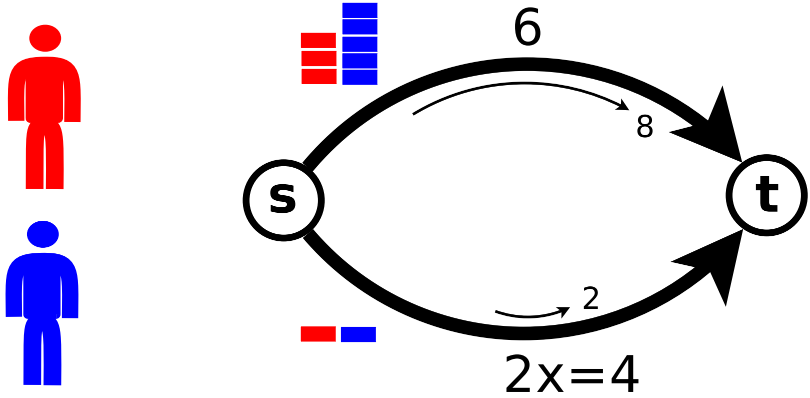

- Delay to red player $= 3 \times 6 + 1 \times 4 = 22$ (earlier 24)

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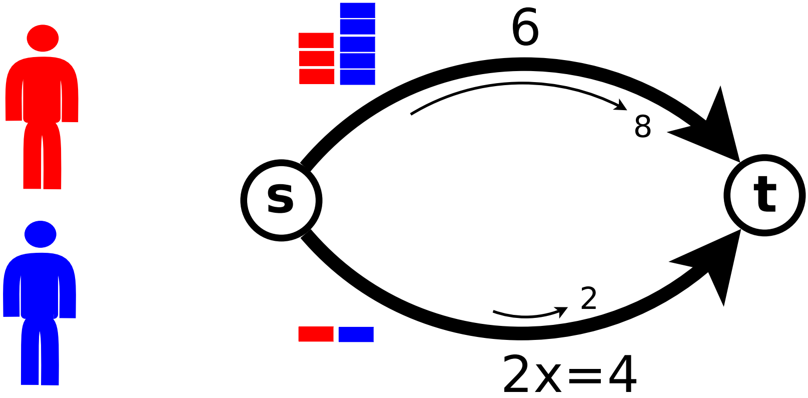
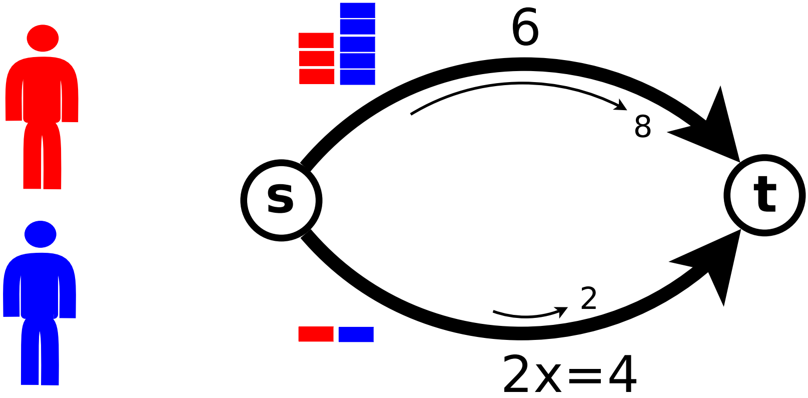


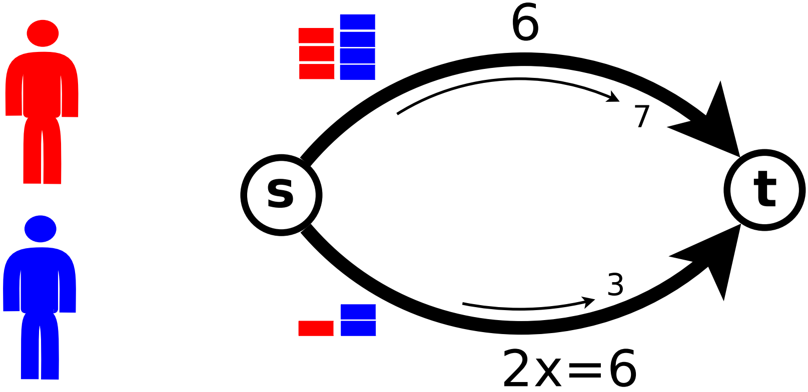
- Delay to red player $= 3 \times 6 + 1 \times 4 = 22$ (earlier 24)
- Delay to blue player $= 5 \times 6 + 1 \times 4 = 34$ (earlier 36)

Routing with Multiple Users



- Delay to red player $= 3 \times 6 + 1 \times 4 = 22$ (earlier 24)
- Delay to blue player $= 5 \times 6 + 1 \times 4 = 34$ (earlier 36)
- Neither player can reduce delay by changing its flow

Routing with Multiple Users



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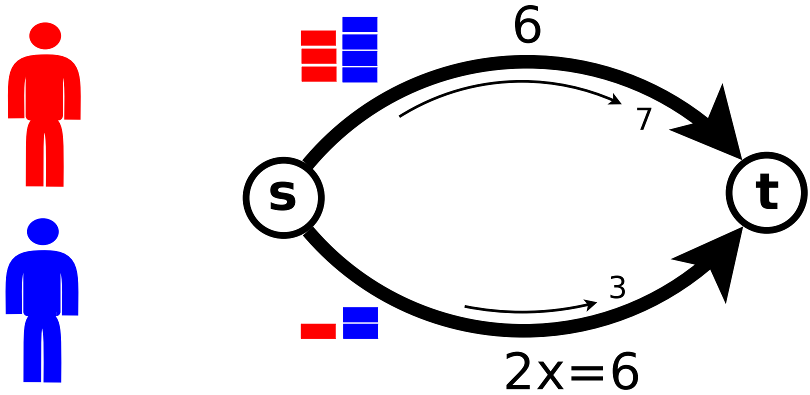

- Delay to blue player $= 4 \times 6 + 2 \times 6 = 36$

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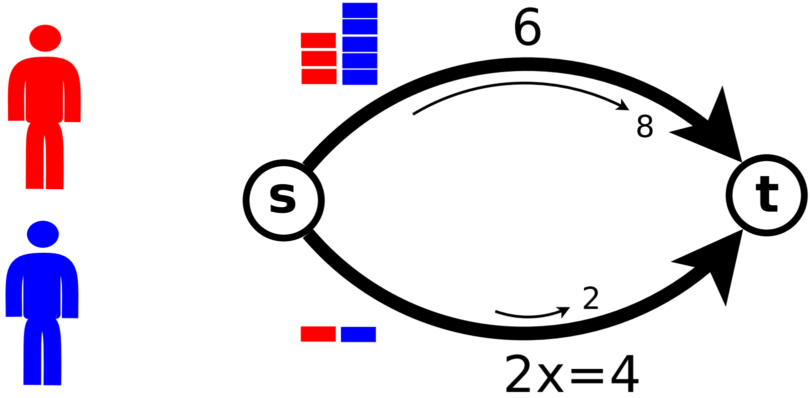

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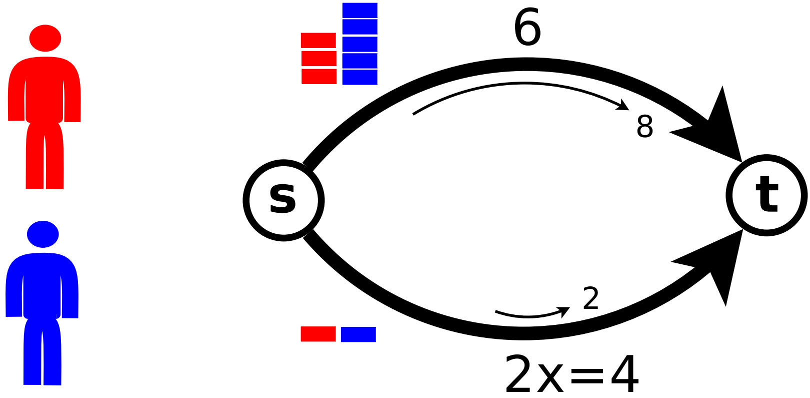
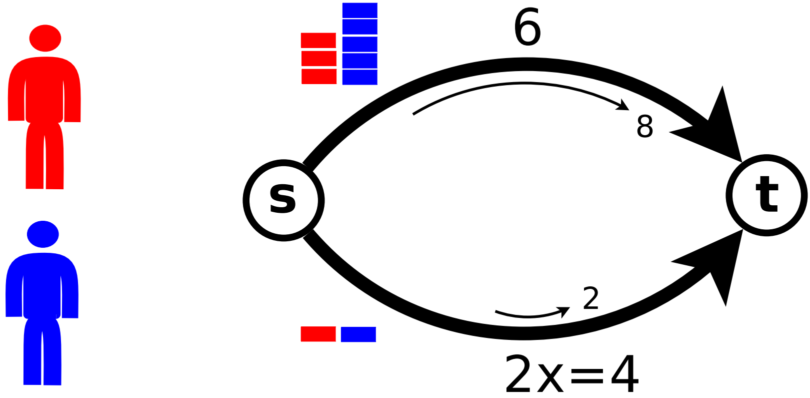


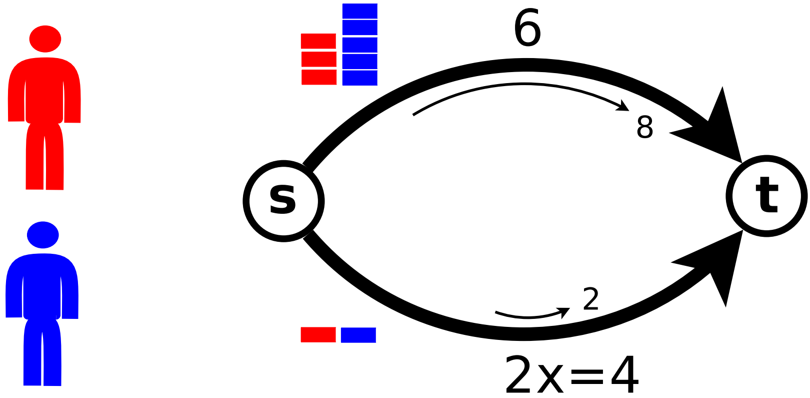
- Delay for red player = 22, delay for blue player = 34
- No user can reduce delay by changing its flow

Routing with Multiple Users



- Delay for red player = 22, delay for blue player = 34
- No user can reduce delay by changing its flow
- The flow is in "equilibrium"

Equilibrium



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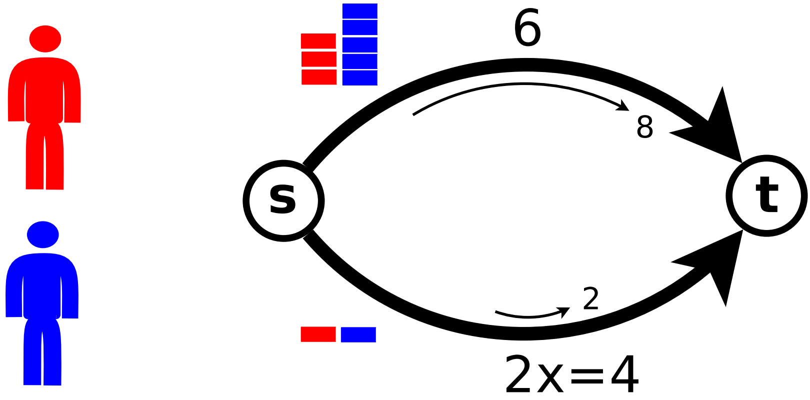

- A flow is in "equilibrium" if no user can reduce its delay by changing its own flow
- Equilibrium represents "stability"

Questions about Equilibrium

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  - Depends...

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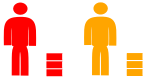
Players are of the same *type* if they have the same flow value



Two players of the same type

Four players, two types

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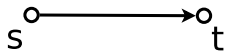
  - For players of more than 2 types, equilibrium is unique if and only if the network is *generalized nearly-parallel*

Series-parallel graphs

Defined inductively:

- Base case: a single edge is a series-parallel graph



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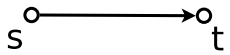

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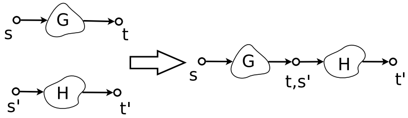

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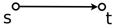

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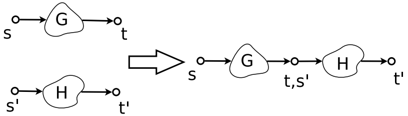
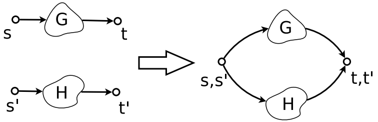
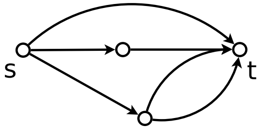


- or in *parallel*

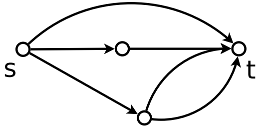


Series-parallel Graphs



A series-parallel graph

Series-parallel Graphs



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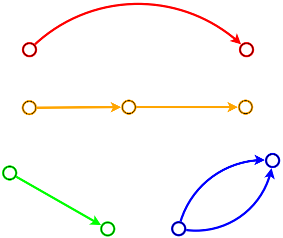

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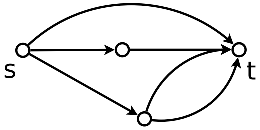

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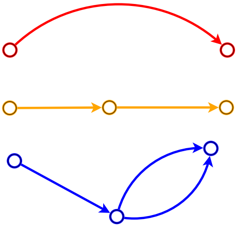

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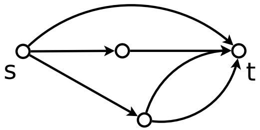

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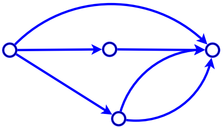

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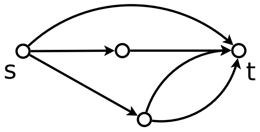

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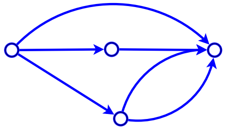
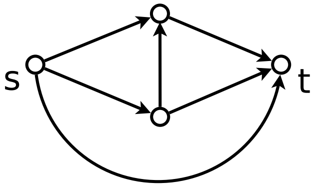


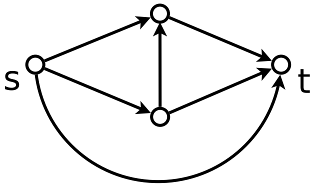
- *Generalized series-parallel graphs* are a slightly bigger class of graphs.

Series-parallel Graphs



A non-series-parallel graph

Series-parallel Graphs



A non-series-parallel graph

- This is the smallest non-generalized series-parallel graph

Uniqueness of Equilibrium

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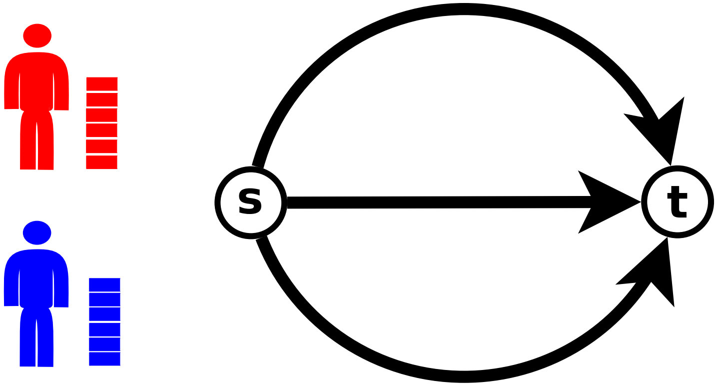
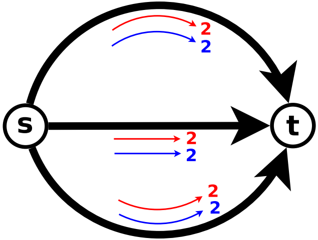
- Key idea: properties arising from difference of flows

The Difference of Two Flows



The Difference of Two Flows



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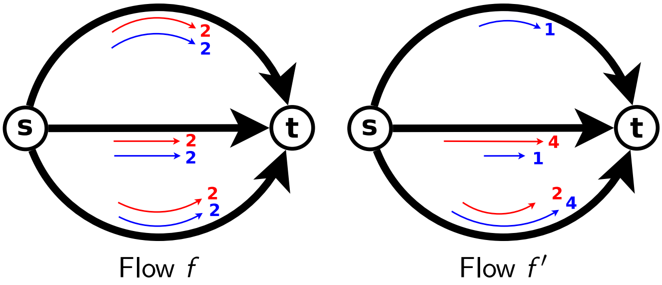

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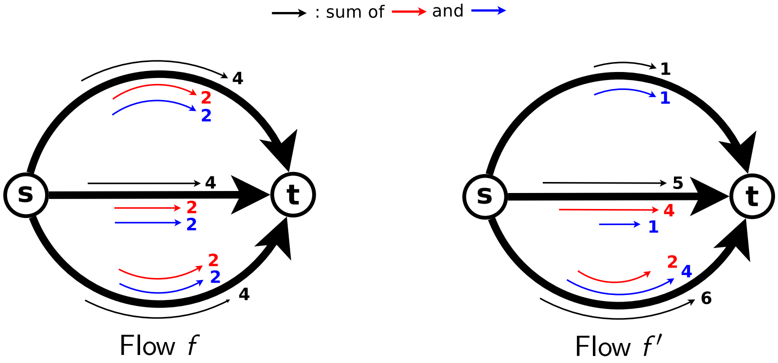
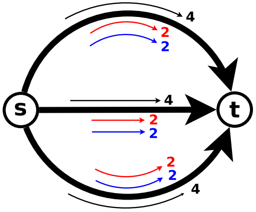
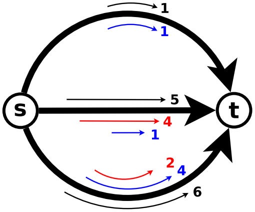


The Difference of Two Flows

→ : sum of → and →



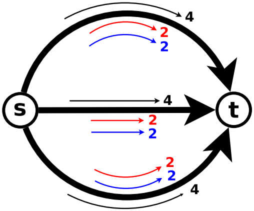
Flow f



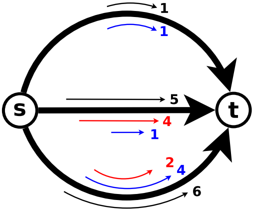
Flow f'

The Difference of Two Flows

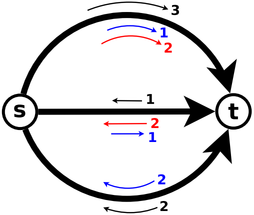
$\longrightarrow$: sum of $\longrightarrow$ and $\longrightarrow$



Flow f



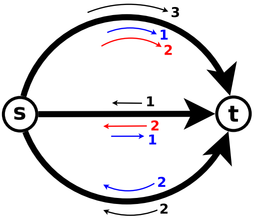
Flow f'



The difference of flows, $f - f'$

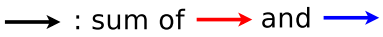
Agreeing Cycles

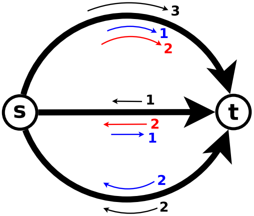
$\longrightarrow$: sum of $\color{red}\longrightarrow$ and $\color{blue}\longrightarrow$



The difference of flows, $f - f'$

Agreeing Cycles





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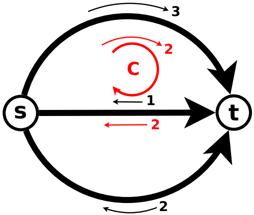
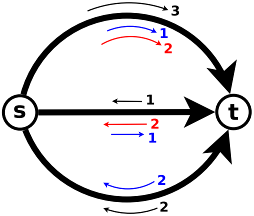


A red agreeing cycle

Agreeing Cycles

→ : sum of → and →



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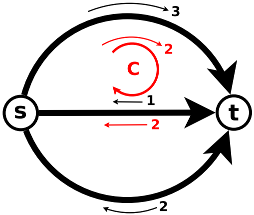

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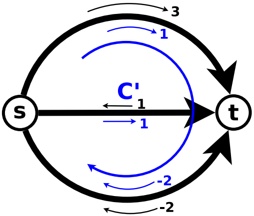
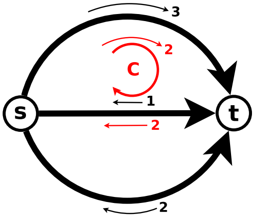


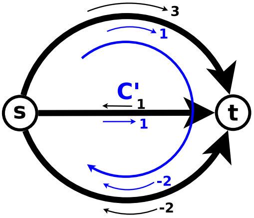
A blue agreeing cycle

Agreeing Cycles

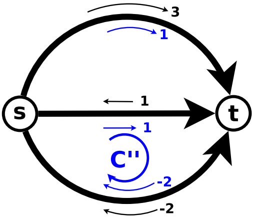
$\longrightarrow$: sum of $\longrightarrow$ and $\longrightarrow$



A red agreeing cycle

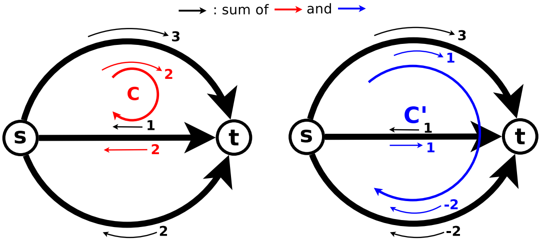


A blue agreeing cycle



Not an agreeing cycle

Agreeing Cycles



A red agreeing cycle　A blue agreeing cycle

A cycle C is a *p-agreeing cycle* if on every edge of cycle C, the direction of the total change in flow is the same as the direction of change in flow for player p.

Agreeing Cycles on Generalized Series-parallel Graphs

We show the following results in our paper:

- If f and f' are equilibrium flows, then the flow difference $f - f'$ cannot contain an agreeing cycle

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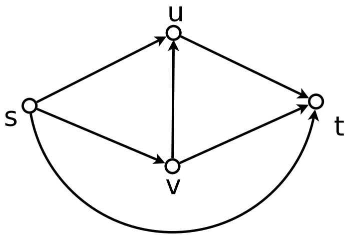
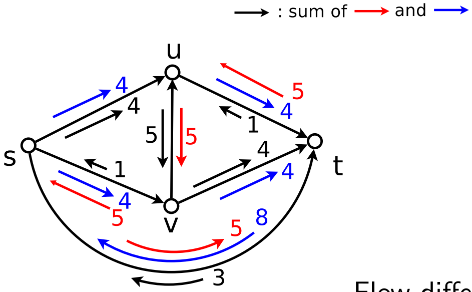
- For 2 players on generalized series-parallel graphs, the difference in two flows must contain an agreeing cyle
- Hence, for 2 players on generalized series-parallel graphs, there must be a unique equilibrium
- We explicitly construct an example of multiple equilibria on a non-generalized series-parallel graph for 2 players

Multiple Equilibria on a Non-generalized series-parallel Graph



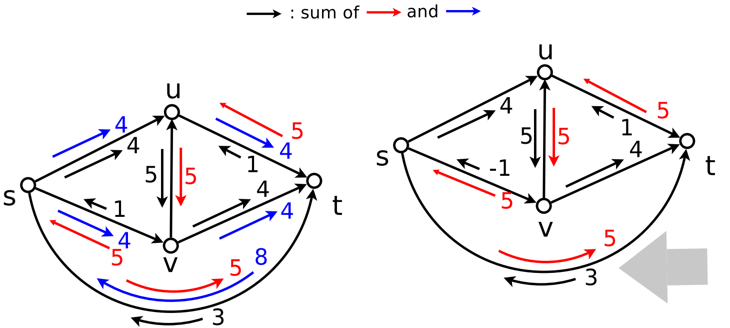
A non-generalized series-parallel graph

Multiple Equilibria on a Non-generalized series-parallel Graph



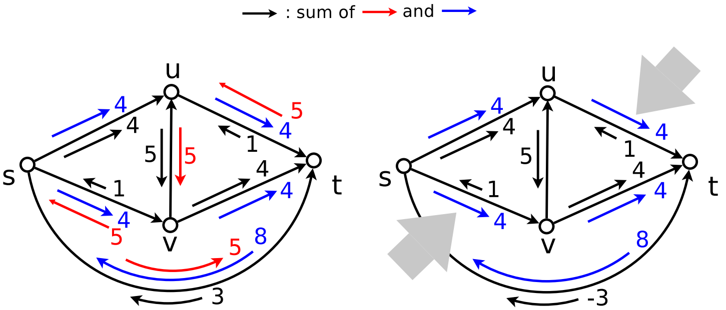
Flow difference on the graph

Multiple Equilibria on a Non-generalized series-parallel Graph



Red player does not have an agreeing cycle

Multiple Equilibria on a Non-series-parallel Graph



Blue player does not have an agreeing cycle

Multiple Equilibria on a Non-series-parallel Graph

- No agreeing cycles in the given flow difference

Multiple Equilibria on a Non-series-parallel Graph

- No agreeing cycles in the given flow difference
- We use this as basis to construct example of multiple equilibria in paper

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Summary

- Difference of two equilibrium flows cannot contain agreeing cycle
- For 2 players on generalized series-parallel graphs, difference of two flows contains agreeing cycles; hence equilibrium unique
- For more than 2 types of players on nearly-parallel graphs, difference of two flows contains agreeing cycles; hence equilibrium unique
- Give examples of multiple equilibria on non-generalized series-parallel graphs and non-nearly-parallel graphs

Questions?